Reconstructing with Moments

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Abstract

The relatively old problem of image reconstruction from a finite set of its geometric moments (inverse moment problem) is examined. The application of the information-theoretic method of Maximum Entropy (ME) is proposed and compared to the Legendre Moments (LM) approach. Several computer simulations are presented that demonstrate the superiority of the proposed technique. The comparative study is extended to include the case of noise-corrupted images.

1: Introduction

The mathematical concept of moments has been around for many years and has been utilized in many diverse fields ranging from mechanics and statistics to pattern recognition and image understanding. Very early, it was recognized that a truncated set of moment values could offer a more convenient and economical representation of an image segment than a pixel-format representation.

Hu [1] first presented results on how to achieve fundamental image transformations (ie, translation, rotation, scale change, etc) with the image moment representation. Dudani et al [2] used moment invariants, defined by Hu, to represent the shape of aircraft images. Later, Teague [3] extended Hu's moment invariants and introduced the concept of orthogonal moment sets to reconstruct the image from moments. Because of the wide applicability of image moments as features, there has been a lot of effort, recently, in designing fast computational algorithms [4,5] as well as VLSI implementations of moment-generating algorithms for real-time operation [6].

Referring back to the basic problem of image compression/reconstruction, it is well-known that an image can be fully reconstructed from the infinite number of its moments (since they are related to the coefficients of the power expansion of its characteristic function). Clearly this is of little use, unless it can be shown that a finite, **small** set of moments, usually available, can reconstruct the original image to within an adequate degree of accuracy. The interesting question, then, is how much information does a finite set of moments retain and what method or methods can be used to make a **best** reconstruction of the original image. An answer is sought after in this paper, where the proposed Maximum Entropy (ME) method is compared to the Legendre Moments (LM) method proposed by Teague [3]. The simulation experiments for noise-free and noise-corrupted images reveal the superiority of the first.

2: Basic Moment Concepts and Notation

The two-dimensional geometric moments (GM) of order $(p + q)$ of the image intensity function $f(x,y)$ are conventionally defined in terms of Riemann integrals as

$$M_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\,dxdy, \qquad (2.1)$$

$$(p,q = 0,1,2,..)$$

If $f(x,y)$ is piecewise continuous with bounded support (ie, has non zero values only in the finite part of the x-y plane), then moments of all orders exist. The double moment sequence $\{M_{pq}\}$ is uniquely determined by $f(x,y)$ and conversely $f(x,y)$ is uniquely determined by $\{M_{pq}\}$ [1].

When integrals are replaced by sums, eq (2.1) gives the moments of order $(p+q)$ for a digitized image segment $f(x,y)$, ie,

$$M_{pq} = \sum_x \sum_y x^p y^q \; f(x,y) \qquad (2.2)$$

A complete moment set (CMS) of order n consists of all the moments of order n and lower and can be represented by a triangular matrix of $(n+1)(n+2)/2$ moment values, as shown below:

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$$
\begin{matrix}
M_{00} & M_{01} \ldots & M_{0n} \\
M_{10} & & \cdot \\
\cdot & \cdot & \\
\cdot & \cdot & \\
M_{n0} & \cdot &
\end{matrix} \tag{2.3}
$$

3: The Legendre Moments (LM) Method

The definition of the geometric moment, as given by eq (2.1), has the form of the projection of the intensity function f(x,y) onto the monomial $x^p y^q$. However, the basis set $\{x^p y^q\}$, while complete (Weierstrass approximation theorem), is not orthogonal. Orthogonal moment forms [3] may be defined by using Legendre polynomial basis functions $P_n(x)$ rather than conventional monomials.

The Legendre polynomials of degree n, defined by

$$
P_n(x) = \sum_{j=0}^{n} C_{nj} x^j = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{3.1}
$$

$$
\text{where } \left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, & \text{for n even} \\ \frac{n-1}{2}, & \text{for n odd} \end{cases}
$$

are a complete orthogonal basis set over the range -1 to +1, ie,

$$
\int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2m+1} \delta mn \tag{3.2}
$$

where δmn is the kronecker delta.

The Legendre moments (LM) of order (p+q) are defined by

$$
L_{pq} = \frac{(2p+1)(2q+1)}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_p(x) P_q(y) f(x,y)\, dxdy
$$

$$
\text{where } p,q = 0,\ 1,\ 2,\ \ldots.. \tag{3.3}
$$

For the moments to be orthogonal, the image must be scaled to be within a 2 x 2 square centred at the origin.

Legendre moments may be obtained directly from geometric moments by

$$
L_{pq} = \frac{(2p+1)(2q+1)}{4} \sum_{j=0}^{p} \sum_{k=0}^{q} C_{pj} C_{qk} M_{jk} \tag{3.4}
$$

By using Legendre rather than geometric moments, an approximate inverse transform - to obtain f(x,y) from $\{L_{pq}\}$ - may be achieved by moment matching, ie,

$$
f_N(x,y) \simeq \sum_{j=0}^{N} \sum_{k=0}^{j} L_{j-k,k} P_{j-k}(x) P_k(y) \tag{3.5}
$$

which is a truncated series, with N the maximum order of Legendre moments available. In this method, obviously, all unknown moments ($n > N$) are assumed to be zero, which is not very legitimate, indeed.

4: Reconstructing with Maximum Entropy (ME)

The entropy maximization approach to the solution of underdetermined inverse problems (eg, the classical moment problem) has roots in the works of Shannon [7] and Jaynes [8].

The **maximum entropy formalism**, exploiting the concept of the **entropy** of a random variable, casts the problem of determining a pdf into the form of an optimization problem:

"Suppose a set of constraints on a probability distribution is known, but the constraints do not completely determine the distribution and nothing more about the distribution is known. Then the least prejudiced or biased assignment is the pdf of maximal (Shannon) entropy satisfying the given constraints."

This principle can be easily applied to the image reconstruction problem from a finite set of moments. Any irradiance distribution f(x,y), being non-negative, can be considered as a pdf, because it can be, also, easily normalized to integrate to unity. The given moments are the constraints in our optimization problem, which can be stated mathematically as:

Maximise:

$$
H = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \log[f(x,y)] dxdy \tag{4.1}
$$

Subject to:

$$\int_{-\infty}^{\infty} x^p y^q f(x,y)dxdy = M_{pq}, \tag{4.2}$$

$$\text{for } p+q = 1,2,\ldots,N$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 = M_{oo} \tag{4.3}$$

Solution of this standard variational problem yields

$$f_N(x,y) = \exp\left[-\lambda_o - \sum_{p,q} \lambda_{pq} x^p q^q\right] \tag{4.4}$$

where the λ's are Lagrange multipliers determined from the constraint conditions. The symbol $f_N(x,y)$ is used to point out that it is an estimate of the unknown intensity function f(x,y), based on moments up to order N.

The most important difference between the two methods is that, while the LM approach assumes all the unknown (or not given) moments to be zero, the ME method results in an estimate which is maximally non-committal with regard to missing information (ie, unknown moments).

The ME reconstruction of moment-compressed images, first proposed in 1985 by Papademetriou [9], may produce more accurate estimates than the LM method using appreciably fewer moments. The need for more computational time in the ME case becomes continuously less restrictive, because of the steadily increasing computing power available today.

5: Computer Simulations

A: Reconstruction of Noise-Free Images

Figures 1 and 2 show examples of reconstruction of images (binary-valued figures) from their moments, using both methods. In Figure 1 the assumed test image is one-dimensional, the step function, in the interval [-1,+1], made up from four hundred and forty-one discrete samples. For ease of visual comparison, the original step function is shown on top of all estimates. In Figure 2, however, the test image is a two-dimensional capitalised letter, defined across a 21 x 21 pixel array. Although the presented reconstructions are thresholded versions of the continuous ones, all mean square error calculations were developed by comparing the original and the actual reconstructed image (not the threshold image).

The normalized mean square reconstruction error between an image f(x,y) - defined over a region D of the xy-plane - and its reconstructed version $\hat{f}(x,y)$ from a finite set of its moments (up to order N), defined by

$$\overline{e}^2(N) = \frac{\int\int_D [f(x,y) - \hat{f}(x,y)]^2 dxdy}{\int\int_D [f(x,y)]^2 dxdy} \tag{5.1}$$

is considered as a good measure of the image reconstruction ability of the moments; so, it has been adopted, here, for comparing the performance of the two methods. Figures 1c and 2c present this error as a function of N. The superiority of the entropic method is obvious; an ME reconstruction even with moments up to only the 8th order is much better than an LM reconstruction based on moments up to the 18th order.

B: Reconstruction of Noisy Images

Figure 3 shows simulation results of the reconstruction of a noisy image (the 1-d letter F) by including increasingly higher-order moments. As measure of the amount of noise present on the image, the signal-to-noise ratio (SNR) is used, which is defined, here, as the ratio of the image energy per unit area to the noise variance, ie,

$$\mathrm{SNR} = \frac{\frac{1}{S_D}\int\int_D [f(x,y)]^2 dxdy}{\sigma_n^2} \tag{5.2}$$

where S_D is the area of the region D.

The normalized reconstruction error, given in Figure 3 as a function of N (the order of moments used) with the SNR as a parameter, is calculated by averaging over five noisy realizations generated for each SNR value.

From the simulation results we come to a conclusion pertaining to both methods: For each SNR value, there is a certain optimal order of moments, which leads to the best image reconstruction. Since higher-order moments are generally more sensitive to image noise, using moments of order higher than the optimal will result in larger reconstruction errors. This is clearer at high noise level, in both methods, but slightly more in the ME method.

Moving, however, down to lower noise levels, which is of practical interest here, the ME approach proves

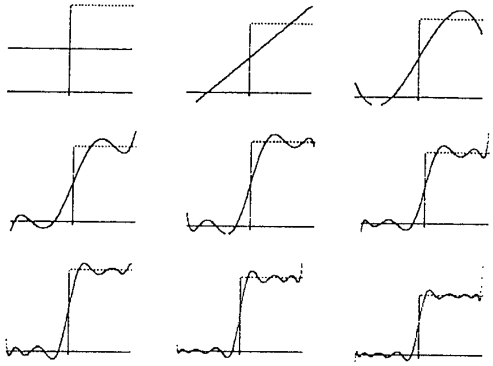

Figure 1a: LM Reconstructions of the Step Image

Top row, left to right: zeroth, first and third order moments. Centre row, left to right: sixth, eighth and tenth order moments. Bottom row, left to right: twelfth, fifteenth and finally eighteenth order moments.

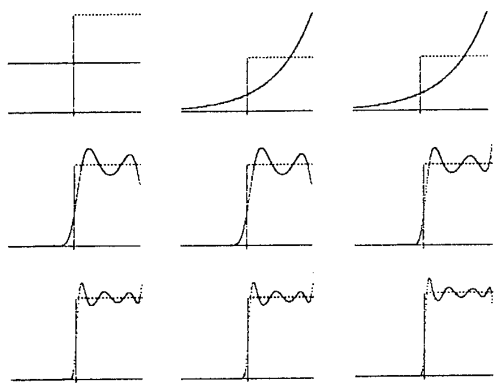

Figure 1b: ME Reconstructions of the Step Image

Top row, left to right: zeroth through second order moments. Centre row, left to right: third through fifth order moments. Bottom row, left to right: sixth through eighth order moments.

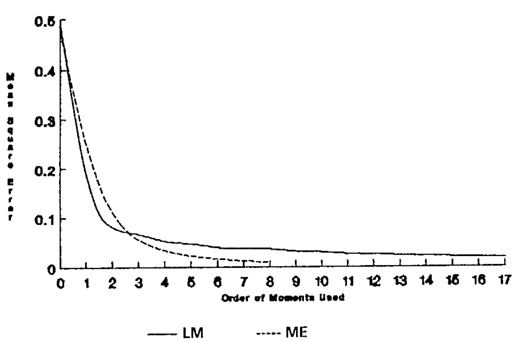

Figure 1c: MSE vs order of retained moments

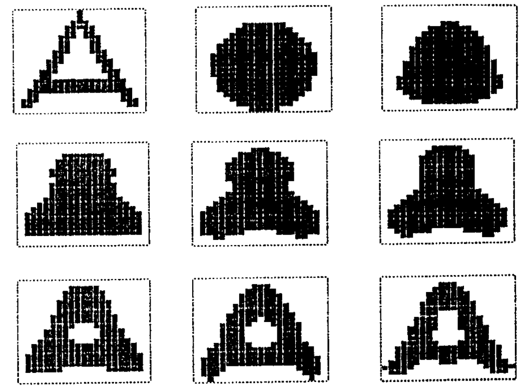

Figure 2a: LM Reconstructions of the Two-Dimensional Letter A

Threshold Images. Top row, left to right: second and forth order moments. Centre row, left to right: sixth, eighth and ninth order moments. Bottom row, left to right: eleventh, fifteenth and eighteenth.

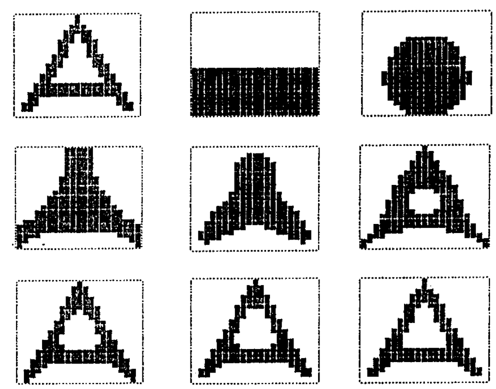

Figure 2b: ME Reconstructions of the Two-Dimensional Letter A

Threshold Images. Top row, left to right: original, first and second order moments. Centre row, left to right: third through fifth order moments. Bottom row, left to right: sixth through eighth order moments.

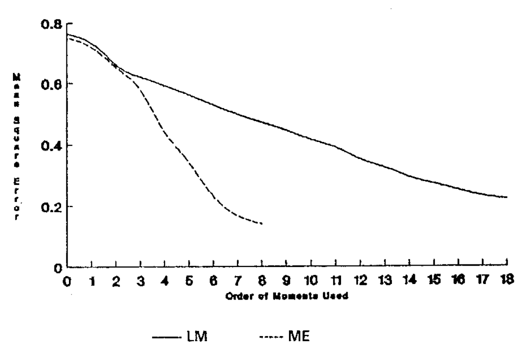

Figure 2c: MSE vs order of retained moments

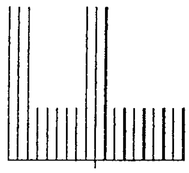

Figure 3a: Original 1-d letter F

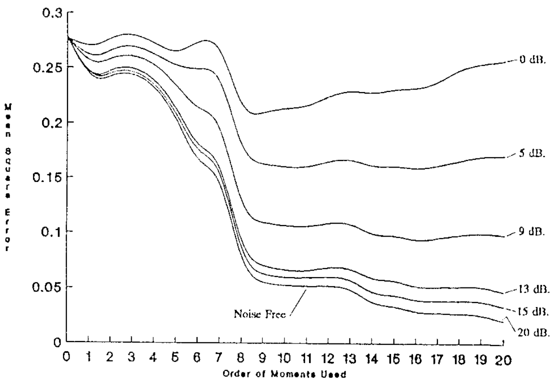

Figure 3b: Effect of Gaussian noise on the reconstruction MSE for the LM method

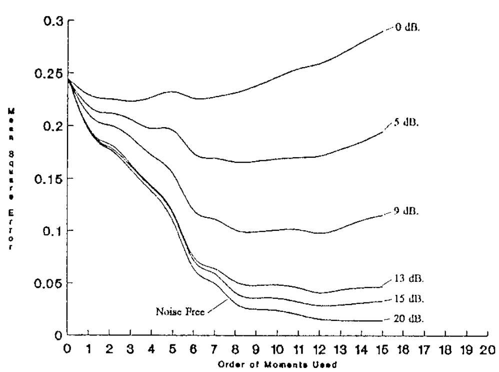

Figure 3c: Effect of Gaussian noise on the reconstruction MSE for the ME method

again its superiority over the LM method, by giving the same reconstruction error with a much smaller number of retained moments, eg, at 20 dB, an MSE = 0.05 is achieved by using moments of up to 7th order in ME and up to 13th order in LM, or an MSE = 0.025 with 9th order moments in ME and 19th order in LM.

6: Conclusions

The above comparative analysis shows that the ME method outperforms the LM approach even under noisy-image conditions. This efficiency of the entropic method does not come without any cost; and this is the computational cost. However, because of the success of the method in leading to appreciably higher compression ratios, this drawback can be overlooked, since it can be eventually eliminated by designing more efficient, fast algorithms and special purpose VLSI architectures.

7: References

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