Closest Normal Matrix Finally Found!

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December 1986
Revised August 1987

Dedicated to the memory of Peter Henrici (1923-1987).

Abstract

A method of finding the closest normal matrix in the Frobenius matrix norm is developed. It is shown that if a matrix is represented in those coordinates where its closest normal matrix is diagonal, its restriction to any pair of coordinate directions is a multiple of a real diagonal and skew nondiagonal 2×2 matrix. A convergent algorithm to bring an arbitrary matrix into that form is described and results of numerical tests are reported.

1 Introduction and summary.

In the present contribution we develop a complete solution to the problem of finding the closest normal matrix to a given square complex matrix. A normal matrix is one which can be diagonalized by a unitary transformation, and we start by giving a characterization of those matrices for which the diagonal part actually is the closest normal matrix. Then we describe a convergent algorithm that brings an arbitrary matrix into the form prescribed by this characterization.

Our problem is :

Problem 1 *Given an $n \times n$ matrix A, find $X \in \mathcal{N}$ minimizing $\|A - X\|_F$.*

Here $\mathcal{N}$ stands for the set of normal matrices.

We regard the set of all complex $n \times n$ matrices as a real vector space of $2n^2$ dimensions. The reason to choose a real vector space instead of a complex one is that this makes the sets $\mathcal{H}$ and $\mathcal{S}$ of Hermitian and skew Hermitian matrices into two linear subspaces each of n^2 real dimensions, together spanning up the entire space. $\mathcal{N}$ is now a homogeneous variety of $n(n+1)$ real dimensions containing both $\mathcal{H}$ and $\mathcal{S}$.

Define an inner product by,

$$(A, B) \equiv \mathrm{Re}\{\mathrm{Trace}(B^H A)\} \quad , \tag{1}$$

and get the Frobenius norm $\| \quad \|_F$ as its induced norm,

$$\|A\|_F \equiv (A, A)^{1/2} \quad .$$

For any matrix A, the components H and S in the decomposition,

$$A = H + S \qquad \begin{cases} H = (A + A^H)/2 \in \mathcal{H} \\ S = (A - A^H)/2 \in \mathcal{S} \end{cases} , \tag{2}$$

are its closest Hermitian and skew Hermitian matrices. The closest unitary matrix can be computed by means of the Singular Value Decomposition as

$$Q = UV^H \text{ where } A = U\Sigma V^H.$$

No corresponding expression for the closest normal matrix is known. In his thesis Causey [1] found a solution to the 2×2 case which is the same as the one developed here. Later Gabriel [2,3,4] has made a thorough investigation of optimality conditions for Problem 1, and his results, unknown to me when this study started, overlap those reported here until quite close to the final result.

The closest Hermitian matrix is invariant under translation, i.e. replacing A by $A - mI$, while the closest unitary matrix is invariant under rotation, replacing A by $e^{i\theta}A$. (By invariant we mean that the approximations are changed in the same way as A). It is easy to understand that the closest normal matrix must be invariant under both of these operations, and that there is a close relation between the field of values of the two matrices. The field of values of a matrix is defined by

$$W_A \equiv \{x^H A x, \quad \|x\|_2 = 1\}. \tag{3}$$

It is a convex set in the complex plane which gets translated and rotated in the same way as A. For a normal matrix it is the convex hull of the set of eigenvalues. See e. g. the text book by Householder [8].

Now we will specify the special form a matrix takes in those coordinates where the closest normal matrix is diagonal. Following Gabriel ([2]), we use the term ΔH-matrix:

Definition 1 *A is a ΔH-matrix if the obesity matrix*

$$Y \equiv (y_{jk}) \quad , \quad y_{jk} = \begin{cases} 0 & if \quad a_{jk} = 0 \\ a_{jk}/(a_{kk} - a_{jj}) & if \quad a_{jk} \neq 0 \end{cases} \tag{4}$$

exists and is Hermitian.

We see that for each pair j, k , we can find a shift m and phase θ so that,

$$e^{i\theta} \begin{pmatrix} a_{jj} - m & a_{jk} \\ a_{kj} & a_{kk} - m \end{pmatrix} = \begin{pmatrix} d & s \\ -\overline{s} & -d \end{pmatrix} \tag{5}$$

with $d \geq 0$ and $s = 0$ whenever $d = 0$, a sum of a real diagonal and a skew nondiagonal matrix. Note that different shifts m and phases θ are given for each of the $n(n-1)/2$ subblocks. Also note that the this property is invariant under shift and rotation of the whole matrix.

Theorem 1 *Every $n \times n$ matrix A has a closest normal matrix N. In the coordinates of the eigenvectors of N, A is a ΔH-matrix.*

We will prove this theorem in Section 2. We further have:

Theorem 2 *There is a convergent algorithm that transforms an arbitrary $n \times n$ matrix A into a ΔH-matrix by means of a sequence of unitary similarities.*

This algorithm, which is actually the Jacobi algorithm for normal matrices as described by Goldstine and Horwitz [5], will be described in Section 3, and its convergence when applied to a general matrix will be proved.

It is true that the diagonal part of this ΔH-matrix actually is the closest normal matrix under an additional condition stated in:

Theorem 3 *If A is a ΔH-matrix and the spread of the obesity matrix Y (4) is less than 1, then its closest normal matrix is $N = \mathrm{diag}(a_{kk})$.*

The spread is the distance between the largest and the smallest eigenvalue. This condition is not the sharpest possible, but it is easy to test when the Jacobi algorithm has been executed. It says that the Lagrange function of the minimization problem 1 has a local minimum at N. Actually it is only necessary that it must have a local minimum with respect to feasible perturbations. We get:

Theorem 4 *For a ΔH-matrix A to have its diagonal as closest normal matrix it is necessary (sufficient) that the tangential projection G_{tang} (15) of the Hessian matrix G (14) is positive semidefinite (positive definite).*

As will be explained in Section 2, this condition is also simple, but involves large $(2n^2 \times 2n^2)$ matrices. All ΔH-matrices with larger obesity are just saddle points, and we have not encountered any case when the Jacobi algorithm has converged towards such a saddle point.

We will conclude the paper in Section 4 by giving a few numerical examples and demonstrating a way to visualize matrices graphically. The reader just interested in seeing the closest normal matrix may move there directly.

We know of no industrial application of closest normal matrices, but it may have some interest for building up a comprehensive theory of matrices. Our results may give sharper tools to measure non-normality of a matrix. The distance from the variety of normal matrices,

$$d_{\mathcal{N}}(A) \equiv \inf_{X \in \mathcal{N}} \|A - X\|_F,$$

is always smaller than the departure from normality as defined by Henrici [7],

$$\Delta_F(A) \equiv (\|A\|_F^2 - \|D\|_F^2)^{1/2} \text{ where } D = \mathrm{diag}(\lambda_k(A)).$$

The obesity Spread(Y) where Y is given by (4), may also give interesting insights. The condition of the minimization problem 1 is $\kappa(G_{\text{tang}})$. If Spread$(Y) < 1$ it can be estimated by,

$$\kappa(G_{\text{tang}}) = \frac{\lambda_{\max}(G_{\text{tang}})}{\lambda_{\min}(G_{\text{tang}})} \leq \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} = \frac{1+\text{Spread}(Y)}{1-\text{Spread}(Y)} \quad . \tag{6}$$

It can be compared to the condition number of a diagonalizing similarity

$$\kappa(S) \text{ where } A = SDS^{-1}$$

2 Conditions for minimum.

Here we prove theorems 1 and 3. First we decompose the perturbation of a diagonal matrix into components tangential and normal to the variety of normal matrices. Using this we investigate first and second order conditions for the constrained minimization problem.

Lemma 1 *Any $n \times n$ matrix A can be decomposed into*

$$A = D + H + S \tag{7}$$

where

$$\begin{array}{ll} D & \textit{is diagonal} \\ H & \textit{is tangential to } \mathcal{N} \\ S & \textit{is normal to } \mathcal{N} \end{array}$$

all in the scalar product (1). *Their elements are given by*

$$\begin{aligned} d_{kk} &= a_{kk} \\ h_{jk} &= \begin{cases} a_{jk} & \textit{if } a_{jj} = a_{kk} \\ (a_{jk} + \exp(2i\theta_{jk})\,\overline{a}_{kj})/2 & \textit{otherwise} \end{cases} \\ s_{jk} &= \begin{cases} 0 & \textit{if } a_{jj} = a_{kk} \\ (a_{jk} - \exp(2i\theta_{jk})\,\overline{a}_{kj})/2 & \textit{otherwise} \end{cases} \quad . \end{aligned}$$

Here the phase θ_{jk} is determined as

$$\theta_{jk} = \arg(a_{kk} - a_{jj}) \quad . \tag{8}$$

We will prove this lemma by producing an orthonormal basis of the space of all complex $n \times n$ matrices, regarded as a $2n^2$ dimensional real vector space. Lemma 1 will then simply be the decomposition of an arbitrary matrix in this basis.

To find a basis vector in the tangential direction of $\mathcal{N}$, we consider a perturbation C to the diagonal (and hence normal) matrix D. $D + C$ is normal if

$$(D + C)^H (D + C) - (D + C)(D + C)^H = 0 \quad .$$

Disregarding second order terms in C we see that this implies:

$$C^H D + D^H C - DC^H - CD^H = 0 \quad ,$$

or for the jk element

$$\overline{c}_{kj}(d_{kk} - d_{jj}) + c_{jk}(\overline{d}_{jj} - \overline{d}_{kk}) = 0 \quad ,$$

which is satisfied for e.g. $c_{jk} = c_{kj} = \exp(i\theta_{jk})$ and $c_{jk} = -c_{kj} = i\exp(i\theta_{jk})$, with θ_{jk} defined by (8).

This gives us $n \times (n-1)$ basis vectors for the tangential subspace, as those corresponding to,

$$\begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} j \\ k \end{array} & \begin{pmatrix} 0 & \varrho \\ \varrho & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} j \\ k \end{array} & \begin{pmatrix} 0 & i\varrho \\ -i\varrho & 0 \end{pmatrix} \end{array} \quad , \tag{9}$$

with $\varrho = \exp(i\theta_{jk})/\sqrt{2}$, and H will be the component of A in this subspace.

The normal subspace is the orthogonal complement of this and will have a basis given by,

$$\begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} j \\ k \end{array} & \begin{pmatrix} 0 & \varrho \\ -\varrho & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} j \\ k \end{array} & \begin{pmatrix} 0 & i\varrho \\ i\varrho & 0 \end{pmatrix} \end{array} \quad , \tag{10}$$

and S will be the corresponding component of A.

All diagonal perturbations, as well as perturbations corresponding to equal diagonal elements, are tangential.

This proves Theorem 1, since if we use the coordinates where the closest normal matrix is diagonal, the difference must be orthogonal to $\mathcal{N}$, which means that $H = 0$ and $A = D + S$ a ΔH-matrix.

In order to prove Theorem 3 we need to formulate our problem as a minimization problem with constraints:

Problem 2

$$\begin{array}{ll} \textit{Minimize} & \phi(X) \equiv \|A - X\|_F^2 \\ \textit{subject to} & g(X) \equiv X^H X - X X^H = 0 \end{array} \quad .$$

The Lagrange function of Problem 2 is

$$L(X, Y) \equiv \phi(X) - \mathrm{Trace}(Y^H g(X)) \quad , \tag{11}$$

where the Lagrange multipliers build up the Hermitian matrix Y and the scalar product is defined by (1) as usual.

Differentiating $L(X, Y)$ is slightly tricky. It is a real valued function of complex matrices. Let us make a perturbation expansion of L, and identify the first and second derivatives with the coefficients of the first and second order terms.

We get

$$\begin{aligned} L(X + H, Y) &= \mathrm{Trace}\{(X + H - A)^H (X + H - A)\} \\ &\quad - \mathrm{Trace}\{Y^H ((X + H)^H (X + H) - (X + H)(X + H)^H)\} \\ &= \mathrm{Trace}\{(X - A)^H (X - A) - Y^H (X^H X - X X^H) \\ &\quad + H^H (X - A) + (X - A)^H H - Y^H (H^H X + X^H H - H X^H - X H^H) \\ &\quad + H^H H - Y^H (H^H H - H H^H)\} \quad . \end{aligned}$$

Using the identities $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$ and $\mathrm{Trace}(A^H) = \overline{\mathrm{Trace}(A)}$, we get the first order term equal to

$$\begin{aligned} & 2\mathrm{Re}(\mathrm{Trace}((X - A)^H H - Y^H X^H H + X^H Y^H H)) \\ = \quad & 2\mathrm{Re}(\mathrm{Trace}((X - A - XY + YX)^H H)) \quad . \end{aligned}$$

The first order condition for a stationary point is

$$X - A - XY + YX = 0 \quad .$$

For diagonal X, we get

$$\begin{array}{lrcl} jj \text{ element}: & x_{jj}-a_{jj}-x_{jj}y_{jj}+y_{jj}x_{jj} & =0 & \Rightarrow \quad x_{jj}=a_{jj} \\ jk \text{ element}: & -a_{jk}-x_{jj}y_{jk}+y_{jk}x_{kk} & =0 & \Rightarrow \quad y_{jk}=a_{jk}/(x_{kk}-x_{jj}) \end{array} \quad .$$

We see that the Lagrange multipliers build up the obesity matrix Y (4). A has to be a ΔH-matrix for this to yield a Hermitian matrix.

The second order condition for a minimum of Problem 2 is that

$$\mathrm{Trace}(H^H H - Y^H(H^H H - HH^H)) \geq 0 \tag{12}$$

for all perturbations H in the tangential direction.

In order to be able to represent the Hessian matrix of L (11), let us first replace each matrix by a long vector,

$$\mathrm{vec}(X) \equiv \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \\ x_{12} \\ \vdots \\ x_{nn} \end{pmatrix} \quad , \tag{13}$$

and get the Hessian from (12) as

$$G = I - (Y \otimes I) + (I \otimes \overline{Y}) \quad . \tag{14}$$

(See [6] or [10] for a discussion on how to use Kronecker products $\otimes$. Note that the second Y is conjugated here, due to the fact that the straightforward definition of $\otimes$ is not appropriate in this complex case.) G is a Hermitian $n^2 \times n^2$ matrix with eigenvalues

$$\lambda_{jk}(G) = 1 - \lambda_j(Y) + \lambda_k(Y) \quad .$$

It is positive definite when

$$\mathrm{Spread}(Y) \equiv \max_{j,k}(\lambda_j(Y) - \lambda_k(Y)) < 1 \quad .$$

This proves Theorem 3.

This condition for a minimum is not the sharpest possible. G need only be positive semidefinite over the tangential subspace. We have found no way to express this in terms of small $(n \times n)$ matrices. On the other hand it is reasonably simple to project the large matrix G into the tangential subspace.

We computed it by applying $n(n-1)/2$ unitary rotations to G (14), with the rotation matrices given by,

$$\begin{array}{c c} & \begin{array}{cc} jk & kj \end{array} \\ \begin{array}{c} jk \\ kj \end{array} & \begin{pmatrix} \varrho & i\varrho \\ \varrho & -i\varrho \end{pmatrix} \end{array} \quad ,$$

for each pair j, k with $j \neq k$. See the basis given in (9) and (10) to understand how this comes about.

After that we separate G into real and imaginary parts,

$$G = G_1 + iG_2 \quad ,$$

with G_1 real symmetric and G_2 real skew symmetric. The real Hessian is

$$\begin{pmatrix} G_1 & -G_2 \\ G_2 & G_1 \end{pmatrix} \quad ,$$

and we get its projection into the tangential subspace by taking

$$G_{\text{tang}} = \begin{array}{c c} & \begin{array}{cc} n^2 & n \end{array} \\ \begin{array}{c} n^2 \\ n \end{array} & \begin{pmatrix} G_1 & -G_{21} \\ -G_{21}^T & I \end{pmatrix} \end{array} \tag{15}$$

where G_{21} are those columns of G_2 that correspond to diagonal elements of A.

A necessary condition for a minimum is now that G_{tang} is positive semidefinite, and a sufficient condition is that G_{tang} is positive definite.

3 The Jacobi algorithm.

We have actually used the Jacobi algorithm for normal matrices as described by Goldstine and Horwitz [5], but using the notion of translation and rotation in the complex plane we can simplify the description considerably. It will also be evident how the algorithm will converge into a ΔH-matrix when applied to a non-normal matrix. Moreover the exceptional case of stationarity that bothered Goldstine and Horwitz will never occur; it corresponds to the algorithm prematurely landing at a ΔH-matrix.

Let us formulate our algorithm:

ALGORITHM J: *For* $s = 1, 2, \ldots$ *until* convergence

1. *Choose pivots* j, k

2. *Determine shift* m *and phase* θ *so that pivot block gets zero trace and real determinant*

$$
\begin{aligned}
m &= (a_{jj} + a_{kk})/2 \\
\theta &= \frac{1}{2} \arg\det \begin{pmatrix} a_{jj} - m & a_{jk} \\ a_{kj} & a_{kk} - m \end{pmatrix}
\end{aligned}
$$

3. *Find rotation angle* φ *that diagonalizes the tangential part of pivot block*

$$
\begin{aligned}
h_{jk} &= e^{-i\theta} a_{jk} + e^{i\theta} \overline{a}_{kj} \\
\varphi &= \frac{1}{2} \arctan \left(\frac{|h_{jk}|}{\mathrm{Re}(e^{-i\theta}(a_{jj} - a_{kk}))} \right) \\
\alpha &= \arg(h_{jk})
\end{aligned}
$$

4. *Perform unitary rotation*

$$A^{(s+1)} = R_s^H A^{(s)} R_s$$

where the nontrivial elements of R_s *are given by*

$$
R_s = \begin{array}{c c} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} j \\ k \end{array} & \begin{pmatrix} \cos\varphi & -e^{i\alpha} \sin\varphi \\ e^{-i\alpha} \sin\varphi & cos\varphi \end{pmatrix} \end{array} .
$$

Note the difference from the previous section. Now we choose the phase θ so that the pivot block has the ΔH-property *after* the transformation.

It is possible to use also these phases in the decomposition (7) , with the understanding that the directions do not relate to the manifold $\mathcal{N}$ in the same way. This difference is important, since it is the only way for a real matrix to move into the complex field.

Looking at the total sum of squares of nondiagonal elements

$$\tau_s = \|H^{(s)}\|_F^2 + \|S^{(s)}\|_F^2$$

it decreases during the sth rotation to

$$\tau_{s+1} = \tau_s - 2|h_{jk}^{(s)}|^2 \quad .$$

Since it is nonnegative it will converge to some value τ_∞.

The convergence of the sequence $\{\tau_s\}$ implies that the differences between successive values tends to zero. If we seek out the maximum element of $H^{(s)}$ when searching for a pivot, then

$$\|H^{(s)}\|_F^2 \leq n^2|h_{jk}^{(s)}|^2 \quad ,$$

making sure that $H^{(s)}$ tends to zero, and we arrive at a ΔH-matrix. (When $H = 0$ the difference between the two choices of θ disappears).

The rate of convergence has appeared to be linear. The changes of the diagonal elements cause migration from S to H between iterations, so the simple analysis that is applicable to the Hermitian case is not possible now. Only when A is normal is the ultimate convergence rate quadratic [9].

4 Numerical examples.

We have computed the closest normal matrix to several test matrices, and specially studied the relations between the fields of values (3) of the different matrices involved.

It is easy to plot the field of values since it is included in the strip

$$\lambda_{\min}(H) \leq \mathrm{Re}(z) \leq \lambda_{\max}(H) \quad ,$$

where H is the Hermitian part of A (2). We can rotate A into $e^{i\theta}A$, and get a new strip at an angle θ to the previous one, and continue so around the clock.

Figure 1: Field of values of random 2×2 example.

Legend for all figures:

$\star$	λ_k	Eigenvalue
$\circ$	a_{kk}	Diagonal element of normalized matrix
$\vert$	s_{jk}	Nondiagonal elements

Note that h_{jk} are zero and the s_{jk} form mid point normal to line from a_{jj} to a_{kk}.

We plot the diagonal elements as points in the field of values, but it is slightly tricky to get a good representation of the nondiagonal ones. We choose to use the decomposition (7) and plot four vectors $|h_{jk}|$,$-|h_{jk}|$, $i|s_{jk}|$ and $-i|s_{jk}|$ in a complex plane shifted and rotated so that the diagonal elements are real $\pm d$ (5).

In Figure 1, we give this plot for the 2×2 matrix,

$$A = \begin{pmatrix} 0.7616 + 1.2296i & -1.4740 - 0.4577i \\ -1.6290 - 2.6378i & 0.1885 - 0.8575i \end{pmatrix} \quad ,$$

which is transformed by Algorithm J into the ΔH-matrix

$$B = \begin{pmatrix} -1.3170 - 1.5432i & 0.9626 - 0.1996i \\ 0.1651 - 0.9691i & 2.2671 + 1.9153i \end{pmatrix} \quad .$$

To get the closest normal matrix N, take the diagonal of this and transform back to the coordinates of A and get,

$$N = \begin{pmatrix} 1.1449 + 0.8324i & -2.0841 - 0.9958i \\ -1.0695 - 2.0473i & -0.1948 - 0.4603i \end{pmatrix} \quad .$$

In this 2×2 example the field of values is an ellipse with the eigenvalues in the foci. After a Jacobi rotation the diagonal elements mark up the major axis and the s_{jk} vectors the minor axis. The obesity Spread(Y) (4) is 0.3948. The other choice of rotation angle yields a configuration perpendicular to the first one with the diagonals marking up the minor axis. In that case the obesity is larger than 1. If the matrix is defective the field of values is circular, making the direction arbitrary and any pair of opposite points can act as diagonal elements.

Let us now turn to the first nontrivial example. We call it $B7$, and it is built up by the first 98 normally distributed random numbers our Matlab implementation used to give. See Figure 2! We believe it to be typical. Some of the diagonal elements are close to the boundary of the field of values, one is actually hitting it. The simple test for minimality given in Theorem 3 does not apply since Spread$(Y) = 1.26$, and we had to project the Hessian (15) in order to make sure that we had a minimum. It may be instructive to study the eigenvalues of the Hessian and the projected Hessian plotted in Figure 3.

We tested different ways of upsetting the algorithm, but it appeared to be rather robust. Starting far from a ΔH-matrix it always converged to the minimal solution. Making an artificial start at a nonminimal ΔH-matrix, the algorithm starts very very slowly to move away from the saddle point, but finally finding its way to the optimum. Some data on the test matrices are collected in Table 1.

Of course we were curious to see how a matrix with a defective eigenvalue problem was treated, and tested on $J7$, a 7×7 Jordan block with zero

Figure 2: B7 a 7×7 matrix of normal random numbers.

Figure 3: Eigenvalues of Hessian G and projected Hessian G_{tang} for test matrix B7. Each eigenvalue is given as a vertical line at the appropriate position in the horizontal direction.

eigenvalue. Here the solution is not unique, and we get

$$N = \frac{6}{7} \begin{pmatrix} & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ e^{i\alpha} & & & & & & \end{pmatrix}$$

with α arbitrary. N is a multiple of a unitary matrix whose spectrum can be rotated by an arbitrary phase angle, see Figure 4! The obesity Spread(Y) is exactly 1. The condition given in Table 1 is modulo such trivial perturbations.

A general nilpotent matrix, formed by taking a random strictly upper triangular matrix, has still its field of values circular but none of the other special properties of $J7$, see Figure 5!

As a final example consider the 12:th order Frank matrix, that was a favorite stumbling block for eigenvalue algorithm developers in the old days. Its elements are:

$$f_{jk} = \begin{cases} \min(j,k) & \text{if } k \geq j-1 \\ 0 & \text{otherwise} \end{cases}$$

It has real eigenvalues and those close to zero are really ill-conditioned. Surprisingly, it appears that the field of values is thickest at the upper end, but order is restored when we see that our normalization algorithm has its hardest time sorting out the left part, leaving a Figure 6 that resembles a winning configuration of the louse game.

Table 1: Summary of results for test matrices.

Matrix A	Distance $\frac{d_N(A)}{\|A\|_F}$	Departure $\frac{\Delta_F(A)}{\|A\|_F}$	Obesity Spread(Y)	Cond. $\kappa(G_{tang})$	Diag. $\kappa(S)$	Sweeps 10^{-9}
B7	0.4479	0.6382	1.2626	25.0	6.23	46
J7	0.3780	1.0000	1.0000	3.26	∞	16
T7	0.6478	1.0000	1.2075	60.8	∞	
F12	0.4215	0.6491	1.5683	101.9	$4 \cdot 10^{12}$	> 200

Figure 4: J7 a 7×7 Jordan block.

Figure 5: T7 a 7×7 nilpotent matrix.

Figure 6: F12 the 12×12 Frank matrix.

Acknowledgement:

Gene H. Golub was the one who set the author in the right direction to seek the closest normal matrix at a meeting in Copenhagen to start a SIAM Nordic Chapter, November 14, 1986. Wu Wenda stopped him in time from including a too good to be true statement, and Ludwig Elsner made him aware of that he was not the only one in the chase by showing him [2], [3], and [4].

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