GENERATING FUNCTIONS OF ABSTRACT GRAPHS WITH SYSTEMS APPLICATIONS

A thesis presented

by

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SYNOPSIS

This thesis is concerned with the concept, properties and application of generating functions of abstract graphs. Many practical problems like code generation, path enumeration, linear shift register sequences, sampled data systems, discrete Markov processes, and certain connectivity considerations in automata theory can be handled in a unified manner using these techniques. This unified treatment of all these processes using the generating functions of directed graphs is original. This approach is developed in Chapters 1 through 5.

Chapter 1 gives a brief historical introduction to the quantitative aspects of graph theory.

Chapter 2 defines the generating function of a directed graph when its branch transmittance matrix is known.

Chapter 3 deals with some properties of the generating function and its relationship to a set of simultaneous linear equations and to the loop structure of the graph. We prove that the generating function can be expressed as a ratio of two characteristic functions, which depend only on the loop structures of their graphs. One is the original graph and the other is a modified version of the former.

Chapter 4 deals with the interpretation of the generating function in the contexts of simultaneous linear equations, path enumeration, code generation, sampled data systems, discrete Markov processes, and shift register sequences. This unified treatment is original.

Chapter 5 is concerned with the application of generating functions to certain problems in communication and computer systems. A standard graph is taken and interpreted in various systems contexts. Useful information about the appropriate system representation is derived. Much of this chapter is original.

Chapter 6 treats the connectivity of directed graphs as a property of their generating functions. Some new theorems are given. Tests and algorithms are presented to test the well formation of graphs, the determination of redundant nodes, and the detection of loops in graphs. There is also a test for strong connectivity, an algorithm for partitioning the graph into maximal strongly connected subgraphs, and an algorithm for partitioning the graph into disconnected subgraphs. A new theorem and an algorithm are given for complete factoring of determinants.

The whole chapter is original except for the Marimont test and a theorem derived from Mason's work.

Chapter 7 classifies graphs as small, large well-ordered, and large random. For the first two, we show that the topological method of deriving the generating function appears best. The method of node reduction is proposed for large random graphs. We provide a test to find the complexity of a given graph in terms of its loop structure to determine which method to use in deriving the generating function. Our node reduction procedure minimizes the amount of computation at every stage using the connectivity matrix of the graph. We also discuss explicit solutions of the characteristic difference equation from which the coefficients of expansion of the generating function can be found. The classification, the tests, and the node reduction procedure using the connectivity matrix are original.

Chapter 8 gives a model of digital waveforms generally used in discrete recording schemes. We classify the parameters in such systems as cyclical or as non-cyclical. Since synchronization and other control information is indicated by certain transitions in these parameters, we develop techniques of enumeration of waveforms according to the specific types of parameter transitions they contain. We apply these techniques to transitions in multi-level, multi-parameter systems and to certain prefixed comma-free systems. The whole chapter is original.

Chapter 9 illustrates the power of the graph generating function approach to complex combinatorial problems. We define certain constrained codes, and show that these can be the basis of many practical recording and communication schemes. We develop some very general types of codes, and give their generating functions explicitly. We give an application of a constrained code in an actual recording scheme. We also develop a technique for "minimum" redundancy encoding of these codes. The whole chapter is original.

Chapter 10 develops the path enumeration viewpoint of linear shift register sequences. The generating function method is much simpler than the existing methods of finding the outputs of any stage of a shift register generator (SRG). We give the analysis of SRG's with specified initial conditions and extend it to the case where some inputs are injected externally and are functions of time. We develop equivalent SRG's by path inversion and develop the generating functions of complementary sequences. We propose and design maximal symmetric sequence generators. The whole chapter is original.

In Chapter 11, we compare our methods with those developed by Gilbert for enumerating prefixed comma-free codes and show that our methods are simpler and more general. We extend our methods to enumerate prefixed comma-free encodings with distance-2 property. We derive a necessary condition for the existance of such multi-distanced prefixed codes. Since the number of sequences with multi-distanced prefixes are small, we propose methods of locally reinforcing the prefix in the vicinity of its occurance. We find the generating functions and enumerate some sequences. Except for the comparison with Gilbert's work, the whole chapter is original.

GENERATING FUNCTIONS OF ABSTRACT GRAPHS WITH SYSTEMS APPLICATIONS

ABSTRACT

This thesis is concerned with the concept, properties and applications of generating functions of abstract graphs. Many practical problems like code generation, path enumeration, shift register sequences, samples data systems, discrete Markov processes and certain connectivity considerations in automata can be handled in a unified manner using these techniques.

The generating function of a graph is a function of the complex variable z which has the property that interesting attributes of the graph can be extracted from it by numerical operations.

The computation of the generating function involves either matrix inversions or application of formulas that take into account the topological characteristics of the graph. When the graph has certain orderly topological features, the topological method gives the generating function explicitly even when the graph contains an extremely large number of nodes. However, if the graph has been chosen at random, often no advantage can be derived from a topological characterization. The matrix method can then be applied, but computational complexity restricts the size of graphs which can be analyzed.

I. INTRODUCTION

The origin of graph theory can be traced back to Euler and Kirchhoff who utilized it for visualizing and elucidating certain relationships in physical science as an aid towards their mathematical formulation.

Purely algebraic formulations of physical quantities generally ignore the peculiar and interesting situations existing in a particular system. Such is the case when one sets up a set of linear equations to describe the current-voltage relationships in an electrical network and solves the same by matrix methods. An ideal method would take into consideration the particular physical situations associated with the system at each stage of manipulation. Also when the physical relations in a system are represented by a graph, certain topological aspects look evident, thereby enabling one to characterize the problem also from a topological (graph-theoretic) viewpoint.

Mason [1, 2] developed the signal flow graph primarily as a tool to solve steady-state linear network problems. He showed that a linear circuit may be conveniently represented by a signal flow graph which can be analyzed in detail for its topological structure. The major result of his work was the formula that calculated the gain between two nodes in a flow graph.

Coates [3] modified the conventions involved in Mason's flow graph theory, to obtain some simplification in the final computation of the signal gain. More innovations were later added either to accomodate special situations or to "unify" the Mason and Coates approaches. These attempts are variously known as N-graphs, matrix graphs, etc. [7, 8]. We shall only use Mason's conventions in this report.

Widrow [9] applied Mason's signal flow graph theory to linear time invariant sampled-data systems. Howard, Lorens and primarily Sittler [4, 5, 6] recognized the analogy between the linear sampled data systems and discrete Markov processes with constant transition probabilities and extended the range of applicability of the signal flow graphs. Since discrete Markov processes can be represented by transition or state diagrams with a set of constant transition probabilities between various states, one can surmise the applicability of the results of signal flow graphs to the study of sequential machines and automata theory. Indeed such an approach [11] was iniated only recently.

1.1 Main results of this thesis

Upto now, quantitative graph-theoretic approaches have been applied in solving certain special physical problems (like electric networks, and discrete Markov processes with constant probabilities). We shall take the opposite viewpoint in this report. We will study the quantitative properties of abstract graphs by means of generating functions and characteristic functions. We shall derive these functions without regard to any physical processes they may represent. Next, we shall interpret the physical meaning of the graph, its generating function and its characteristic function in terms of actual processes.

An abstract graph through its generating and characteristic functions enables us to evaluate or study a variety of attributes like the following: the transfer function of a sampled data system, the transient or steady-state probabilities in a discrete Markov process, the number of distinct ways of of reaching from one set of nodes to another set of nodes in exactly n steps, the output of a delay element (or an adder) in a shift register generator, the average execution time and variance of a computer program, the degree of asynchronism of a multiplier unit in a computer, the number of distinct waveforms in a multi-level and multi-parameter discrete recording (or communication) system in which each parameter is separately constrained; the maximum number distinct code words of exactly n bits in which no more than k_o but not less than m_o "zeroes" can occur consecutively and no more than k_1 but not less than m_1 "ones" can occur consecutively in any sequence of the code words; and the characterization of strongly connected automata by their generating functions, and the generation and the enumeration of prefixed, easily synchronizable codes with certain comma-free properties.

Since a particular graph can be given different interpretations in different physical processes, one can apply the body of knowledge in one system (e.g., synthesis procedures in sampled data systems) to problems in another (e.g., design of sequential machines). Also, solutions to a complex systems problems in one area can be obtained by simulating their easily-obtained analog in another area.

2. GENERATING FUNCTION AND GRAPH TRANSMISSION

2.1 Definition

An abstract graph is a set of nodes connected by directed branches and such that a complex number called the branch transmission is attached to each branch. If the graph has n nodes, an n x n transmission matrix $\underline{T}$ can be constructed which exhibits the relationships between the nodes. This matrix is unique to the graph upto a permutation of the entries due to the renumbering of the nodes.

A branch is represented by a directed line joining two nodes with an arrow. The branch transmission generally specified by an attached number and its direction, specified by the arrow completely determine the relationship between the two nodes.

In this report, we will call t_{ij} as the branch transmission from node i to node j

Example

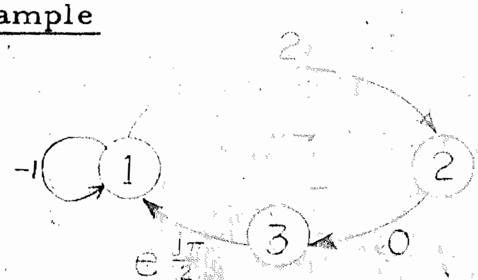

FIG. 2.1 A DIRECTED GRAPH

$$\underline{T} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ e^{\frac{j\pi}{2}} & 0 & 0 \end{bmatrix}$$

Two rules govern the manipulation of transmissions:

1. Addition Rule

The branches in parallel between two nodes can be replaced by a single branch with a transmission which is the sum of the transmissions of individual branches, if the branches are of the same direction.

$$t_{ij} = \sum_{s=1}^{r} t_{ij}^{(s)}$$

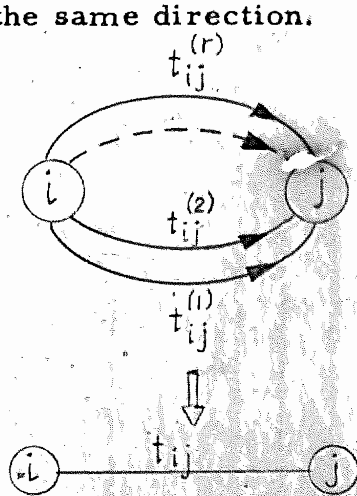

FIG. 2.2 ADDITION RULE

2. Multiplication Rule

Branches in series can be represented by a single branch with a transmission which is equal to the product of the original branches.

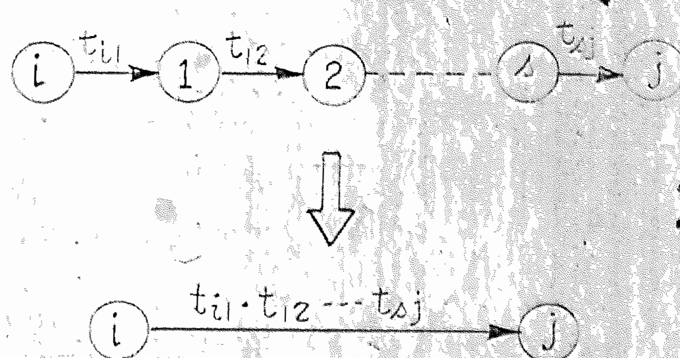

FIG. 2.3 MULTIPLICATION RULE

2.2 Interpretation of graph transmission

The use of the name "transmission" to express the relation between two nodes should not convey the impression that we are restricting our discussion to electrical networks and signals. It has a purely abstract, non-dynamic interpretation.

For example, in Fig. 2.2, let $t_{ij}^{(s)}$ for any s be equal to 1; then $t_{ij} = \sum_{s=1}^{r} t_{ij}^{(s)} = r$ is a positive integer. If each $t_{ij}^{(s)}$ represented a mutually exclusive way of reaching node j from i, then t_{ij} is the total number of different ways of reaching j from i. Similarly if the number of mutually exclusive ways of reaching node 1 from node i, node 2 from node 1; node 3 from node 2; ..., and, node j from node s are t_{i1}, t_{12}, $t_{23} \ldots t_{sj}$, respectively, then the total number of different paths of reaching node j from node i by passing through nodes 1, 2, ... s in sequence, is given by the product of the transmissions namely $t_{i1} \cdot t_{12} \cdots t_{sj}$. Thus the addition and the multiplication rules indeed apply to the paths in a graph. This interpretation is valid only in the case in which branch transmissions are integers.

The applicability of the addition and multiplication rules also exhibit themselves in electrical systems where a complex system can be formed from its components a) by cascading thereby multiplying the transfer functions and b) by paralleling thereby adding transfer functions. Similarly in a discrete

Markov process, the probability of a compound event can be built-up in the same way, viz., if the component events are statistically independent the component probabilities are multiplied; and if the component events are mutually exclusive, the component probabilities are added.

2.3 The concept of generating function

Discrete probability theory [10] makes use of the generating function concept to solve a number of problems. This function is similar to the ones used in generating Bessel, Legendre or hypergeometric functions. Its use in solving combinatorial problems is well known [22].

Given an abstract graph, whether representing sequential processes, signal flows or connectivities, there arises a need to know the total transmission from one set of nodes to another set as a function of number of branches traversed. Or, from the path enumeration viewpoint, it is required to know the number of different ways of reaching from one set of nodes to another set as a function of number of branches traversed in series. Such a transmission (e.g., path enumeration) function is called the generating function between the given set of nodes (terminal nodes) and is of considerable interest in the study of communication and transport networks, information retrieval and biophysics [14, 15, 16 and 17]. Indeed, the fundamental study of abstract set theory is concerned with specification of subsets of a given set and enumeration of elements in them. The generating function is a storehouse of information about the given graph and one can get from it many interesting properties of the graph by certain operations.

3. PROPERTIES OF THE GENERATING FUNCTION

3.1 Definition of the generating function

Let $(\underline{e}^i)$ be a row vector of n elements such that its jth element is

$$(\underline{e}^i)_j = 1 \quad \text{if } i = j \qquad (3.1)$$
$$= 0 \quad \text{otherwise}$$

Let $(\underline{e}_k)$ be a column vector of n elements such that

$$(\underline{e}_k)_j = 1 \quad \text{if } k = j \qquad (3.2)$$
$$= 0 \quad \text{otherwise}$$

We define the real numbers g_j^{ik} as follows:

$$g_0^{ik} = (\underline{e}^i)\,[\underline{T}]^0\,(\underline{e}_k)$$
$$g_1^{ik} = (\underline{e}^i)\,[\underline{T}]^1\,(\underline{e}_k)$$
$$\vdots$$
$$g_P^{ik} = (\underline{e}^i)\,[\underline{T}]^P\,(\underline{e}_k) \qquad (3.3)$$

in which $[\underline{T}]^0$ is the identity matrix $\underline{I}$, and $[\underline{T}]^n = \underbrace{\underline{T}\,\underline{T}\,\underline{T}\,\ldots\,\underline{T}}_{n \text{ terms}}$.

In the case of a sampled data system g_j^{ik} can be interpreted as the transmission from node i to node k (the sum of all different transmissions traversing j branches in series from node i to node k), which is defined to be equal to the signal appearing at the node k per unit of external signal injected into some specified node i. Analogously in the case of path enumeration g_j^{ik} is the total number of ways of reaching node k from node i by traversing in exactly j steps in series.

The generating function from node i to node k is defined by

$$G_{ik}(z) = \sum_{j=0}^{\infty} g_j^{ik} z^j$$

$$= \sum_{j=0}^{\infty} (\underline{e}^i)[\underline{T}]^j z^j (\underline{e}_k)$$

$$= \underline{e}^i \sum_{j=0}^{\infty} [\underline{T}z]^j \underline{e}_k \qquad (3.4)$$

in which z is a complex valued independent variable such that $|z| < \frac{1}{M}$. We shall next show that $G_{ik}(z)$ is well-defined.

Let M be the largest row sum of $\underline{T}$. That is,

$$M = \max_i \sum_{j=1}^{n} |t_{ij}| \qquad (3.5)$$

Then, by the definition of M, $\max_i |t_{ij}| \leq M$ and,

$$\max_{i,j} |t_{ij}^{(K)}| \leq M^K \qquad (3.6)$$

in which $t_{ij}^{(K)}$ is the ijth element of $[\underline{T}]^K$

Therefore, if $|z| < \frac{1}{M}$ then $\sum_{j=0}^{\infty} [\underline{T} z]^j$ converges to a limiting n x n matrix which implies that $G_{ik}(z)$ is well-defined.

3.2 Representation of the generating function

If $\underline{I}$ is an n x n identity matrix then

$$[\underline{I} - \underline{T} z] \left\{ \sum_{j=0}^{\infty} [\underline{T} z]^j \right\}$$

$$= \sum_{j=0}^{\infty} \underline{T}^j z^j - \sum_{j=0}^{\infty} \underline{T}^{j+1} \cdot z^{j+1} = \underline{I} \qquad (3.7)$$

Thus $$\sum_{j=0}^{\infty} [\underline{T} z]^j = [\underline{I} - \underline{T} z]^{-1} \qquad (3.8)$$

therefore $$G_{ik}(z) = \underline{e}^i [\underline{I} - \underline{T} z]^{-1} \underline{e}_k \qquad (3.9)$$

Let $[\underline{I} - \underline{T} z] = \underline{A}$ then $\underline{A}^{-1}$ is the matrix as shown below

$$\underline{A}^{-1} = \begin{bmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \cdots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \cdots & \frac{A_{n2}}{|A|} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & \cdots & \frac{A_{nn}}{|A|} \end{bmatrix} \qquad (3.10)$$

where A_{ij} is the co-factor of the ij th element of $\underline{A}$ and $|\underline{A}|$ is the determinant of $\underline{A}$.

Thus

$$G_{ik}(z) = \frac{A_{ki}}{|\underline{A}|} \qquad (3.11)$$

We have proved then the following result.

The generating function of an abstract graph from node i to node k is given by the i k th element of the matrix $[\underline{I} - \underline{T} z]^{-1}$ where $\underline{T}$ is the transmission matrix of the graph; $\underline{I}$ is an identity matrix and z is a complex valued variable restricted to values $|z| < \frac{1}{M}$.

Later we find it useful to define the generating function from one subset of nodes to another subset of nodes. The terminal nodes in all cases will be evident from the context.

3.3 Relationship between the generating function of an abstract graph and a set of linear equations

We shall next show that finding the i k th element of the matrix $[\underline{I} - \underline{T} z]^{-1} = \underline{A}^{-1}$ is equivalent to solving for a particular variable in a set of linear algebraic equations in n unknowns.

$$\text{i k the element of } [\underline{I} - \underline{T} z]^{-1} = \frac{A_{ki}}{|\underline{A}|} \qquad (3.12)$$

Let $\underline{B} = [\underline{I} - \underline{T}^T z]$ where $\underline{T}^T$ is the transpose of $\underline{T}$.

$$\text{Then } \underline{B} = [\underline{I}^T - \underline{T}^T z] = [\underline{I} - \underline{T} z]^T = \underline{A}^T \qquad (3.13)$$

$$\text{Thus } |\underline{A}| = |\underline{A}^T| = |\underline{B}| \qquad (3.14)$$

And we then have the following relationships among the co-factors:

$$\underline{A}_{ki} = \underline{A}^{T}_{ik} = \underline{B}_{ik} \tag{3.15}$$

Let $X_i [\underline{e}_i]$ be a column vector whose ith component is X_i and all others zero. Define another column matrix $\underline{x}$ with n elements.

Consider now the set of linear simultaneous equations given by

$$[\underline{B}][\underline{x}] = X_i [\underline{e}_i] \tag{3.16}$$

Solving for a particular variable x_k and using (3.15) we get

$$\frac{x_k}{X_i} = \frac{B_{ik}}{|\underline{B}|} = \frac{A_{ki}}{|\underline{A}|} = G_{ik}(z) \tag{3.17}$$

Thus $G_{ik}(z)$ corresponds to the explicit solution for the kth unknown in a set of simultaneous equations in n unknowns which are inhomogeneous in the ith equation only. It should be noted that the coefficient matrix and hence x_k also, are functions of z.

Let us write down (3.16) explicitly

$$\begin{bmatrix} (1-t_{11}z) & -t_{21}z & -t_{n1}z \\ -t_{12}z) & (1-t_{22}z) & -t_{n2}z \\ \hline -t_{1i}z & -t_{2i}z & -t_{ni}z \\ -t_{1k}z & -t_{2k}z & -t_{nk}z \\ -t_{1n}z & -t_{2n}z & (1-t_{nn}z) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_i \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_i \\ 0 \\ 0 \end{bmatrix} \tag{3.18}$$

One can rewrite the above as follows

$$x_1 = [t_{11}\, x_1 + t_{21}\, x_2 \ldots\ldots t_{n1}\, x_n]\, z$$

$$x_2 = [t_{12}\, x_1 + t_{22}\, x_2 \ldots\ldots t_{n2}\, x_n]\, z$$

$$x_i = [t_{1i}\, x_1 + t_{2i}\, x_2 \ldots\ldots t_{ni}\, x_n]\, z + X_i$$

$$x_k = [t_{ik}\, x_1 + t_{2k}\, x_2 \ldots\ldots t_{nk}\, x_n]\, z$$

$$x_n = [t_{1n}\, x_1 + t_{2n}\, x_2 \ldots\ldots t_{nn}\, x_n]\, z$$

(3.19)

We note that x_1 is the sum of all different products of branch transmissions that arrive at node 1 multiplied by z and by appropriate x's of the originating nodes. Thus, one can write down the set of linear simultaneous equations (3.19) by inspection of the graph itself (Fig. 3.1). The n variables represented by $\underline{x}$ can now be interpreted as <u>signals</u> as in the signal flow graph theory.

Thus $\dfrac{x_k(z)}{X_i}$ is the signal appearing at node k per unit external signal X_i injected at node i, which can also be interpreted as the <u>signal flow-graph transmission</u> $T_{ik}(z)$ of the modified graph from node i to node k (cf. Fig. 3.2).

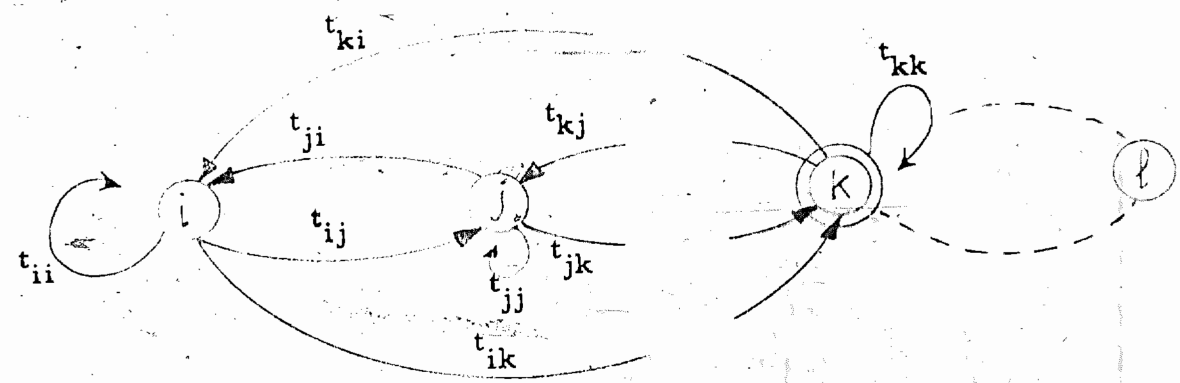

FIG. 3.1 ABSTRACT DIRECTED GRAPH

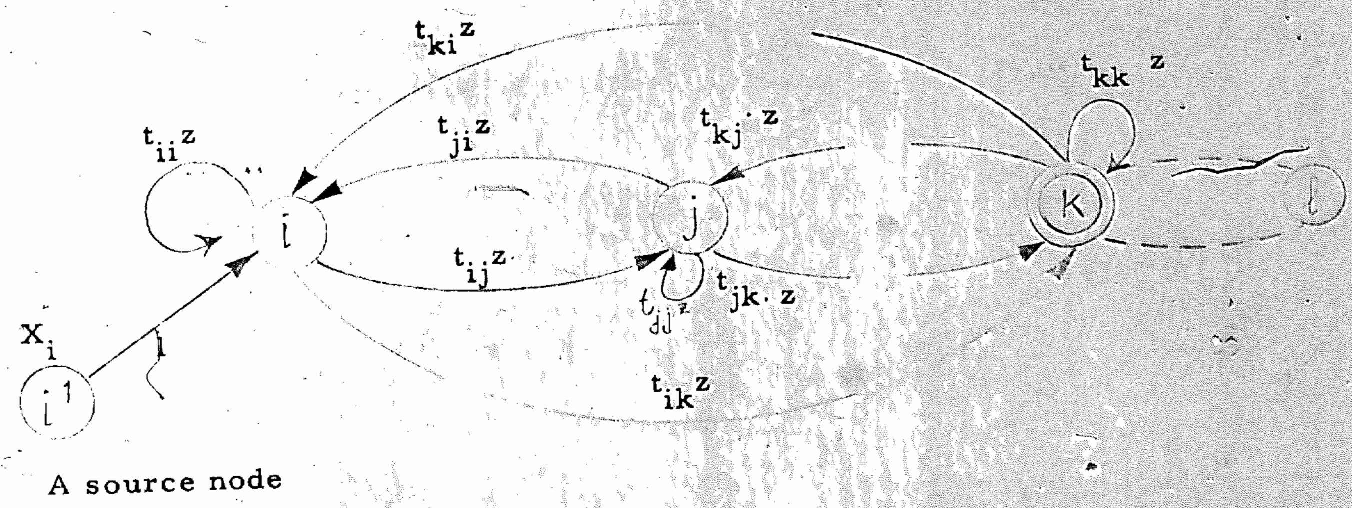

FIG. 3.2 MODIFIED GRAPH

Applying Cramer's rule in (3. 18), we get

$$T_{ik}(z) = G_{ik}(z) = \frac{x_k}{X_i} =$$

$$= \frac{\begin{vmatrix} (1-t_{11}z) & -t_{21}z & -t_{k-1,1}z & 0 & -t_{n1}z \\ -t_{12}z & (1-t_{22}z) & -t_{k-1,2}z & 0 & -t_{n2}z \\ -t_{1,i}z & -t_{2,i}z & -t_{k-1,i}z & 1 & -t_{ni}z \\ -t_{1n}z & -t_{2n}z & -t_{k-1,n}z & 0 & -t_{nn}z \end{vmatrix}}{\begin{vmatrix} (1-t_{11}z) & -t_{21}z & & -t_{k1}z & -t_{n1}z \\ -t_{12}z & (1-t_{22}z) & & -t_{k2}z & -t_{n2}z \\ -t_{1i}z & -t_{2i}z & & -t_{ki}z & -t_{ni}z \\ -t_{1n}z & -t_{2n}z & & -t_{kn}z & (1-t_{nn}z) \end{vmatrix}} \qquad (3.\ 20)$$

3. 4 Relationship between loop-structure and the generating function

We shall next show that the value of $\frac{x_k}{X_i}$ from the set of linear equations can be obtained from considering only the loops in the given graph and the equivalent numerator graph. We shall next define a few terms.

A source node is a node which has only outgoing branches. A sink node has only incoming branches.

A loop is a sequence of nodes 1, 2, 3, ... n, 1 such that the transmission from node 1 to itself viz. $t_{12} \cdot t_{23} \cdots t_{n,1}$ is not zero. Conventionally all nodes 1 through n in a loop are distinct. In a later chapter we shall also consider loops with repeated nodes. Unless it is explicitly states that latter is the case, a loop will always refer to the conventional case.

A path is a sequence of distinct directed branches connected in series between two nodes. A loop is thus a path which begins and ends at the same node.

A self-loop is a loop with exactly one node.

A non-touching (non intersecting) loop set is a set of loops not having any node or branch in common.

A graph is strongly connected if any node can be reached from any other.

The determinant of the denominator in (3. 20) is known as the characteristic function of the graph. It is given by

$$|D(z)| = 1 - \sum P_{a_1} + \sum P_{a_2} - \sum P_{a_3} \cdots \qquad (3.21)$$

where P_{a_m} is the product of transmittances of the a-th set of m non-touching loops; the summation is taken over all possible non-touching loops when taken m at a time.

Determination of denominator determinant:

Expanding the determinant in the denominator of (3. 20), we find that is is equal to $1 + \sum_{\alpha} P_\alpha(z)$ where each $P_\alpha(z)$ is the product of certain branch transmissions. Concerning ourselves with a particular product $P_\beta(z)$, we can make a few observations regarding the subgraphs G_β whose branch transmissions are involved in $P_\beta(z)$.

(1) Since any product term of the determinant can contain only one term from a row and only one element from a column, its subgraph G_β cannot have more than one branch emanating or terminating at any of its nodes. Thus G_β will not have the situations illustrated in Fig. 3. 3.

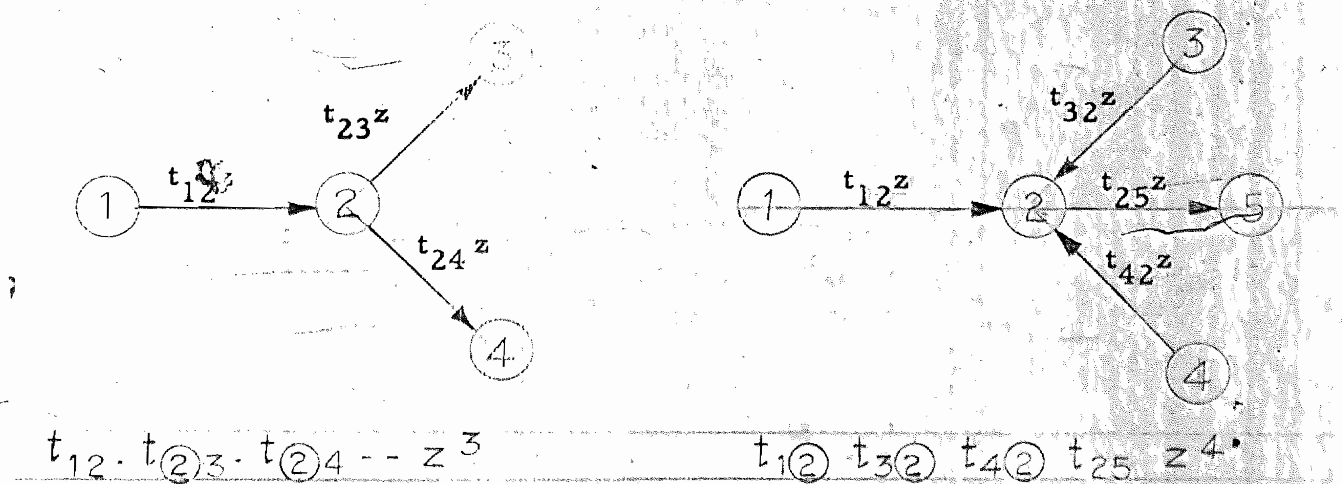

FIG. 3. 3 EXAMPLES OF SUBGRAPHS NOT POSSIBLE

(2) $P_\beta(z)$ is a product of loop transmissions of non-touching loops. To prove this, we argue as follows.

a) If G_β is made up of loops, they must be non-touching since by the previous argument, no two loops can have a node or branch in common.

b) Let G_β contain a connected part that is not a loop. We shall show that this is impossible.

Let G_β consist of a path between two nodes and some non-touching loops. For simplicity let this path have a transmission $t_{12} \cdot t_{23} z^2$. Since this is a path which starts at node 1 and ends at node 3, node 1 cannot have any other branch coming into it. Similarly node 3 can have no branches leaving it. Then the product P_β of the branch transmissions of the branches in G_β <u>cannot</u> look as follows

$$t_{\ell 1} \left\{ t_{12} \cdot t_{23} \right\} t_{3\ell} \qquad z^m$$

since $t_{\ell 1}$ and $t_{3\ell}$ will continue the path $t_{12} \cdot t_{23}$ for any values of $\ell = 1, 2, \ldots n$ contrary to our initial assumption.

Let us look at the determinant, which is reproduced in part below

$$\begin{vmatrix} (1-t_{11} z) & -t_{21} z & -t_{31} z \\ -t_{12} z & (1-t_{22} z) & -t_{32} z \\ -t_{13} z & -t_{23} z & (1-t_{33} z) \end{vmatrix}$$

Since $P_\beta(z)$ is a term of the above determinant, $P_\beta(z)$ must contain an element from row 1 (which cannot be $t_{\ell 1}z$) and an element from column 3 (which cannot be $t_{3\ell}z$). Only options are the 1's in locations (1, 1) and (3, 3). This is impossible because the "1" in (1, 1) belongs to the same column as $t_{12}z$ and the "1" in (3, 3) belongs to the same row as $t_{23}z$.

Therefore, $t_{\ell 1}z$ must be there in P_β for a value of $\ell \neq 3$ and similarly $t_{3\ell}z$ must be present for values of $\ell \neq 1$, and so on.

Therefore, G_β cannot contain a connected part which is not a loop. Thus G_β can be either a single loop or a number of non-touching loops.

(3) The product of the transmissions in $P_\beta(z)$ has a negative sign if G_β consists of an odd number of loops and a positive sign if the number of loops is even. To prove this, we first note that any even number of row and column interchanges does not change the value of the determinant. Consider two identical graphs G and G' whose nodes are identically numbered except for the nodes i and j of G are labeled j and i respectively in G'. The transmission matrix of graph G' can be obtained from that of G by interchanging rows i and j as well as columns i and j. Hence, their determinants are identical. Thus relabeling of the node names does not change the determinant. Suppose G_β consisted of two non-touching loops. Let the nodes of each loop be numbered $1, 2, \ldots \ell_1$; $\ell_1 + 1, \ldots \ell_2$, respectively. The gain products associated with the loops are

$$[t_{12} \cdot t_{23} \cdots t_{\ell_1, 1}]\,[t_{\ell_1+1, \ell_1+2} \cdots t_{\ell_1+\ell_2, \ell_1+1}]\, z^{\ell_2} \qquad (3.22)$$

This product will be assigned a sign according to the parity of the permutation P:

$$(2, 3, 4, \ldots \ell_1, 1; \ell_{1+2}, \ell_{1+3}, \ldots, \ell_1+\ell_2, \ell_{1+1})$$

Since ℓ_1th element, 1, can be brought to the position 1 by $\ell_1 - 1$ interchanges of adjacent symbols and since the $(\ell_1 + \ell_2)$ element, ℓ_{1+1} can be brought to the (ℓ_{1+1}) position by ℓ_{2-1} interchanges of consecutive symbols the sign of the permutation is

$$\text{sgn } P = (-1)^{\ell_1 - 1} \cdot (-1)^{\ell_2 - 1}$$

Since each t_{ij}'s have a minus sign in (3. 20), the sign of the product P is

$$(-1)^{\ell_1} (-1)^{\ell_2} \text{ sgn } P = (-1)^2 .$$

If G_β had consisted of r loops, the reasoning above would have led to $(-1)^r$. Therefore, any term of the expansion of D is either "1" or a product $P_a(z)$ of loop gains of non-touching loops times $(-1)^{r_a}$ where r_a is the number of loops corresponding to the product $P_a(z)$.

Conversely, any product of loop transmission of non-touching loops of the graph G with appropriate sign will appear as a term of the expansion of $|D(z)|$. This follows directly from considerations almost identical with those outlined above. This completes the proof of our assertion that

$$|D(z)| = 1 - \sum P_{a_1}(z) + \sum P_{a_2}(z) \ldots .$$

Determination of the numerator determinant:

Every determinant can be considered as the characteristic function of some graph. Thus given an arbitrary nth order determinant, one can find an n-node graph whose characteristic function is the given determinant. We shall call this the equivalent graph of the given determinant.

To derive the transmission matrix $\underline{T}'$ of the equivalent graph of a given determinant $|\underline{A}\,z|$, we proceed as follows.

By definition of the equivalent graph

$$|\underline{A}| = |\underline{I} - \underline{T}'^{T}|$$

Thus

$$T' = I - |\underline{A}|^{T}$$

so that

$$t'_{ij} = \delta_{ij} - a_{ji} \qquad (3.23)$$

where

$$\underline{\delta}_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ otherwise.}$$

Where a_{ij} and t_{ij} are corresponding elements of $\underline{A}$ and $\underline{T}'$ respectively. Thus there is a 1-1 mapping between the elements of $\underline{A}$ and the elements of $\underline{T}'$.

Consider the numerator determinant in (3.20). This determinant can be mapped to an equivalent graph whose transmission matrix T' is given by

$$
\begin{aligned}
t'_{pq}z &= t_{pq}z && \text{if } p \neq k \\
&= 0 && \text{if } p = k \text{ and } q \neq i, k \\
&= -1 && \text{if } p = k \text{ and } q = i \qquad (3.24) \\
&= 1 && \text{if } p = k \text{ and } q = k
\end{aligned}
$$

where t'_{pq} and t_{pq} are corresponding elements of the numerator and denominator graph transmission matrices. Thus the numerator determinant is the characteristic function of the graph whose transmission matrix is T'. Thus we have the following result:

The generating function $G_{ik}(z)$ of a graph whose transmission matrix is $\underline{T}$ is the ratio of the characteristic functions of the graph whose transmission matrices are T' and T respectively defined by the relation:

$$
\begin{aligned}
t'_{pq} &= t_{pq} && \text{if } p \neq k \\
&= 0 && \text{if } p = k \text{ and } q \neq i, k \\
&= -1 && \text{if } p = k \text{ and } q = i \\
&= 1 && \text{if } p = k \text{ and } q = k
\end{aligned}
$$

This result is equivalent to Mason's gain rule [Reference 1] but not the same and its usefulness lies in the uniform treatment of both the numerator and denominator of the generating function.

Since only characteristic functions are involved, no paths need be considered.

Given a directed graph, the path enumerating generating function from an arbitrary node i to another node k is given by the gain or the transmission function corresponding to those nodes when the given graph is treated as a signal flow graph after each branch transmission of the original graph is multiplied by a $\neq$ which is a seperator variable.

The generating function of a directed graph from a set of starting nodes S to a set of end nodes E is given by $G_{S,E}(z) = \sum_{\forall m} \sum_{\forall n} G_{m \in S, n \in E}(z)$, since this is a linear process where the principle of superposition applies.

This can be seen by using the representation of the generating function of Section 3.3. More specifically a set of starting nodes can be considered by allowing more than one non-zero element on the right-hand side of (3.18) and more than one end node can be considered by solving (3.18) for more than one unknown. Thus, $G_{S,E}(z)$, the generating function from a subset of nodes S to another subset of nodes E is the sum of solutions for the unknown corresponding to the end nodes of E in (3.18) when the right hand side is vector $\underline{X}$ having non-zero components in positions corresponding to the starting nodes.

Let $\underline{S} = (S_1, S_2 \ldots S_p)$ be a vector of starting nodes receiving external signals represented by the vector $\underline{X} = (X_1, X_2 \ldots X_p)$. Let $\underline{E} = (E_1, E_2 \ldots E_q)$ be a set of end nodes. The directed graph is given in Fig. 3.4

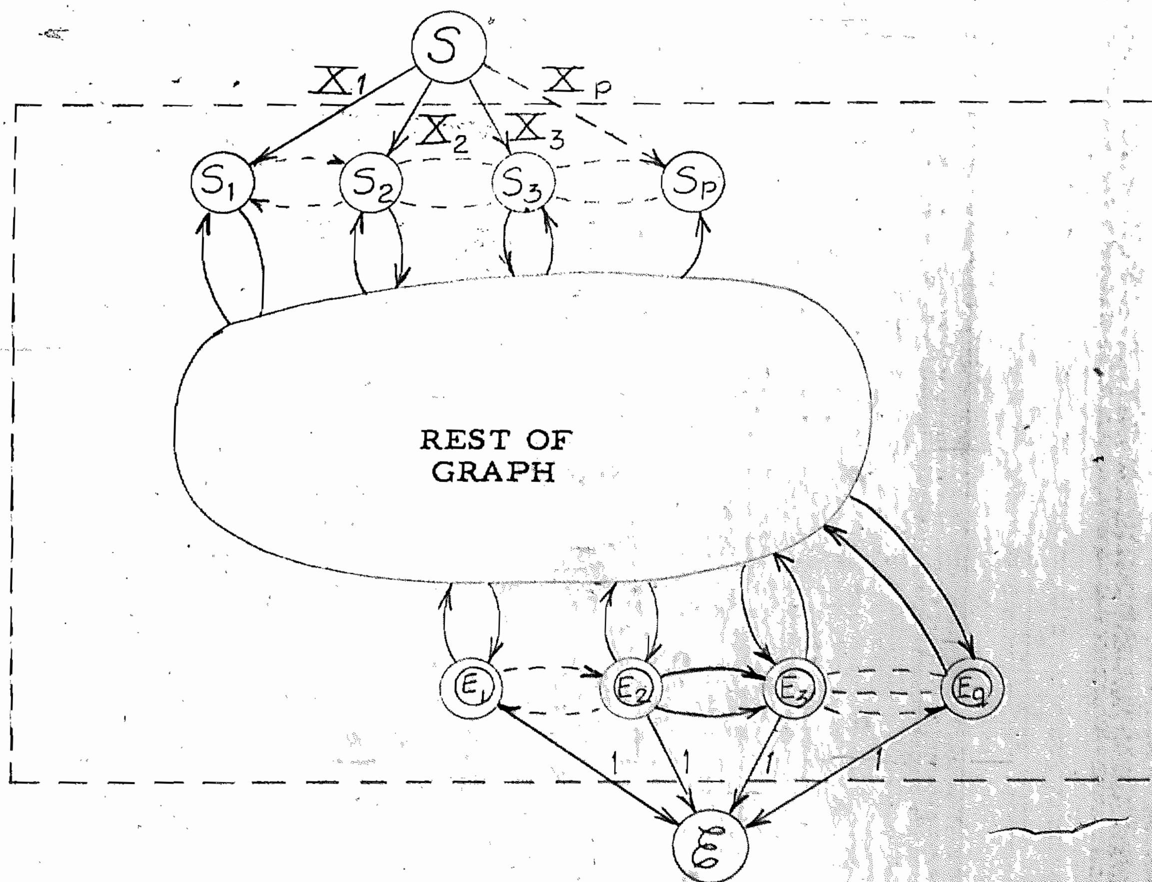

FIG. 3.4 GIVEN GRAPH

Let us introduce a source node $\mathcal{S}$ as shown in Fig. 3.4 such that it generates a unit of signal and its branch transmissions to nodes in $\underline{S}$ is given by $\underline{X}$. Let $\mathcal{E}$ be a sink node connected from nodes in $\underline{E}$ by branches with unit transmissions. It is easy to see $G_{\underline{S},\underline{E}}(z)$ is the same as $G_{\mathcal{S},\mathcal{E}}(z)$ from the arguments introduced previously.

3.5 Derivation of the characteristic difference equation

The characteristic function of the graph is defined as the determinant of $[I - \underline{T} z]$. It is also the denominator of the generating function between any set of terminal nodes of the graph (before any cancellation of common factors with the numerator). From the topological viewpoint, it is a function of loop transmissions only.

Since $G_{S,E}(z)$ is a rational function of z, its expansion into an infinite series of the form $\sum_{n=0}^{\infty} g_n z^n$ by division is cumbersome even for simple graphs. A superior method is to compute the coefficient of z^n by recursion by using the characteristic difference equation of the graph.

Let

$$G_{SE}(z) = \frac{N(z)}{D(z)} \qquad (3.25)$$

where

$$N(z) = n_o + n_1 z^1 + n_2 z^2 \qquad n_\ell z^\ell$$

and

$$D(z) = d_o + d_1 z^1 + d_2 z^2 \qquad d_m z^m \qquad (3.26)$$

where p and ℓ are finite positive integers, and the coefficients n_j $(0 \leq j \leq \ell)$ and d_p $(0 \leq p \leq m)$ are all known.

From (3.25)

$$G_{SE}(z)\, D(z) - N(z) = 0 \qquad (3.27)$$

If $G_{SE}(z) = \sum_{n=0}^{\infty} g_n z^n$ for $|z| < 1$, we get on substitution in (3.25)

$$\left[\sum_{n=0}^{\infty} g_n z^n\right] \cdot \left[\sum_{p=0}^{m} d_p z^p\right] = \sum_{j=0}^{\ell} \left[n_j z^j\right] \qquad (3.28)$$

Since the numerator is a polynomial in z of degree $\leq \ell$, the coefficients of z for exponents greater than ℓ must vanish in (3.28).

Setting the coefficient of z^n for $n > \ell$ in the left hand expression to zero, we get

$$d_o g_n + d_1 g_{n-1} + d_2 g_{n-2} + d_p g_{n-p} = 0 \qquad (3.29)$$

for $n > \max [\ell, p]$

$$g_n = -\frac{1}{d_o}\left[\sum_{i=1}^{\ell} d_i g_{n-i}\right]$$

for $n > \max [\ell, p]$ (3.30)

The equation (3.30) is called the characteristic difference equation or recursive enumerator of the graph. This equation is unique for the graph and does not depend on the choice of starting or ending nodes for a particular generating function. However, the particular set of terminal nodes of the generating function determines the initial values that starts the recursion of (3.30).

In subsequent work, we will not explicitly draw the nodes generating the external signals when the system representation contains no such source nodes. The presence of such nodes is assumed implicitly, however, in line with the general conventions of signal flow graph theory.

NOTE ON CONNECTIVITY MATRIX:

The connectivity matrix $\underline{C}$ can be derived from a given transmission matrix $\underline{T}$ by simply setting

$$\underline{C}_{ij} = 1 \text{ if } \underline{T}_{ij} \neq 0$$

$$\underline{C}_{ij} = 0 \text{ if } \underline{T}_{ij} = 0 \ .$$

3.6 An Example of Deriving The Generating Function

Find the generating function of the following graph from node 1 to nodes 3, 5.

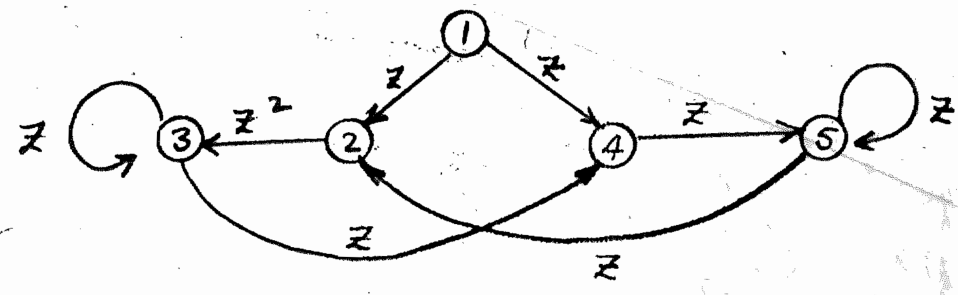

FIG. 3.5: A GIVEN GRAPH

The transmissions of the three loops are: z, z and z^5.

Using (3.21), we get the denominator determinant

$$|D(z)| = 1 - (2z + z^5) + z^2.$$

We next form the graph which is used to evaluate the numerator determinant (see Eq. 3.24)

Consider node 3, as the end node.

Equivalent numerator graph for node 3 is in Fig. 3.6

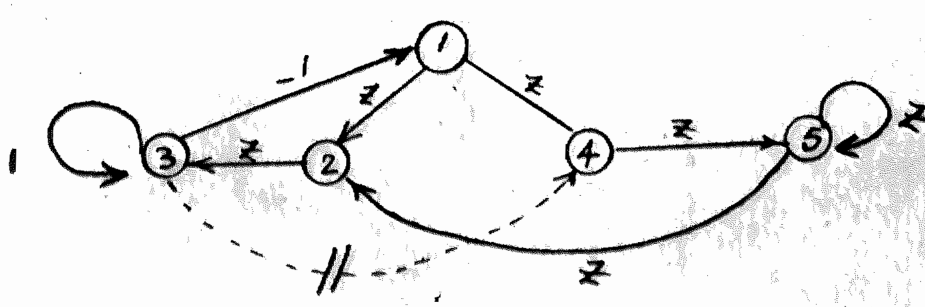

FIG. 3.6: NUMERATOR GRAPH (END NODE 3)

The loop transmissions are:

$$1,\ z,\ -z^3,\ -z^5$$

therefore,

$$|N_3(z)| = 1 - \left[1 + z - z^3 - z^5\right] + z - z^4$$

$$= z^3 - z^4 + z^5$$

Similarly considering node 5 as the end node, we get the equivalent numerator graph for node 5 in Fig. 3.7.

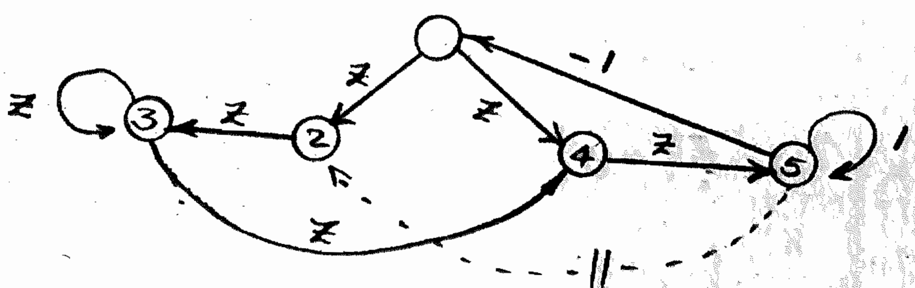

FIG. 3.7: NUMERATOR GRAPH (END NODE 5)

The transmissions of loops are:

$$1,\ z,\ -z^2,\ -z^5$$

therfore,

$$|N_5(z)| = 1 - \left[1 + z - z^2 - z^5\right] + z - z^3$$

$$= z^2 - z^3 + z^5.$$

$$G(z) = \frac{|N_3(z)| + |N_5(z)|}{|D(z)|}$$

$$= \frac{z^2 - z^4 + 2z^5}{1 - 2z + z^2 - z^5}$$

We could have simplified the above, had we connected a new end node ξ to nodes 3 and 5 (Fig. 5.10a) and computed $|N_{\xi}(z)|$ directly.

6. CONNECTIVITY OF DIRECTED GRAPHS

6.1 Some Properties of Generating Functions

Definition: A node j is said to be reachable from node i if there is at least one directed path from node i to node j. Equivalently, we can say that the $G_{ij}(z) \neq 0$. A node p is irredundant with respect to the generating function $G_{ij}(z)$ if a) it is reachable from node i and b) it can reach the terminal node j. Nodes which are not irredendant are redundant.

In Figure 6.1, for $G_{SE}(z)$, nodes S, A, E are irredundant.

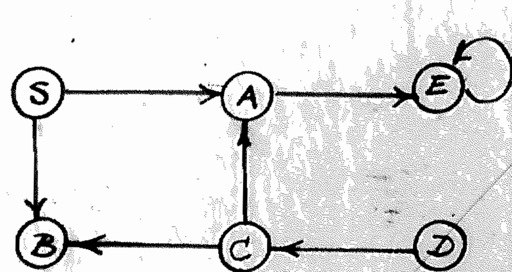

FIG. 6.1 GRAPH WITH REDUNDANT NODES

Theorem 6.1

The generating function $G_{SE}(z)$ from node s to node e can be obtained by considering only its irredundant nodes.

Proof:

Though this theorem can be proved in more than one way, a proof based on the linear sampled-data interpretation appears to be simple.

a) Consider the case where the redundant nodes (R_1 and R_2 in Figure 6.2) are reachable from node S, but which cannot reach end node E.

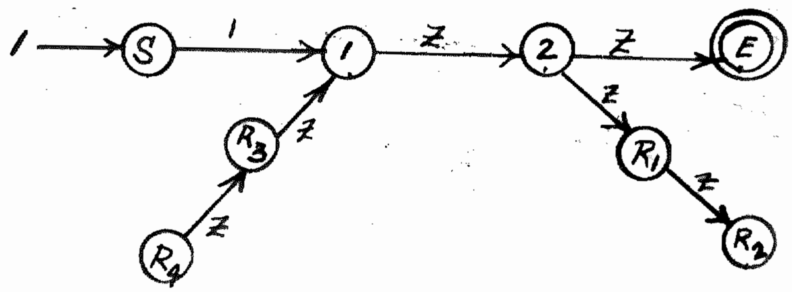

FIG. 6.2 GRAPH WITH REDUNDANT NODES

If the branch transmittances represent appropriate delays, removal of redundant nodes does not change the signal magnitude at node E caused by a unit impulse at S at t = 0.

b) Consider the case where the redundant nodes are not reachable from S but can reach end node E (R_3 and R_4 in Figure 6.2). We argue that the unit impulse at S at t = 0 can never enter the redundant nodes and hence the contribution of the latter to the signal at node E is zero.

Thus $G_{SE}(z)$ can be calculated by considering only its irredundant nodes. In Section 6.6 we develop an algorithm to remove the redundant nodes.

The denominator $\phi(z)$ of the generating function is known as <u>the characteristic function</u> of the graph. It depends entirely on the loop structure of the graph and not on the sources and sinks.

When studying <u>only</u> the characteristic function of the graph without any reference to a particular set of terminal nodes, one must include <u>all</u> nodes (i.e., redundant nodes with respect to any set of terminal nodes must not be discarded). For example, one can lose parts of the characteristic function by cancellation, if it is obtained by a preliminary calculation of a generating function in which redundant nodes are removed.

a) $G_{S,E}(z) = 0$ if and only if there is no path from node S to node E.

b) The largest exponent n in $G(z) = \sum_{i=0}^{n} g_i Z^i$ is a finite positive integer if and only if there are no loops in the graph, otherwise, the sum is infinite.

6.2 A Note on $G_{ii}(z)$ (Self-Transmittance)

When $i \neq j$, the coefficient A_0 in $G_{ij}(z) = \sum_{n=0}^{\infty} A_n . z^n$ is identically zero because i and j nodes are distinct. When $i = j$, it is convenient to define A_0 to be zero i.e., we assume there are no paths of zero length from node i to itself.

Let us redefine $G_{ij}(z) = \sum_{n=1}^{\infty} A_n z^n$ as the generating function. Thus $G_{ij}(z) \neq 0$ implies there is at least one path from node i to node j for $i \neq j$. For $i = j$, this implies a loop through i. We shall use without any loss of generality the revised definition of $G_{ij}(z)$ for the rest of the chapter.

6.3 Equivalent Graphs

When only the transmission from a node S to a node E is of interest in a directed graph, the later can be represented by a directed graph with only two nodes (Fig. 6.3) such that the transmittance between the two nodes is same as the transmittance from S to E in the original graph.

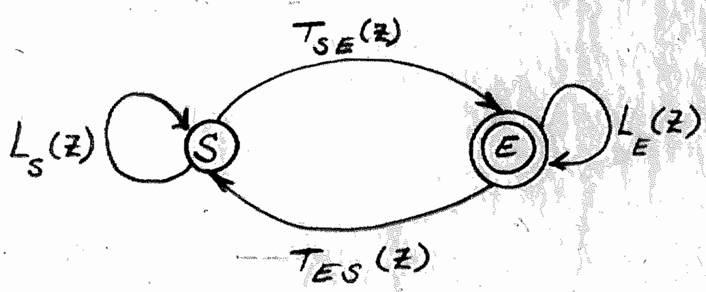

FIG. 6.3 EQUIVALENT GRAPH

$$G_{SE}(z) = \frac{T_{SE}(z)}{1 - \left[T_{SE}(z)\ T_{ES}(z) + L_S(z) + L_E(z)\right] + L_S(z)\ L_E(z)}.$$

Here $T_{SE}(z)$ can be interpreted the generating function from S to E when other transmittances shown in the graph are zero. One similarly interprets other transmittances.

We shall call this two node-graph as the equivalent graph for the given generating function = $G_{SE}(z)$. Equivalent graphs are very usefull in studying the characteristics of generating functions.

Characteristics of Equivalent Graphs and Their Generating Functions

Case 1. Fig. 6.4

$G_{SE}(z) \neq 0$ implies there is at least a path from node S to E.

Case 2.

$$\text{Let } G_{SE}(z) = \frac{T_{SE}(z)}{1 - \left[T_{SE}(z) \,.\, T_{ES}(z) + L_E(z) + L_S(z)\right] + L_E(z) \,.\, L_S(z)} \tag{6.1}$$

as in Fig. 6.4

$G_{SE}(z) \neq 0$ & $G_{ES}(z) = 0$ implies $T_{ES}(z) = 0$ and $T_{SE}(z) \neq 0$.

Case 3.

$G_{SE}(z) \neq 0$ & $G_{ES}(z) \neq 0$ implies $T_{ES}(z) \neq 0$; $T_{SE}(z) \neq 0$

Theorem 6.2

If $G_{SE}(z)$ and $G_{ES}(z)$ are not equal to zero, then nodes E and S lie on the same loop possibly with repeated nodes.

Proof:

Construct the equivalent graph for the two nodes S and E with respect to the generating functions $G_{ES}(z)$ and $G_{SE}(z)$ (Fig. 6.3). $G_{ES}(z) \neq 0$ and $G_{SE}(z) \neq 0$ implies that $T_{ES}(z) \neq 0$ and $T_{SE}(z) \neq 0$.

Thus E is reachable from itself through S. They therefore lie on the same loop which may contain repeated nodes.

Theorem 6.3

If $G_{SE}(z)$ and $G_{ES}(z)$ exist $(\neq 0)$ then any node irredundant (redundant) for $G_{SE}(z)$ is also irredundant (redundant) for $G_{ES}(z)$.

Proof:

Let i be an irredundant node for $G_{SE}(z)$. This implies that i is reachable from S and it can reach E. Since $G_{ES}(z) \neq 0$, there is at least one path from E to S. If i is on this path then it is irredundant. If not, let P_{ES} be a path from E to S. Then i is reachable from E because one can go from E to S (via P_{ES}) and S to i. i can reach S, by the route i to E and E to S. By definition i is a irredundant node of $G_{ES}(z)$.

6.4 Connectedness in Directed Graphs

We shall now examine certain basic notions in automata theory and interpret them in the light of our results.

A strongly connected machine has a state diagram in which any node is reachable from any other [19]. Thus:

Theorem 6.4

A necessary condition that a state diagram is strongly connected is that for all i, $G_{ii}(z) \neq 0$.

This only states that every node in a strongly connected state diagram is a member of some loop and reachable from itself in more than one step.

Theorem 6.5

A graph is strongly connected if and only if $G_{ij} \neq 0$ for all i and j.

Proof:

$G_{ij}(z) \neq 0$ for all i and j, if and only if i can reach j and vice-versa.

Theorem 6.6

A graph is strongly connected if and only if there exists a node i such that $G_{ii}(z) \neq 0$ and all nodes are irredundant nodes with respect to $G_{ii}(z)$.

Proof:

1) Assume $G_{ii}(z) \neq 0$ and all nodes are irredundant with respect to $G_{ii}(z)$. Starting at any node k any other node m is reachable from k via node i.

2) Assume graph is strongly connected. Therefore we can reach node i from node i in one or more steps. Therefore $G_{ii}(z) \neq 0$. In addition all nodes can reach node i and can be reached from it. Hence, all nodes are irredundant with respect to $G_{ii}(z)$.

We note that a strongly connected graph is a loop with possible repetitions of nodes. The last theorem is important, since it provides a test for strong connectivity. In addition, it can be used to find the largest subgraph that is strongly connected and which includes a given node. (Section 6.63). The size of the graph is measured here by the number of nodes.

In the above discussion we stressed the fact that $G_{ij}(z)$ not being equal to zero. To test if $G_{ij}(z)\neq 0$, one only has to find if node j is reachable from node i, which is a simple matter in view of the well-formation test of Section 6.61. Detection of irredundant nodes again is provided by the algorithm in Section 6.61. Thus, the location of the largest strongly connected subgraph that includes a particular node is quite simple. Theorem 6.7 below immediately follows Theorem 6.6.

Theorem 6.7

A directed graph is strongly connected if and only if, for every node i, $G_{ii}(z)\neq 0$ and all the other nodes are irredundant with respect to $G_{ii}(z)$.

Definition:

Let $N = n_1, n_2, \ldots n_R$ and $M = m_1, m_2, \ldots m_E$ be two sets of nodes in a graph. N and M are said to be unconnected (disconnected, disjoint) if no node from N can reach a node in M and vice-versa.

Theorem 6.8

The subgraphs M and N are unconnected if and only if $G_{m,n}(z) = G_{n,m}(z) = 0$ for all $m \in M$ and $n \in N$.

Proof:

$G_{ij}(z) = G_{ji}(z) = o$ if and only if node i cannot be reached from node j and vice versa. The result follows immediately.

In Section 6.65 we shall give an algorithm to partition a graph into its unconnected subgraphs.

6.5 Reverse Graphs

Definition: The reversal of a directed graph (state diagram) is obtained by transposing its transmission matrix. Schematically, the process of reversal implies the change of direction of the directed branches of the graph.

The process of reversal changes simple sources to sinks but leaves the loops intact. The characteristic function of the original graph is invariant under reversal.

Theorem 6.9

For any directed graph $G_{SE}(z) = \hat{G}_{ES}(z)$ where G and $\hat{G}$ are generating functions of the original graph and its reverse.

Proof:

$$G_{SE}(z) = \frac{\underline{A}_{ES}}{|\underline{A}|} \quad \text{where } \underline{A} = \left[\underline{I} - \underline{T}z\right]$$

by (2.52)

$$\hat{G}_{ES}(z) = \frac{\underline{A}'_{SE}}{|\underline{A}'|} \quad \text{Where } \underline{A}' = \left[\underline{I} - \underline{T}^T z\right]$$

$$\underline{A}' = \left[I^T - \underline{T}^T z\right] = \left[\underline{I} - \underline{T}z\right]^T = \underline{A}^T$$

6.7 Factoring of Determinants

(An application of graphs to to numerical analysis.)

In Theorem 6.10, we proved that the characteristic function of a graph is the products of characteristic functions of its maximal strongly connected subgraphs. We also note that the characteristic function is a special type of determinant.

We can consider that every determinant is the characteristic function of some graph. However, the matrix of the determinant is not the transmission matrix. Our purpose will be to derive the equivalent graph of a given determinant and from it, its connectivity matrix. We can next partition its loops into maximal strongly-connected subgraphs rather simply by our tests. The partitioned subgraphs will then represent the component determinants into which the determinant can be factored.

$\underline{A}_i$ is a component matrix of a matrix $\underline{A}$ if $\underline{A}_i$ is obtained from $\underline{A}$ by non-trivial deletions of certain rows and the corresponding columns. Determinant of $\underline{A}$ = $|\underline{A}|$ is said to be factorable into k factors if $|\underline{A}|$ can be expressed as a product of the determinants of k component matrices of $\underline{A}$.

Thus, $|\underline{A}| = |\underline{A}_1| \cdot |\underline{A}_2| \cdots |\underline{A}_k|$. We shall call $|\underline{A}_1|$ through $|\underline{A}_k|$ component determinants of determinant of $\underline{A}$.

If $|\underline{A}| = |\underline{A}_1|\ |\underline{A}_2| \ldots |\underline{A}_k|$, where $|\underline{A}_i|$'s are component determinants of $|\underline{A}|$ and no $|\underline{A}_i|$ can be factored further, then we say that $|\underline{A}|$ is completely factored into k component determinants.

We also note that if $|\underline{A}|$ is factored into component determinants, the order of $|\underline{A}|$ is the sum of the orders of its component determinants.

Determination of the transmission and connectivity matrices of the equivalent graph of a determinant:-

Let $\underline{A}$ be the matrix of the given determinant. Let us consider this as the coefficient matrix of a set of linear simultaneous equations. (6.3)

$$\underline{A}\ \underline{x} = \underline{r} \qquad (6.3)$$

$$[\underline{I} - \underline{A}]\ \underline{x} + \underline{r} = \underline{x} \qquad (6.4)$$

Equation (6.4) represents the flow graph where $\underline{x}$ is the signal vector of the nodes and $\underline{r}$ is the vector of external signals injected into the system.

The transmission matrix of the equivalent graph is $[\underline{I} - \underline{A}]^T = [\underline{I} - \underline{A}^T]$, that is, the branch transmittance t_{ij} of the equivalent graph is given by

$$t_{ij} = -a_{ji} + \delta_{ij} \qquad (6.5)$$

where a_{ji} is the ji th component of the matrix $\underline{A}$, and δ_{ij} is one if and only if i = j and otherwise zero. The connectivity matrix $\underline{C}$ will be determined by its components.

$$C_{ij} = 1 \text{ if } |a_{ji}| + \delta_{ij} \neq 0$$
$$= 0 \text{ otherwise} \qquad (6.6)$$

We next observe that the determinant of the $\underline{A}$ and the determinant of its transpose, $\underline{A}^T$, is the same. Since the loop structure is invariant to transpositions of connectivity matrices, one can use the transposed connectivity matrix for loop determination. Thus given a matrix $\underline{A}$, its equivalent connectivity matrix for loop determination is also given by

$$C_{ij} = 1 \text{ if } |a_{ij}| + \delta_{ij} \neq 0$$
$$= 0 \text{ otherwise} \qquad (6.7)$$

Theorem 6.11:

The determinant of matrix $\underline{A}$ is <u>completely factorable</u> into k component determinants if and only if its equivalent graph given by its connectivity matrix

$$C_{ij} = 1 \text{ if } [|a_{ij}| + \delta_{ij}] \neq 0$$
$$= 0 \text{ otherwise}$$

contains exactly k maximal strongly connected subgraphs. Each component determinant corresponds to a maximal strongly connected subgraph and each of it elements correspond to a node of that graph.

We shall only sketch the proof.

Let the equivalent graph contain exactly k maximal strongly-connected subgraphs. Let $\underline{T}$ be the transmission matrix of the equivalent graph. From the construction of the equivalent graph

$$\underline{T} = [I - \underline{A}]^T$$

$$[\underline{I} - \underline{T}] = |\underline{A}| = \emptyset$$

Thus, $|A|$ is the characteristic function of the graph , which depends only on the loops. Since there are exactly k maximal strongly connected subgraphs, by Theorem 6.10, $\emptyset$ is factorable into k factors where each factor is the characteristic function of a maximal strongly connected subgraph.

We also note that each such characteristic function is a determinant depending on the nodes of that subgraph. Since there is a 1 - 1 correspondence between the nodes and the rows (columns) of the matrix $\underline{A}$, and the transmittance t_{ij} corresponds to matrix component a_{ji}, it follows that $\emptyset = \prod_{i=1}^{k} \emptyset_i = \prod_{i=1}^{k} |\underline{A}_i|$. The nodes of the maximal strongly connected subgraph identify the rows (columns) of the component determinant.

Assume that the determinant of $\underline{A}$ is completely factorable into k component determinants $|\underline{A}_1|$ through $|\underline{A}_k|$. Then draw equivalent graphs for the component determinants using the rule

$$t_{ij} = -a_{ji} + \delta_{ij}.$$

Since no two $\underline{A}_i$'s can contain same row (column) of the matrix $\underline{A}$, the non-intersecting property of the subgraphs is assured. Since by construction $|\underline{A}_i|$ is equal to $|\underline{I} - \underline{T}_i|$, $|\underline{A}_i|$ is the characteristic function of the subgraph $\underline{T}_i$, which depends on the loop structure of $\underline{T}_i$. If $\underline{T}_i$ has a non-intersecting loop within it, then by Theorem 6.10, it is factorable, which implies $|\underline{A}_i|$ is factorable by the first part of the theorem. Since this is not true, the theorem is proven.

An important result of the above theorem is that the factorization of a determinant into a complete set of component determinants is unique upto a permutation of rows (columns) within each component determinant.

Example:

We shall next use this result to solve for X_3 in the following linear simultaneous equation.

$$\begin{matrix} (1) & (2) & (3) & (4) & & & (5) \end{matrix}$$

$$\begin{bmatrix} A & 0 & a & 0 \\ d & D & e & g \\ b & 0 & B & 0 \\ c & f & h & C \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = - \begin{bmatrix} J \\ K \\ L \\ M \end{bmatrix} \qquad (6.8)$$

Solution:

$$\frac{X_3}{1} = \frac{\overset{(1)\quad(2)\quad(5)\quad(4)}{\begin{vmatrix} A & 0 & -J & 0 \\ d & D & -K & g \\ b & 0 & -L & 0 \\ c & f & -M & C \end{vmatrix}}}{\underset{(1)\quad(2)\quad(3)\quad(4)}{\begin{vmatrix} A & 0 & a & 0 \\ d & D & e & g \\ b & 0 & B & 0 \\ c & f & h & c \end{vmatrix}}} \qquad (6.9)$$

Which imply from Theorem 6.11 that the connectivity matrices of numerator and demominator graphs

$$\frac{\begin{array}{cccc} \textcircled{1} & \textcircled{2} & \textcircled{5} & \textcircled{4} \\ \end{array} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} \begin{array}{cccc} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \end{array}} \qquad (6.10)$$

Applying our strong connectivity test to both the numerator and the denominator connectivity matrices, we partition them into maximal strongly connected subgraphs as follows

$$\frac{\left\{ \left\{ 1, 5 \right\} \quad \left\{ 2, 4 \right\} \right\}}{\left\{ \left\{ 1, 3 \right\} \quad \left\{ 2, 4 \right\} \right\}}$$

$$X_3 = \frac{\Delta_{15} \circ \Delta_{24}}{\Delta_{13} \circ \Delta_{24}} = \frac{\Delta_{15}}{\Delta_{13}}$$

$$= \frac{\begin{vmatrix} A & -J \\ b & -L \end{vmatrix}}{\begin{vmatrix} A & a \\ b & B \end{vmatrix}} \qquad (6.11)$$

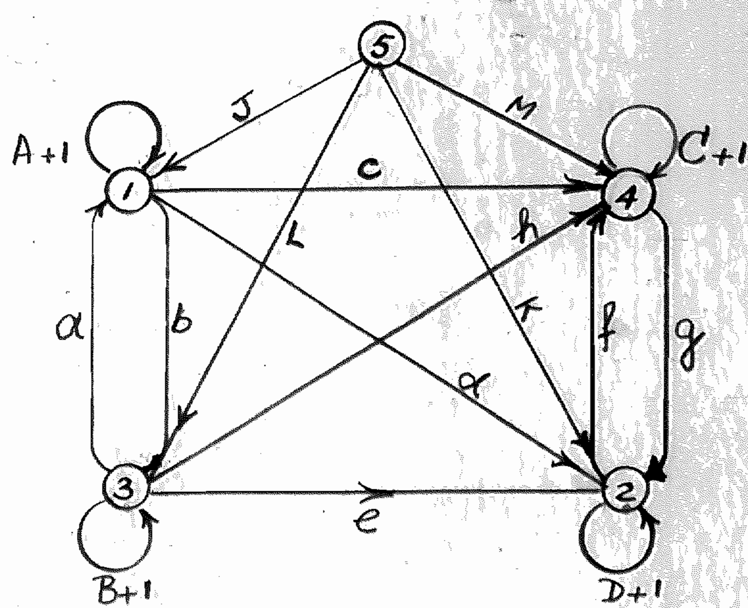

FIG. 6.16 EQUIVALENT GRAPH OF THE DENOMINATOR (EQN. 6.9)

Note: In this example, we have treated the numerator and the denominator of (6.9) in an identical manner, i.e., mapping the numerator and denominator into equivalent graphs and finding their characteristic functions. Since finding the generating function is the same as solving a set of simultaneous linear equations for some unknown, this method of mapping the numerator and denominator determinants into equivalent graphs and finding the ratio of their characteristic functions, avoids the problem of finding paths between terminal nodes etc., as required in Mason's formula. This is essentially what we proposed in Section 3.4.