Ising models and neural networks

H.G. Schaap

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Beoordelingscommissie:	Prof. dr. A. Bovier Prof. dr. H. W. Broer Prof. dr. W. Th. F. Den Hollander

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Chapter 1

Introduction

Imagine a river, continuously flowing on, in a constant way running its course; the water in the river always tends to flow in the direction with the minimal resistance possible. Suppose during heavy rain the water in the river becomes higher and higher. Then, suddenly, the water becomes so high that the river overflows its banks, allowing the water to flow in the nearby meadows. Then it takes some time before the river is adjusted to this new situation.

Statistical physics is an attempt for modelling natural processes. It tries to connect the microscopic properties with the macroscopic properties. For the river the movement of the water molecules is connected to the overall flow. The global properties of the river are given by some universal laws for some characteristics, notwithstanding the huge number of involved water molecules. Because of this huge number we have to apply probabilistic methods (stochastics) on the underlying microscopic differential equations defining the movement of all of the water molecules. Then if we look at the macroscopic properties, we know almost for certain its global behavior. This global behavior can be described by equations only depending on macroscopic properties. The underlying microscopic part is removed by the performed stochastics.

Let us take a die to demonstrate some of the involved stochastic principles. As we know, we have the same probability to throw a 1 or a 6. However, experience tells us that after a small number of throws, the resulting *relative frequencies* of individual numbers can significantly differ from each other. Only when we throw a die a large number of times, the resulting relative frequencies of the numbers become more and more equal. The same result can be arranged when we throw not one die many times, but rather when we throw a lot of dice at once and then look at the relative number frequencies. Obviously throwing one die 1000 times is equivalent to throwing 1000 dice one time. Eventually all of the relative frequencies approach $\frac{1}{6}$: on average every number appears

once if we throw a die six times. This relative frequency of a number is sometimes identified as the *probability* of the number to appear. To shorten notation it is denoted by $P(i)$. For every number i on our die, $P(i) = \frac{1}{6}$. The function which assigns to each number i the corresponding probability, we call the probability *distribution* of the property.

Suppose we have two dice. Then the combined probability equals $P(n_1, n_2)$, where n_i represents the numbers on the two dice. For instance $P(1, 1)$, the probability of throwing with both dice a 1 equals $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. Note that the throw of one die does not depend on the outcome of the throw of the other die. We say that the outcome of die 1 is *independent* of the outcome of die 2. When two properties are *dependent* on each other the expression for the combined probability is in general more complicated.

All matter around us is made out of atoms. Every gram of matter contains around 10^{23} atoms. Often one considers a collection of some global (bulk) properties in addition to the atomic properties. In the description of matter, all the atoms together with the mentioned global properties define the *system*. Any particular realization of the corresponding atomic values is called a *configuration* of the system. Determining the configuration resembles the throwing of 10^{23} dice at once.

Suppose one wants to measure a macroscopic property, for instance the average density. Because of the large number of atoms, in the probability distribution of the atomic values there is no need of the possibility of tracking the locations of the single atoms. In case of the river: during the heavy rainfall, the river has a way of flowing which does change in time. After some adjustments are made, the changes do stop; the river flow becomes stationary. The time scale of the adjustments is extremely large compared to the scale of the local movements of the water molecules. The way in which the system properties are evolving we call the *dynamics* of the system. In the global flow of the river the microscopic movements of the water molecules have been averaged out.

In the next chapter we give a general overview of the part of statistical physics which is important for the two particular kind of models we study in this thesis: neural networks and Ising models.

Neural networks

The first subject of the thesis is about a model originating in the theory of neural networks. In particular we like to understand the concept of memory. Our brain is built up out of billions of neurons connected in a highly non-trivial way. This structure we call a *neural network*. It is difficult to study it directly, because of the huge number of neurons involved in a relatively small area. In order to understand how the memory works, a common approach is to build a simpler model which captures its main features. Just as the neural network of the brain, the model should be sufficiently robust: in transmit-

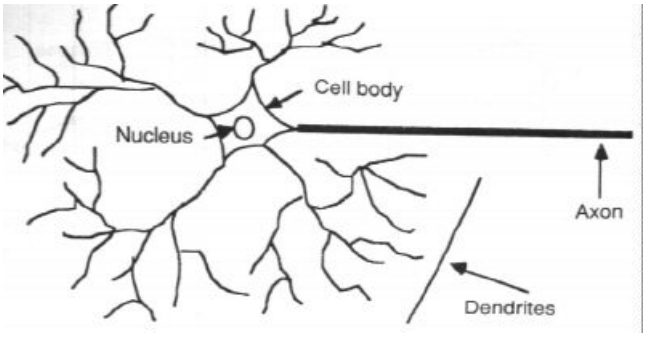


Figure 1.1: *Components of a neuron (taken from [19])*

ting signals between neurons there are always some small errors involved. Given this slightly deformed signal, the brain is able to remove the noise and to reconstruct the pure signal. For a good general overview of neural networks see [19].

A neuron is build up of three parts: the cell body, the dendrites and the axon, see Figure 1.1. The dendrites have a tree-like branched structure and are connected to the cell body. The axon is the only outgoing connection from the neuron. At the end of the axon it branches and it is connected to the dendrites of other neurons via synapses. The end of any branch of the axon is separated from a dendrite by a space called the synaptic gap.

Neurons communicate with other neurons via electric signals. The electric signal of a neuron i transfers to a neuron j in the following way, see Figure 1.2. First it travels from the cell body of neuron i into the axon which is connected to neuron j. This is the *output* signal of neuron i. When the signal of the neuron arrives at the end of the axon it transmits neurotransmitters into the synaptic gap. Then by receptors on the dendrite of neuron j the neurotransmitters are transformed back to an electric signal. There are several types of neurotransmitters. Some of transmitters amplify the incoming signal before transmitting it to the dendrites of other neurons, whereas others weaken it.

This resulting signal originating from the receptors of neuron j we call the *input* signal from neuron i to neuron j. Finally the signal arrives at the cell body of neuron j.

In the cell body of neuron j all the inputs come together. The cell processes the inputs (as we will model mathematically by performing a *weighted sum*), what we call the *total input* h_j of neuron j. Then, depending on the outcome, the cell produces a new signal which is transported to the axon of the neuron j in order to be transferred to other neurons. This is called the output or the *condition* of neuron j.

For making a useful model based on these neural processes we need to make some

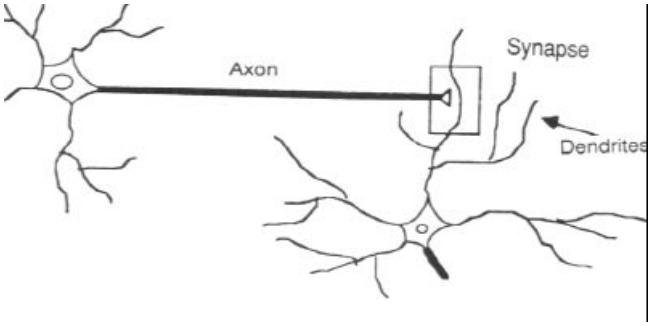


Figure 1.2: *The synapse (taken from [19])*

simplifications. As a first simplification we assume that every neuron interacts with every other neuron. We say that the neural network is *fully connected*. Further we assume that each neuron can have only two possible outputs, i.e. it can only be in two conditions. For reference we denote by σ_i the condition of neuron i: $\sigma_i = +1$ if it is *excited* and $\sigma_i = -1$, when it is at *rest*.

We also assume that no alteration of the signal takes place when it travels across a synaptic gap. As result the input to neuron j which comes from neuron i is equal to the output σ_i from neuron i which is send to neuron j.

For modelling the dynamics of our model we introduce the time t. At every time step Δt (with Δ very small) every neuron output is changed simultaneously. The processing of the cell body of every neuron j we model by two steps:

1. At time t we multiply every input $\sigma_i(t)$ coming from the other neurons with a weight. To obtain the total input $h_j(t)$ at time t we sum the result over all of the neurons (except neuron j).

2. For the output $\sigma_j(t + \Delta t)$ of neuron j at time $t + \Delta t$ we take the outcome of a probability distribution over the two possible neuron conditions. This distribution is formed by a stochastic rule on $h_j(t)$.

We assume that the connections are treated by the neuron cell bodies in a symmetrical way: the weight given in neuron j to the input of neuron i is equal to the weight given in neuron i to the input of neuron j. In realistic neural networks in general this interaction symmetry does not hold.

The dynamics of our model is summarized by Figure 1.3. The stable configurations under this dynamics form the *memory* of the system. Stability means that starting from a stable configuration the system only reaches configurations which are very much alike

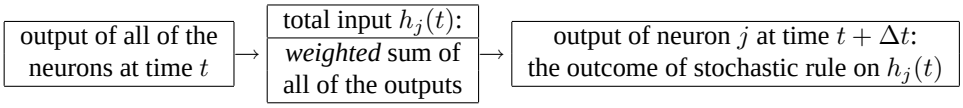


Figure 1.3: *Dynamics of the neural network model*

(in the sense of neuron configurations). By choosing appropriate weights we can tune the dynamics such that the memory is formed by a finite number m of preselected neuron configurations $\xi^{(m)}$, also called *patterns*.

The stochastic rule depends on a parameter β. The inverse $T = 1/\beta$ of parameter β is called *temperature*. If parameter β is large, the neuron has a strong tendency (high probability) to become equal to the sign of its total input. As β approaches infinity the stochastic rule turns into a deterministic one. Then, if we put in a configuration which is close enough to e.g. $\xi^{(1)}$, the system evolves to configurations equal to the pattern $\xi^{(1)}$. In other words the neural network *remembers* the configuration $\xi^{(1)}$ of its memory. This means that the neuron configuration becomes equal to $\xi^{(1)}$ and, afterwards, the system stays in this configuration. This is the so-called zero-temperature dynamics of the Hopfield model, see Section 2.3.2. It is e.g. very useful for information transmission. The dynamics defines algorithms to remove noise from the received signals. Often it is advisable to allow the parameter β to be finite. Then, when we perform the dynamics of Figure 1.3, we have excluded the probability of getting trapped in undesired so-called metastable configurations.

In order to increase the capacity of the memory, obviously one can make the generalization of increasing the number of possible conditions to a finite larger number q. In information transmission if one takes $q = 26$, every neuron state corresponds to a letter. Of course, for $q < 26$ one can also deal with words by a more carefully encoding, but then the encoding becomes less clear. If we make in the above model this generalization to have more possible neuron conditions than two, the resulting model is also known as the Potts-Hopfield model.

In Chapter 3 we choose the weights in the total inputs in a different way. For this we need to define first for each neuron a set of p continuous variables $\xi_i^{(p)}$ which we refer to as patterns. We take at random a realization of these variables. They have a Gaussian distribution. This special distribution is often used in statistics. Then with the values of the introduced patterns $\xi_i^{(j)}$ we determine the weights for the total input. What will be the memory of the resulting model? Are there any stable neuron configurations? We will also look what happens when we increase the total number of neurons. What will be the effect on the memory?

We form the weights of the total input by two Gaussian patterns. The possible number of conditions of a neuron we set to three. When we increase the number of neurons,

for large numbers the following will happen. For a fixed number of neurons, the memory is concentrated around six neuron configurations. These configurations are related to each other by a discrete symmetry. Every neuron configuration is associated to a point in macroscopic space formed by some macroscopic variables. We can differentiate the six stable neuron configurations into pairs of diametrical opposite points. When one increases the number of neurons the discrete symmetry always occurs, however the six configurations tend to rotate on three circles. This we will see in Chapter 3. If we look at the sequence of increasing number of neurons, then in the macroscopic space of above, the appearing stable neuron configurations do fill up the three circles in a regular uniform way.

Ferromagnets

In Chapter 4 we consider a famous model for magnetic materials: the Ising model. In general there are several kinds of magnetism. For the so-called *paramagnets* only when we are applying an external field to it, the metal is magnetized. Otherwise there is no magnetization. Another important type of metal are the *ferromagnets*. These metals retain their magnetization, once they have been exposed to an external field. Initially the ferromagnetic metals have no magnetization. This is comparable with what happens when we magnetize pieces of iron with the help of a magnet. When we heat the material, then eventually this effect disappears: the metal behaves like a paramagnet. For more general information about magnetism we refer to e.g. [46]. We will use the Ising model as a model for ferromagnetism.

Ising models

For justification of the Ising model as a model for a ferromagnet, we need to make some assumptions. We assume that the unpaired electrons of the outermost shell of the atoms are *localized*: i.e. closely bound to the corresponding atoms. Only these unpaired electrons are responsible for the magnetization. For the Ising model we assume that for every atom only one unpaired electron is in the outermost shell.

Every electron has an intrinsic angular momentum which we call *spin*. This spin generates a magnetic moment. Due to quantum mechanics the spin of the electron can have only two orientations with respect to this magnetic moment, which we call *up* and *down* [5]. With a bit abuse of notation we mostly refer to these orientations as the *values* of the spin. Because we have assumed every atom only has one unpaired electron in the outermost shell, we also have only two orientations for the total spin per atom. Most of the metals do consist of atoms with more than one unpaired electrons in the outermost shell. For these metals there can be more than two orientations of the total

spin per atom. Most solid materials are crystalline. The atoms, or ions or molecules do lay in a regular repeated 3 dimensional pattern. This makes some finite number of spin orientations energetically favorable.

In magnetizable metals the metal is divided into domains which have net magnetic moments. The boundaries between these domains are called *domain walls* [46]. The Ising model only allows for configurations in which the spins of two neighboring electrons are parallel or anti-parallel with respect to each other. If there is a domain wall present, the thickness of the domain wall is automatically zero.

The interactions between the localized electrons are also called the *Weiss interactions*. In general two types of interactions do frequently occur: *nearest neighbor* and *mean-field*. When we restrict ourselves to nearest-neighbor interactions, we assume that all of the remaining interactions between the electrons, which are not nearest neighbors, are zero. When the interaction is mean-field, then the interaction between the moments of any pair of sites is non-zero and all of them are equal.

For the Ising model we restrict ourselves to the nearest-neighbor interactions. For the lanthanide series (a particular series of elements) this is a good approximation. Although the model is simple and is for other magnetic metals at most only a rough approximation it is and has been very useful model. It is the first model (and for long time the only model in statistical physics) which displays the phenomenon of phase transition (e.g. think about the liquid $\rightarrow$ gas transition). Furthermore it is exactly solvable in 1 and 2 dimensions. Nowadays the Ising model (and generalizations of it) appears in several places, e.g. all kinds of optimization problems, voter problems, models for gas versus liquid, etc.

Now we give a mathematical description of the model. Take a piece of a lattice. Every point where a vertical line does cross a horizontal one we refer to as a *site*. The horizontal and vertical line-pieces starting from a site and ending by the nearest next crossing we refer to as *bonds*. On every site i there is an atom which has a net spin magnetic moment to which the spin can have only two orientations. We denote the spin-value by $\sigma_i = +1$ when the spin is oriented up and $\sigma_i = -1$ when the orientation of the spin is down. We refer both to the atom as to the spin orientations as *spin*. The configuration σ of the spins is in our case an array, which contains the spin-values σ_i of every site.

Between each pair of nearest-neighbor spins (i.e. every pair of spins associated with a single bond) there is an *interaction*

$$E_{ij}^{\text{ex}} = -\beta\sigma_i\sigma_j \equiv J\sigma_i\sigma_j \tag{1.1}$$

We call often this interaction also the *exchange energy* between the atoms on site i and j. The *energy* of a configuration is the total of these exchange energies. The variable β

is the inverse of the temperature times a constant, which depends on the type of material considered.

The probability of the configurations are determined by these interactions. In ferromagnets the nearest-neighbor spins tend to have equal orientations, i.e. they tend to be *aligned*. Therefore we have chosen the interactions in the model such that it becomes more probable for spins to be aligned: we have set $J < 0$. When $J > 0$, the model behaves like an antiferromagnet. Then it becomes more probable for spins to be anti-aligned. The higher the energy is the less probable the configuration becomes. The probability of a single configuration equals

$$P(\sigma) = \frac{\exp\left(\beta \sum_{i,j} \sigma_i \sigma_j\right)}{Z(\sigma)} = \frac{\exp\left(-\sum_{i,j} E_{ij}^{\text{ex}}\right)}{Z(\sigma)} \tag{1.2}$$

where $Z(\sigma)$ is the sum of the numerator over all configurations. We see that when the temperature gets lower, the interaction (1.1) becomes stronger. Then it becomes more probable for nearest-neighbor spins to be aligned. From (1.2) we immediately see that for zero temperature only the two configurations which minimize the energy do appear with positive probability: i.e. every spin has the same orientation. For very high temperatures every configuration becomes almost equally probable. Then the model behaves like a paramagnet. The temperature is thus a measure of the disorder in the system. For low temperatures most of the spins do align with each other, for high temperature the orientations of the spins are more or less randomly up or down. In Chapter 4 we will consider the most interesting part, the low-temperature ferromagnetic region of the Ising model.

Until this moment we did not bother about the environment. When the energy of the system is independent of this environment, we say that the system has *free boundary conditions*. But what happens when this environment is formed by a different material with a particular chosen configuration of spins? The values of the spins next to the boundary not only tend to align the internal spins but also feel the nearest-neighbor spins in the environment.

In general a piece of metal contains a lot of atoms. Already one gram contains around 10^{23} atoms. One likes to consider volumes which are of the order of the size of the piece of metal. The volume size is measured in the number of atoms, thus also of the order of 10^{23}. In the mathematical description of the model we approximate this huge number by infinity. First one takes a large finite-volume version of the Ising model. Then one tries to extrapolate the resulting expressions to an 'infinite' volume size model.

What will happen to the system when we increase the volume size, and choose for each step the orientations of the external spins arbitrarily up or down, i.e. we take *random boundary conditions*? How does the alignment of the spins change in the process

of increasing the volume? It turns out to be dependent on the way we let the volume size increase. Our results depend on letting it increase fast enough. Furthermore we need to choose the temperature very low so that there is a strong tendency for the spins to align.

Then, in the long run, by (1.2), in the appearing configurations almost every spin has the same orientation. However, because we have chosen the boundary conditions randomly, for half of the volumes the appearing configurations will have almost all of the spins up and for the other half of the volumes the configurations have almost all of its spins down.

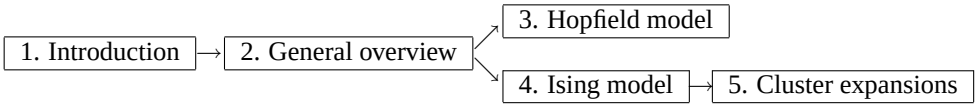
But now, if we look into the volume but far away from the boundary? Do we still see an effect of the boundary conditions? We prove that the local volume density of the area's of aligned spins becomes asymptotically independent of the boundary conditions. However, even for very large volumes, there is a significant effect on the density of spin values. If we look at a fixed (very large) volume, then with probability one, either all configurations have all the spins up or have all the spins down. Almost all of the orientations becomes equal to the orientation of the majority of the external spins which are involved in the boundary condition. Because of the non-zero temperature a small part of the spins has an opposite orientation.

Because we have increased our volumes fast enough the so-called *mixtures* do not appear. This means we do not have with nonzero probability both type of configurations: i.e. having configurations with most of the spins up and configurations with most of the spins down.

This is the subject of Chapter 4. There as a technical tool we need to introduce non-trivial expansion techniques, called *multi-scale cluster expansions*. Our multi-scale expansion method is inspired by the ideas of Fröhlich and Imbrie [35]. The multi-scale expansion is a generalization of the more familiar 'uniform' cluster expansion technique. To simplify our estimates we choose to use a different representation of the expansions from the one used in [35], the so-called *Kotecký-Preiss representation*, which was developed just two years later [50].

In order to have useful expansions, one needs to prove certain *criteria*: we need the convergence of some summations related to the expansions. For cluster expansions it is crucial to check the Kotecký-Preiss criterion. However, in our expansions it is impossible to prove it directly. Therefore we introduce a new criterion, which we prove to be equivalent. This new criterion enables us to obtain useful estimates even for our expansions. In the final chapter the uniform and multi-scale cluster expansions are explained more thoroughly.

Schematically the thesis is built up as follows:


1. Introduction
2. General overview
3. Hopfield model
4. Ising model
5. Cluster expansions

Chapter 2

General overview

2.1 Gibbs measures: Ising model

2.1.1 The Ising model

In some metals, some fraction of the atoms becomes spontaneously magnetized, when the temperature is low enough. This happens for instance in iron and nickel. The magnetized spins, which are the intrinsic magnetic moments of the atoms, tend to be polarized in the same direction (e.g. all up) which gives rise to a macroscopic magnetic field. We call this ferromagnetic behaviour. However, when the temperature is above some T_c then all spins are oriented randomly and there is no macroscopic magnetic field anymore [44]. The interaction between the magnetic moments is short-range. However these short-range interactions do provoke long-range ferromagnetic behavior in the system. These metals have a rather homogeneous-crystalline structure with the atoms fixed, apart for some minor moving. This makes that the short-range interactions are typically homogeneous ones.

The Ising model tries to model this transformation of typically-homogeneous short-range interactions into long-range phenomena in physical ferromagnets. In this model we look only at the basic features of a ferromagnet. We assume that the metal atoms are on a regular crystalline lattice Λ, which is in general a subset of $\mathbb{Z}^d$. Every point of the lattice contains precisely one atom.

Furthermore this atom is fixed and the only degree of freedom is its spin i.e. its magnetic moment. In reality the atom moves a bit around its lattice point, but because of strong crystalline binding this movement is limited. In this model we have neglected the effect of these movements. We assume that the environment outside the metal changes adiabatically slowly. For real ferromagnets this is indeed typically the case when we compare the microscopic changes in the crystal with the macroscopic

exterior environment. Without any harm we consider this environment to be fixed: the so called boundary condition to Λ. On every point i of Λ there is precisely one particle which has only its spin-value σ_i as degree of freedom. The spins can only point up or down, or equivalently its spin-values σ_i are restricted to $\sigma_i = \pm 1$. Here is some re-scaling involved, but for the total picture this pre-factor is not important. There are only nearest-neighbor pair interactions between the spins.

In reality crystals are never perfect, and because of thermal excitations some points of the lattice are empty and other parts of the lattice are deformed. Also more spin-values are allowed. Despite its serious restrictions compared to reality, the Ising model still shows the long-range ferromagnetism it was designed for (if the dimension $d \geq 2$ and the temperature T is low enough) e.g. [20]. This is in contrast to Ising's claim; he found no ferromagnetism in dimension 1 and he conjectured wrongly that the same was true for $d \geq 2$.

As usually happens to simple models, all sorts of generalizations to the Ising model have been done. The reality connection with the ferromagnets is often not so clear or not even there at all. However, we see Ising models in various places to explain many phenomena; Ising models are equivalent to lattice gases, closely related to many percolation problems and useful for optimization problems as well.

We can generalize the Ising model by allowing the spins to have more spin-values. The result is a so-called Potts model when this amount of spin-values is finite. It was proposed by Domb as a subject for his student Potts. Using duality arguments Potts was able to determine for the standard Potts model for $d = 2$ the critical points β_c for all values of q. Further on, in Chapter 3, we will see these Potts spins of the standard Potts model. Another way of generalizing is to allow the spins to have continuous values on a sphere: e.g. the Heisenberg model.

We return to the Ising model and make things more concrete. So -in other words- let's put the model into math. We use the canonical-ensemble description from statistical physics. It describes systems for which their exterior functions as a heat reservoir. Each member of the ensemble is represented by a point in the phase space. All the possible system behavior is described by this phase space together with a probability distribution on the ensemble. For Ising systems the phase space is discrete because the only freedom of the system are the spins. Because each spin can take only two values the phase space equals $\{-1, 1\}^{\mathbb{Z}^d}$. Each point of the phase space we call a (spin) configuration. Denote by σ the spin configuration $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$. The restriction of σ to the finite-sized Λ we refer to as $\sigma_\Lambda \in \{-1, 1\}^\Lambda$, where

$$\Lambda = \{-L, -L+1, \cdots, L-1, L\}^d \tag{2.1}$$

and $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$.

When the system settles into thermodynamical equilibrium, the probability of the spins to be in the configuration σ is described by the so called (finite-size) Gibbs measure:

$$\mu_\Lambda^\eta(\sigma_\Lambda = \hat{\sigma_\Lambda}) = \frac{\exp\left(-H_\Lambda^\eta(\hat{\sigma_\Lambda})\right)}{Z_\Lambda^\eta} \tag{2.2}$$

We denote by $< . >^\eta$ the expectation of the argument with respect to the Gibbs measure μ_Λ^η. Z_Λ^η is the partition function which we obtain by summing over all configurations the corresponding Gibbs-weight of the configuration.

The free energy of the system per spin equals

$$F_\Lambda^\eta = -\frac{1}{\beta|\Lambda|}\log Z_\Lambda^\eta, \text{ with } \beta = \frac{1}{T} \tag{2.3}$$

For a set $A \subset \mathbb{Z}^d$, the symbol $|A|$ refers to the number of sites contained in A. For more details and the derivation for the particular choice of the Gibbs measure μ_Λ we refer to any statistical mechanics book, for instance [44].

The functions $H_\Lambda^\eta(\sigma)$ are the energy functions or the Hamiltonians of the configurations σ_Λ. For the Ising model they are defined as follows:

$$H_\Lambda^\eta(\sigma_\Lambda) = -\beta \sum_{\langle x,y\rangle \subset \Lambda} (\sigma_x\sigma_y - 1) - \beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\, y\in\Lambda^c}} \sigma_x\eta_y \tag{2.4}$$

where $\langle x, y\rangle$ stands for nearest neighboring sites. This means in particular that $\|x - y\| = 1$, where $\|.\|$ is the Euclidean norm. By η we denote the fixed boundary conditions, i.e. to the spin-values of the spins in Λ^c. When we do not include boundary conditions we speak about free boundary conditions. Equivalently we drop the second term in the Hamiltonian. Indeed, the expression for the resulting free energy then is independent of the boundary condition. For the corresponding Hamiltonian we write $H_\Lambda(\sigma)$.

Note that because the interactions are only nearest neighbor only the η's in the sites $x \in \Lambda^c$ with $d(x, \Lambda) = 1$ are involved. Z_Λ^η is the partition function which we obtain by summing over the Gibbs-weights of all configurations σ_Λ. As we see from (2.4) the spins tend to align to each other.

Mean field: Curie Weiss

In general, the partition function Z_Λ^η is hard to compute for the Ising model. For one dimension this can be treated simply by the so called transfer matrix methods. When $d = 2$ there is the famous, much more involved, Onsager solution which gives an complete analytic expression also by using transfer matrices. For higher dimensions however only partial results are known. So some approximation is introduced: the

mean-field theory (we follow closely [68]). With this approximation we are able to obtain an explicit expression for the Gibbs average of the global magnetization.

We look at free boundary conditions and we rewrite the Hamiltonian to

$$H_\Lambda(\sigma_\Lambda) = \beta N(L) - \beta \sum_{\langle x,y\rangle \subset \Lambda} \sigma_x \sigma_y \tag{2.5}$$

where N(L) is the number of nearest-neighbor bond pairs. Because the first term is not dependent on the spin-variables it drops out in the Gibbs measure. So we are allowed to ignore it.

Then we 'expand' every spin σ_i around its Gibbs mean value $< \sigma_i > \equiv m$ and denote the fluctuations by $\Delta_i = \sigma_i - m$. Rewriting the Hamiltonian (2.4) gives for free boundary conditions

$$H_\Lambda(\sigma_\Lambda) = -\beta \sum_{\langle x,y\rangle \subset \Lambda} (m + \Delta_x)(m + \Delta_y) \tag{2.6}$$

Now we assume that we can neglect the higher order terms in Δ so

$$H_\Lambda(\sigma_\Lambda) = \beta m^2 N(L) - \beta m \sum_{<x,y>} (\sigma_x + \sigma_y) = \beta m^2 N(L) - 2d\beta m \sum_x \sigma_x \tag{2.7}$$

Here we have assumed that every site i has 2d bonds coming out from it. The corners and intersecting planes on the boundary of Λ are of lower dimension and therefore ignored.

With the above the partition function easily follows:

$$Z_\Lambda = \mathrm{Tr}_\sigma \exp\left(-\beta H_\Lambda(\sigma_\Lambda)\right) = \mathrm{Tr}_\sigma \exp\left(\beta m^2 N(L) - 2d\beta m \sum_x \sigma_x\right) = \exp \beta m^2 N(L) (2 \cosh \exp 2d\beta m)^{|\Lambda|} \tag{2.8}$$

By Tr_σ we mean the sum over all possible $2^{|\Lambda|}$ configurations. Now we remember that $m =< \sigma_i >$ which is the Gibbs-expectation of the mean of a single spin-value. When we put it in, we obtain the so called mean-field equation for m:

$$m = \frac{\mathrm{Tr}_\sigma \, \sigma_i \exp\left(-\beta H_\Lambda(\sigma_\Lambda)\right)}{Z_\Lambda} = \tanh 2d\beta m \tag{2.9}$$

This equation has three solutions $m^\star$, 0 and $-m^\star$ whenever $2d\beta > 1$, i.e. when $\beta > 1/2d$. The critical value β_c: $2d\beta_c = 1$ is the value where region ends where there is no global magnetization, i.e. there is no non-zero solution.

It turns out that the above mean-field equation (2.9) (after re-scaling) is the exact solution for $m =< \sigma_i >$ of the infinite range version of the Ising model (see e.g. [68]). This version is also called the Curie-Weiss model which has as Hamiltonian

$$H_N(\sigma) = -\frac{\beta}{N} \sum_{i \neq j} \sigma_i \sigma_j \tag{2.10}$$

where $1 \leq i, j \leq N$. Each spin has an (uniform) interaction with any other spin. We will encounter more mean-field equations in Chapter 3.

2.1.2 Thermodynamical limit

In nature macroscopic systems are extremely large of the order of 10^{23} atoms and more. So it is natural to take the system size limit $L \to \infty$. But when we take this limit the Hamiltonian goes to infinity as well. The infinite limit expression of the Hamiltonian does not make any sense. So how to define an infinite-volume Gibbs measure which depends on this divergent function?

All is settled by defining the infinite-volume Gibbs measure by the condition that all the conditional probabilities to finite-sized volumes are finite-size Gibbs measures in a consistent way. The corresponding equations due to this condition are called the DLR-equations.

Definition 2.1. *An infinite-volume measure μ is a Gibbs measure if it satisfies the so-called DLR-equations:*

$$\mu(\cdot|\eta_{\Lambda^c}) = \mu^{\eta}_{\Lambda}(.) \tag{2.11}$$

for all finite Λ and μ-a.e. every η.

Equivalently: if we condition μ on the configuration η outside Λ we obtain the finite-volume Gibbs measure μ^{η}_{Λ}.

If we look at the finite-size Gibbs measures $\mu^{\eta}_{\Lambda_L}$ and if we take the sequence $L = 1, 2, \cdots$ it depends on the boundary condition η what will happen for very large L. The sequence does not need to settle to a single limit Gibbs measure. For $L \to \infty$ the sequence may oscillate between two or even more infinite-volume Gibbs measures.

To see some limiting structure one can define metastates. These metastates are probability measures over the infinite-volume Gibbs measures. Later on we reveal more details about metastates in Section 2.5.

When we cannot write the Gibbs measure μ as a combination of Gibbs measures, e.g. $\mu = (\mu' + \mu'')/2$, we call μ an extremal Gibbs measure or a pure state. From (2.11) follows when μ' and μ'' are pure states, then all the convex combinations in between are infinite-volume Gibbs measures.

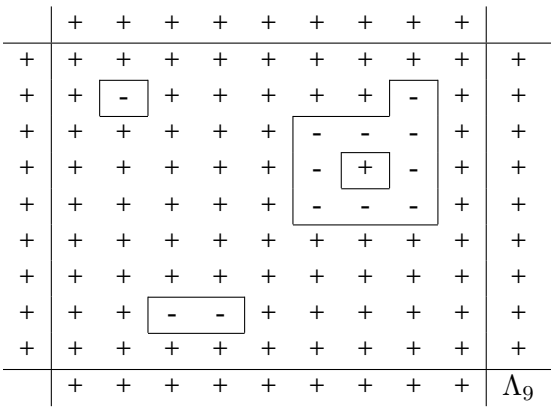


Figure 2.1: *Typical configuration for* μ_β^+

As $T \to 0$ the inverse temperature $\beta \to \infty$. From (2.2) we see that we obtain for infinite volumes only Gibbs measures μ for which configurations of a strictly non-zero weight (with respect to μ) do minimize the corresponding energy function $H^\eta(.)$.

We call these states ground states and the corresponding set of non-zero weight configurations ground-state configurations due to the following property. From the corresponding ground-state configurations σ, for every configuration σ' we can create by flipping any finite number of spins in σ the following holds: the difference of $H^\eta(\sigma') - H^\eta(\sigma) \geq 0$. Note that we need to be careful, because in the infinite volume limit $\Lambda \to \infty$ the energy tends to $-\infty$ for a lot of configurations.

This does not mean that there are no states σ' for which the difference $H^\eta(\sigma') - H^\eta(\sigma) < 0$, where σ is a ground-state configuration. What it does mean in dynamical sense, is that the system will stay in the same state for an infinite amount of time.

2.1.3 Some choices of boundary conditions

For getting a better understanding of the Gibbs measure subjects we just introduced, we consider some examples. All is for the Ising model defined in Section 2.1.1. For simplicity we restrict ourselves mostly to 2 dimensions.

Uniformly agreeing

First we take as boundary condition $\eta \equiv 1$, i.e. every site y has $\eta_y = +1$. Looking at (2.4) we see easily that only the configuration $\sigma \equiv +1$ minimizes the Hamiltonian. This means that there is exactly one ground state μ^+ which equals $\mu^+_{\beta\to\infty}(\sigma) = \delta(\sigma \equiv +1)$.

For β large enough but finite, the Gibbs state μ_β^+ which does appear tends to concentrate around this configuration $\sigma = +1$. The set of configurations σ' which do appear with μ_β^+-measure 1 is of the following structure: σ' has typically small islands of $-$ spins in a sea $+$-spins. The small islands have small lakes of $+$-spins which can contain islands of $-$-spins and so on. This set we will refer to as the $+$-ensemble later on. See Figure 2.1 for an example.

The same is true for the boundary condition $\eta \equiv -1$. Then the configurations has small islands of $+$-spins surrounded by $-$-spins: the $-$-ensemble.

We can make this image plausible by proving the absence of large contours: in literature often referred to as a Peierls bound. Consider all the bonds of the dual lattice $\mathbb{Z}^{2\star}$ between nearest neighbor spins which have opposite signs. When we take the union, the resulting closed curves Γ do form the boundary between $+$ and $-$ spins. Every closed curve we call a contour Γ. The length $|\Gamma|$ of the contour is the number of dual bonds involved. Because of the boundary condition $\eta \equiv +1$ every contour does appear as a closed curve. Every set of non-intersecting contours defines exactly one configuration when we only look at the $+$-boundary condition and vice versa. Later on for different boundary conditions a more general definition is needed and more general curves do appear.

When we look at the definition for the Hamiltonian (2.4) we see that

$$H_\Lambda^+(\sigma = \{\Gamma\}) - H_\Lambda^+(\sigma \equiv +) = 2\beta|\Gamma| \tag{2.12}$$

This means that for the relative probability it holds:

$$\frac{\mu^+(\sigma = \{\Gamma\})}{\mu^+(\sigma \equiv +1)} = \exp\left(-2\beta|\Gamma|\right) \tag{2.13}$$

also called the weight or the cost of contour Γ. Note that the weight of a configuration consisting of more contours factorizes into the weights of the single contours making up the configuration.

Now we can prove the statement:

Peierls bound: *Assume $\beta > (\log 3)/2$ and $+$ boundary conditions. Then for any $\theta > 0$ with μ^+-probability one there are no contours larger than L^θ when $L \to \infty$.*
Proof.

$$\mu_\Lambda^+ \underbrace{(\sigma : \exists\, \Gamma \text{ with } |\Gamma| \geq L^\theta)}_{\theta > 0, \text{ possibly } L^\theta \ll L^d} \equiv \mu_L^+(\sigma : \star) \tag{2.14}$$

	+	-	+	-	+	-	+	
-	.	.	.	.	.	.	.	-
+	.	.	.	.	.	.	.	+
-	.	.	.	.	.	.	.	-
+	.	.	.	.	.	.	.	+
-	.	.	.	.	.	.	.	-
+	.	.	.	.	.	.	.	+
-	.	.	.	.	.	.	.	-
	+	-	+	-	+	-	+	Λ_7

Figure 2.2: *Alternating boundary conditions η for Λ_7*

Because of factorization $H^+_\Lambda(\{\Gamma_1, \Gamma_2\}) = H^+_\Lambda(\{\Gamma_1\}) + H^+_\Lambda(\{\Gamma_2\})$ and therefore

$$\mu^+_\Lambda(\sigma : \star) = \frac{1}{Z^+_\Lambda} \sum_{\sigma:\star} \exp\left(-\beta H^+_\Lambda(\sigma)\right) =$$

$$\sum_{\Gamma:|\Gamma|\geq L^\theta} \exp\left(-2\beta|\Gamma|\right) \frac{1}{Z^+_\Lambda} \sum_{\sigma':\, \sigma=\underbrace{\{\Gamma'\}}_{\sigma'}\cup\Gamma} \exp\left(-\beta H^+_\Lambda(\sigma')\right) <$$

$$L^d \sum_{n=L^\theta}^{\infty} 3^n \exp\left(-2\beta n\right) \leq 2L^d \exp\left(-(2\beta - \log 3)L^\theta\right) \to 0$$

$$\text{for } L \to \infty,\ \theta > 0\ ,\ \beta > \frac{1}{2}\log 3 \quad (2.15)$$

□

Note that the proof of the Peierls bound heavily depends on the uniform exponential size decay of the contour weights.

Alternating

Now we choose the boundary condition η as an alternation of $+$ and $-$ spins, see Figure 2.2. Every boundary spin involved has a sign opposite to its nearest neighbors. Note that this boundary condition gives rise to contours which are not closed curves.

Because the boundary condition does not favor any sign, the ground state $\mu(\sigma) = \frac{1}{2}\left(\delta(\sigma \equiv +1) + \delta(\sigma \equiv -1)\right) = \frac{1}{2}(\mu^+ + \mu^-)$, when we take even volume sizes. We see that this boundary condition gives rise to a mixture; the ground state is a combination of the two pure states $\mu^+ = \delta(\sigma \equiv +1)$ and $\mu^- = \delta(\sigma \equiv -1)$.

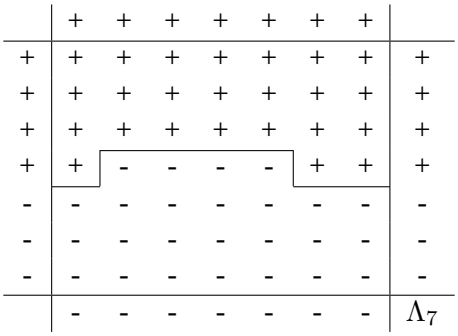


Figure 2.3: *Typical configuration for* $\mu_\beta^{Dobrushin}$

The Gibbs states do concentrate now around both pure states which together do make the ground state: $\mu = \frac{1}{2}(\mu^+ + \mu^-)$. The measure μ^+ is the Gibbs measure which concentrates only on the $+$-ensemble and μ^- concentrates on the $-$-ensemble.

We claim that this means that there are no interfaces involved with probability one. By an interface we mean a contour which crosses the square lattice (so is at least of order L). Of a (vertically-crossing) interface maximally half of the vertical bonds do cancel in considering the weight; the weight of an interface is at most $\exp(-\beta|\Gamma|)$ so we can apply again the Peierls bound for proving the claim.

Dobrushin

Now we create a boundary condition for which interfaces do exist. We choose η as follows. For the upper half of the boundary we take all the spins $+1$ and for lower half we do the opposite: all the spins -1. This boundary condition is also called the Dobrushin boundary condition.

The possible form of the ground states is dimension-dependent. For $d = 2$ ground states and for $d = 3$ also Gibbs states do exist with an interface like in Figure 2.3. This means that the interfaces appear with non-zero probability at a particular position.

Chaotic size dependence

When we choose the boundary conditions carefully we can ensure that the system does not have a limiting Gibbs measure. Take for even system size L the boundary condition $+$ and for odd L the boundary condition $-$. Then the sequence $\mu_{\Lambda_{2L}}$ converges to the unique Gibbs measure μ^+. The restricted odd sequence $\mu_{\Lambda_{2L+1}}$ converges to μ^-. However the full sequence μ_{Λ_L} oscillates between μ^+ and μ^- and never settles to a

limit. It depends on the volume size what the measure looks like even when this size goes to infinity. This limiting dependence we call size dependence. Instead of even and oddness we can also choose the $+$ or $-$ boundary conditions in a random way. Then for very large L the measure still may depend randomly on L. This is called chaotic size dependence.

Quenched-random boundary conditions

The environment around the system can be changing randomly in time. However in reality when this happens then these changes are typically adiabatically slow with respect to the dynamical changes of the system. To model this we assume the external environment is fixed and is an outcome of the random variables making the randomness of the environment. This type of randomness we call *quenched disorder*. However we must be a bit careful when we say that the external environment is fixed. Although the disorder is quenched and therefore fixed, the boundary condition changes randomly when we look at increasing sequences of volumes which are independently chosen of the disorder.

Choose all η_i i.i.d. (=independently identically distributed) according to the following distribution

$$P(\eta_i = \pm 1) = \frac{1}{2} \tag{2.16}$$

What will happen now? This is the question a considerable part of this thesis is all about, in particular Chapter 4. Are there Gibbs states or ground states involved which do contain all the above features: mixtures, interfaces, + and - ensembles? The answer is not obvious from the beginning. Because although the probability of having interfaces goes to zero in the η-distribution (Dobrushin-type configurations), it certainly does not immediately follow that the Gibbs probability μ also goes to zero for the interfaces.

Furthermore there is no such thing as a limiting Gibbs state, because chaotic size dependence is involved. For sparse enough sequences the limiting measure oscillates randomly between measures concentrated on the $+$-ensemble and measures concentrated on the $-$-ensemble. For large enough volumes, with η probability one, neither interfaces nor mixtures will occur. The above model is one of the simplest in which one can study these things rigorously. The concepts have been developed for spin glasses, in which much less is clear even at a heuristic level.

2.2 Spin glasses

2.2.1 Ising spin glasses

In the previous section we have considered the Ising model, which models metals with uniform spin-interactions. The spin glasses we now consider, are modelled by a system with the same concepts but now the spin-interactions will be modelled by random interactions. The atoms do not lay regularly on a crystal but are randomly placed in space. In these spin-glasses these random places do change very slowly in time. Compared to the dynamics of the spins these positions are fixed. After some time the spin values are more or less like random distributed but do not change in time anymore. This rather unusual behavior is seen in some alloys of ferromagnets and conductors like AuFe and CuMn. In these metals the so called RKKY spin interactions are rapidly oscillating and slowly decreasing. Because the atoms are randomly placed the sign of the interactions is also random distributed.

We use the term glass because of the similarity with the glass of windows, which are fluids but where the flow is almost infinitely slow. In the literature the term spin glass is often used for a wider class of models which have a high amount of quenched disorder in common but where the connection to the alloys is often lost.

Spin-glass models with infinite-range interactions turn out to be useful also for explaining pattern recognition in neural networks, in error-correcting codes, image restoration, and in all kinds of optimization problems [68].

For an explanation of the spin-glass phenomena the Edwards-Anderson model has been introduced. This is an Ising spin glass with only nearest neighbor interactions and therefore has only interactions between neighboring pairs of spins. The rapidly oscillating interactions are modelled by i.i.d. Gaussians.

The Hamiltonian is as follows

$$H_\Lambda = -\beta \sum_{<i,j>\in\Lambda} J_{ij}\sigma_i\sigma_j + h\beta \sum_i \sigma_i \tag{2.17}$$

The J_{ij} are quenched i.i.d. non-trivial random variables with common mean $\mathbb{E}[J_{ij}] = \mathbb{E}[J_{12}] \equiv \mathbb{E}[J]$ and h is an uniform magnetic field. The set Λ is a subset of $\mathbb{Z}^d$. Because no boundary conditions do enter in the Hamiltonian the boundary conditions here are free.

In real life when we study spin-systems we expect to observe only quantities which depend on macroscopic properties. The couplings are microscopic and in practice we do not know all the random places of the individual atoms. When we make the setup we do this without knowing the particular realization of the couplings. So for a proper measurement we need that the macroscopic properties we measure are coupling-

independent. Therefore we should observe only states which we can create in a coupling-independent way. These states we call observable states. If a state is not observable we will call it an invisible state [67].

When $h = 0$ it depends on $\mathbb{E}[J]$ and on β for which kind of configurations there is a tendency to order in the system. When β is low enough: $\beta < \beta_c$ then there is no ordering in the system at all. The spins behave approximately independent of each other; the system shows *paramagnetic* behavior. For some systems only this behavior is possible and $\beta_c = \infty$.

When $\beta \geq \beta_c$ there are three possibilities. When $\mathbb{E}[J] > 0$ the system prefers *ferromagnetic* behavior: all spins tend to have equal values. For $\mathbb{E}[J] < 0$ the spin values of nearest neighbor spins tend to be different from each other, which means the system prefers to be *anti-ferromagnetic*. The third possibility, happening typically when $\mathbb{E}[J] = 0$, is *the spin glass phase* which we will consider now.

A good way to see these tendencies is to look at the so called Edward-Anderson order parameter q_{EA}:

$$q_{EA} = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} < \sigma_i >^2 \tag{2.18}$$

where $< . >$ is the Gibbs mean of the argument and $|\Lambda|$ is the total size or the volume of Λ. For paramagnetic behavior $< \sigma_i >= 0$ for every site i, making $q_{EA} = 0$. When the system behaves like a *ferromagnet* or a *anti-ferromagnet* $< \sigma_i >^2 = 1$ for every site i. This makes $q_{EA} = 1$: its maximal possible value.

When we set the average $\mathbb{E}[J]$ of the couplings to zero the systems prefers as many spin pairs for which $\sigma_i \sigma_j = +1$ as for which $\sigma_i \sigma_j = -1$.

With the field h we have some control over the spin-values, when the average $\mathbb{E}[J] = 0$ or small in magnitude compared to h. When we put $h > 0$ the system gives preference to $+$-spins, when $h < 0$, the $-$-spins are more favored. However when the field is not too large there is an intimate interplay between tendency due to the field h and the tendency due to the couplings J_{ij}.

Denote by $[.]$ the coupling average: the average over the disorder J_{ij}. For calculating e.g. the averaged free energy we first calculate the trace as before with a fixed random realization. Then we take the average over the randomness. This because the change in randomness over time is adiabatically small compared to the spin value changes due to thermal activity.

The free energy per spin turns out to be a self-averaging quantity. This means that with probability 1 the free energy per spin for a fixed realization of the couplings is equal to the coupling mean when we take the system size limit $\Lambda \to \infty$. So the limit is independent of the realization of the couplings. This is as it should be, because the free energy is a macroscopic object.

When the temperature is not too low $< \sigma_i >= 0$ for any site i, because of paramagnetism. This makes both the average magnetism $m = [< \sigma_i >] = 0$ and $q = [< \sigma_i >^2] = 0$. For low enough temperature in general $< \sigma_i > \neq 0$ due to the quenched couplings. However when we average over the disorder it can happen due to alternating signs that $[< \sigma_i >] = 0$ although $< \sigma_i > \neq 0$ for the typical realizations of the quenched couplings. But $q = [< \sigma_i >^2] = [q_{EA}] > 0$. This scenario is called *the spin-glass phase* which is the third possibility we have mentioned earlier on.

2.2.2 Mean field: SK-model

Because calculations for the short-range EA-model are extremely hard we can try to do the mean-field approximation like we did in the Ising model. The result is the infinite-range Sherrington-Kirkpatrick model of which the Hamiltonian is

$$H_N = -\frac{\beta}{\sqrt{N}} \sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j + h\beta \sum_i \sigma_i \tag{2.19}$$

again with J_{ij} the outcome of an i.i.d. random distribution. Often one takes J_{ij} as standard Gaussian: $J_{ij} \sim \mathcal{N}(0,1)$ and the external field $h = 0$. It is believed that the infinite dimension limit $d \to \infty$ for the free energy density of the EA-model is equal to the free energy density of the SK-model. As in the SK-model each spin interacts directly with infinitely many other spins.

For the SK-model one can show that the spin-glass phase does occur when the temperature is low enough and the coupling mean not too large [68].

When one tries to take the limit $N \to \infty$ various limit Gibbs states μ seem to appear. In general a Gibbs state then looks like a mixture of infinitely many pure states μ_α:

$$\mu(\sigma) \approx \sum_\alpha w_J(\alpha)\mu_\alpha \tag{2.20}$$

where $w(\alpha)$ is the relative weight of the pure state μ_α. [66]. Note that the decomposition weights as well as the μ_α do depend on the disorder J. Recently it was proven that in the limit $N \to \infty$ each configuration is a ground state [67]. However taking the infinite-volume limit is problematic. A slightly better-behaved class are the Hopfield models.

2.3 Hopfield model

Our brain is a complex structure of many neurons which interact with each other in a non-trivial long-range way. For instance the cerebral cortex consists already about

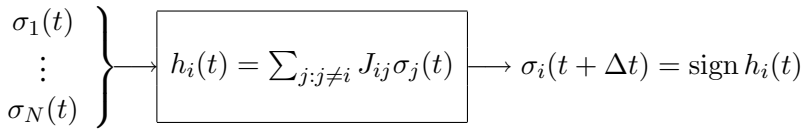


Figure 2.4: *The zero-temperature neuron dynamics*

10^{10} neurons. Nowadays there is a lot of research in this area. It seems that our brain network is scale free. It has the structure of a so called small world network: i.e. small path length between two neurons in the order of the path length in a random edge neural network, but with a relatively high amount of connections (e.g. [18]).

Originally Pastur and Figotin invented the Hopfield model as a model for a special type of spin-glasses. Then Hopfield came up with it independently as a model for neural networks as above. However it is a simplified model and the geometric structure of the neural connections is totally missing.

2.3.1 Setting

Assume that the neural network contains N neurons and that every neuron interacts with any other neuron (i.e. having mean-field like interactions). These interactions are composed out of 2-neuron interactions only. Now consider a neuron i. The state of the neuron is labelled by the variable σ_i. If the neuron is excited then $\sigma_i = +1$. When $\sigma_i = -1$ the neuron is at rest. As a current is going from neuron j into neuron i the signal is altered due to chemical transmitters in the neuron i itself. This synaptic efficacy we denote by J_{ij}. It alters the signal σ_j into $J_{ij}\sigma_j$. The total input h_i of neuron i equals

$$h_i = \sum_{j:\ j\neq i} J_{ij}\sigma_j \tag{2.21}$$

2.3.2 Dynamics and ground states

Let us define the dynamics of this model. If $\sigma_i(t)$ denotes the state of a neuron i at time t then it becomes (or stays) excited at time $t + \Delta t$ whenever h_i exceeds a threshold θ_i. Otherwise it is at rest at time $t + \Delta t$:

$$\sigma_i(t + \Delta t) = \operatorname{sign}(h_i(t) - \theta_i) \tag{2.22}$$

For simplicity we set this threshold to zero. Then the state of neuron i at time $t + \Delta t$ becomes [19]

$$\sigma_i(t + \Delta t) = \operatorname{sign}\Big(\sum_{j:\ j\neq i} J_{ij}\sigma_j(t)\Big) \tag{2.23}$$

so without an extra constant term. See Figure 1.3 and also Figure 2.4.

A particular fixed configuration of neurons we denote by the $N-$dimensional vector $\vec{\xi} \in \{-1,1\}^{\otimes N}$. We call this a pattern. If this pattern $\vec{\xi}$ is a stable fixed point of the evolution defined by (2.23) then we say it is in the system's memory. If we assume that there are no metastable fixed points, then the following happens. If there is only one stable fixed point, then whatever the initial state of the neurons, after long enough time the system will be in the state defined by $\vec{\xi}$, i.e. the system *remembers* the pattern. Of course there can be more patterns in the memory. Then if we pick at random an initial configuration of neurons, at the end we will always end up in one of the patterns. Furthermore every pattern of the memory can be reached with non-zero probability. In general however it can happen that the dynamics get stuck in a metastable fixed point.

To illustrate this process better, imagine you want to answer a question for a quiz. It can happen that you need a bit of time to remember the answer, because the question appears to be difficult. But still you have the feeling that you might know this one. In the process of remembering you try to find associations with the -according to you- presumably right answer. You are fine-tuning your first thought. You are altering your initial condition to a better one, a one with less energy. Then after some time you think you know an answer and you believe it is right. You have recovered something of your memory: either you are in the state of the right answer or in a state of wrongness.

A good way of measuring how well a configuration σ agrees with a given pattern $\vec{\xi}$ is to look at the corresponding order parameter $q_{\vec{\xi}}$:

$$q_{\vec{\xi}} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \xi_i \tag{2.24}$$

When $q_{\vec{\xi}} = 1$ then the configuration σ is equal to the pattern $\vec{\xi}$ and when $q_{\vec{\xi}} = -1$ then it is equal to the opposite: $\sigma = -\vec{\xi}$. The parameters $q_{\vec{\xi}}$ are also called *overlap parameters*. Whenever $\pm q_{\vec{\xi}} > 0$ the configuration $\pm\sigma$ agrees with the pattern $\vec{\xi}$ for more than half of the neurons.

A good way of studying this system is to use the Gibbs description of statistical mechanics. Then the memory of the system is formed by the ground states of the model. The equilibrium features are governed by the Gibbs measure. We choose the Hamiltonian as

$$H_N = -\sum_{i=1}^{N} h_i \sigma_i = -\sum_{i=1}^{N} \sigma_i \sum_{j:\ j\neq i} J_{ij}\sigma_j \tag{2.25}$$

Now the zero temperature dynamics of this system is equivalent to the earlier-defined

neuron dynamics (2.23). We see this as follows. The energy equals

$$H_N(t+\Delta t) = -\sum_{i=1}^{N} h_i(t)\sigma_i(t+\Delta t) \tag{2.26}$$

For zero temperature the Gibbs measure for time $t+\Delta t$ becomes a δ-measure on the configurations $\sigma(t+\Delta t)$ for which the energy $H_N(t+\Delta t)$ is minimal. As we see from (2.26), this is the configuration σ for which for every neuron i $\sigma_i(t+\Delta t) = \operatorname{sign} h_i(t)$. It is easy to see that the energy cannot increase in this operation. However it is not clear that in the end when t is very large, we will end up in a single ground state configuration or oscillate between more.

The above dynamics is deterministic. In reality the neurons might not be deterministic in this way. Furthermore we need to allow for some probability that the energy in the operation can increase. This is to have the ability to get out of the local minima formed by the metastable fixed points. Therefore we introduce a parameter β to control the uncertainty in the model. The smaller β, the more uncertainty. When $\beta = 0$ the neurons behave perfectly random, because the energies of the configurations do not matter. Taking $\beta \to \infty$ makes the system behave like the zero-temperature dynamics of (2.23).

As example we take $J_{ij} \equiv +1$. Then the system transforms into the Curie-Weiss model. The behaviour of this model is well understood. The ground states are $\sigma = \pm 1$. It is also easy to see that the same is true when J_{ij} contains only one pattern $\vec{\xi}$, i.e. $J_{ij} = \xi_i \xi_j$. Indeed the Hamiltonian (2.25) is minimized whenever $\sigma \equiv \pm \operatorname{sign} \vec{\xi}$. In this case the pattern coordinates ξ_i are allowed to have a more general distribution but they need to be i.i.d.

In reality one often knows only the global properties of the memory of the system. Furthermore the memory also changes in time. But these changes in time are very slow compared over the time in which the states of the neurons are changing. A good way of modelling this is that instead of choosing the pattern ourselves we let the patterns be chosen according to a quenched random distribution. All the randomness is i.i.d. and is thus described by a product measure. The measure with respect to this randomness $\vec{\xi}$ we denote by $\mathbb{P}_{\vec{\xi}}$. Usually one takes the randomness as symmetric Bernoulli: $\vec{\xi} \sim \{-1,1\}^{\otimes N}$.

Note an important difference between the 1-pattern case and the SK-model. In the SK-model all the bonds have independent disorder J_{ij}. In the 1-pattern Hopfield model pairs of bonds with a common neuron e.g. (ij) and (jk) have highly dependent disorder.

We generalize from the 1-pattern system to the finite p-pattern system with patterns

$\vec{\xi}^1, \ldots, \vec{\xi}^p$. We take for the J_{ij}

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \tag{2.27}$$

For the patterns we take a random outcome of the uniform distribution on $\left(\{-1,1\}^{\otimes N}\right)^{\otimes p}$. In the literature this choice of the J_{ij} is referred to as the Hebb rule. Because of the scaling the quenched patterns are asymptotically orthonormal to each other

$$\lim_{N\to\infty} \frac{1}{N} \vec{\xi^i} \cdot \vec{\xi^j} = \delta_{ij} + O(\sqrt{1/N}) \tag{2.28}$$

It is easy to see that the states $\sigma = \pm\vec{\xi}^\mu$ are equilibrium states of the system. Indeed if we put in $\sigma(t) = \vec{\xi}^\mu$ into the $\beta \to \infty$-dynamics (2.23) we obtain for $N \to \infty$

$$\sigma_i(t+\Delta t) = \operatorname{sign}\Big(\sum_{j:\ j\neq i} J_{ij}\xi_j^\mu \Big) = \operatorname{sign}\Big(\pm \frac{1}{N} \sum_{\nu=1}^{p} \xi_i^\nu \sum_{j:\ j\neq i} \xi_j^\nu \xi_j^\mu \Big) =$$

$$\operatorname{sign}\Big(\pm \sum_{\nu=1}^{p} \xi_i^\nu \delta_{\nu\mu} \Big) = \operatorname{sign}\left(\pm\xi_i^\nu\right) = \pm\xi_i^\nu \tag{2.29}$$

In other words $\sigma(t) = \sigma(t+\Delta t)$. This makes $\sigma(t) \equiv \pm\vec{\xi}^\mu$ fixed point configurations and therefore equilibrium states. However it is not clear from these calculations whether these states are also ground states. This is because the states can be unstable fixed points. Furthermore we might not be allowed to omit the $O(\sqrt{1/N})$ term in the calculations as we have done. Also it could be possible that the states only can be reached by a set of $\mathbb{P}_{\vec{\xi}}$-measure 0. Or maybe there are more ground states than these fixed points. After more analysis it turns out to be that the $2p$ states $\sigma = \pm\vec{\xi}^\mu$ are indeed the only ground states for this system (e.g. [10]).

2.3.3 System-size-dependent patterns

When the number p of patterns depends on the system size N several things can happen. Denote by α the ratio between the number of patterns and the system size: $\alpha = p/N$. We consider the phase regions in the (T, α) plane, see Figure 2.5. Whenever $T > T_g$ where $T_g = 1 + \sqrt{\alpha}$ the system behaves like a *paramagnet*.

There is a curve T_c such that below this line all of the p patterns are *stable*, i.e. absolute minima of the free energy. Between the curves T_c and T_g there is a different curve T_M which separates between stability and metastability.

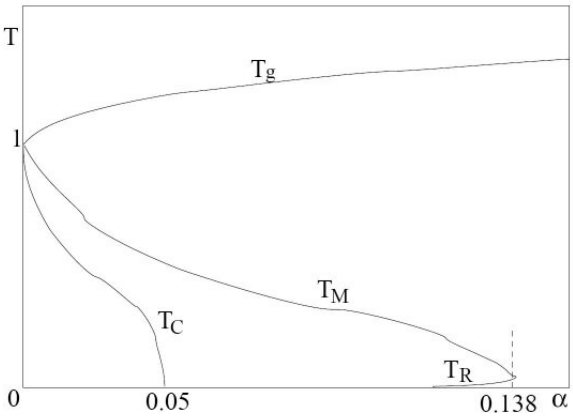


Figure 2.5: *Phase diagram for the Hopfield model (after [3])*

Between the curves T_M and T_c the patterns become *metastable*; they are local minima of the free energy. The global minima correspond to the spin-glass states. These states have vanishingly small overlap q_μ (of order $O(1/\sqrt{\alpha N})$) with all of the patterns μ. So only if the initial configuration is close enough to a pattern the system will remember it.

Above T_M and below T_g the spin-glass states become the only one present. So none of the patterns can be remembered. In this *spin-glass phase* there is presence of ageing. The decay of the energy becomes slower for longer waiting times. According to numerical research the spin-glass properties seem to be closely related to properties of the SK-model, which we obtain by taking the limit $\alpha \to \infty$. Analytic research of the corresponding dynamics is highly complicated; it cannot be described only by the overlap values $q_{\mu,t}$ and the neuron states σ_t at times t [3, 58, 68].

For a more extensive discussion of the Hopfield model, including some history and its relation with the theory of neural networks, see [10, pag. 133 and further] or [12].

2.3.4 Some generalizations

To put in more realism we take into account that not every neuron need to be connected with every other. For this goal we define the matrix Λ_{ij}, which represents the structure of the network. If neuron i is connected with j then $\Lambda_{ij} = 1$ otherwise $\Lambda_{ij} = 0$. When the network is undirected we have a symmetric matrix Λ_{ij}. Now we use the Hopfield dynamics of (2.23) but we replace J_{ij} by the value $\Lambda_{ij} J_{ij}$.

Numerical research seems to suggest that the task of recognition of a finite number

of patterns is better performed (i.e. higher overlap after long time) when we decrease the clustering coefficient of a network [48]. By the clustering coefficient we do mean the following. Take a vertex v of a graph. Suppose v has N neighboring edges. Then at most $N(N-1)/2$ edges can exist between these neighboring vertices. Denote by C_v the actual number of these edges divided by the maximal amount possible. The clustering coefficient C is the mean of C_v over all vertices v.

Another generalization is to allow the neurons to have more values. The neuron-states increase to $\sigma_i \in \{1, \ldots, q\}$ instead of $\sigma_i = \pm 1$. In spin-glass language we say that we have q-state Potts spins instead of Ising spins. For the patterns we can still take the restriction to $\xi_i^\mu = \pm 1$. When the system has only one pattern in its memory then we easily see that the form of the ground states is of the following type. Every site i for which $\xi_i = +1$ has $\sigma_i \equiv j$ and every site for which $\xi_i = -1$ has $\sigma_i \equiv k$, with $j \neq k$.

Of course it is more realistic to consider Potts-patterns, when the neuron states are equivalent to Potts-spins. Then the ground states are $\{\vec{\xi}^\mu\}$ with $\mathbb{P}_{\vec{\xi}}$ probability one whenever the number p of patterns is not too large: $\alpha : 0 \leq \alpha < 1$ arbitrarily, $p < (\alpha / \ln q) \ln N$ [37]. Note that p is allowed to be infinite when $N \to \infty$.

2.4 Scenarios for the spin glass

In the last decades researchers have tried to get an analytic-rigorous grip on the phenomena of short-range spin-glasses. During this process various competing theories were formed which were not at all conclusive. Most theories we can group into three scenarios; the droplet-picture of Fisher and Huse [33], the chaotic-pairs picture of Newman and Stein [66] and the replica symmetic breaking picture which resulted from mean-field theory for the infinite range SK spin-glass developed by Parisi [56].

2.4.1 Droplet-picture short-range spin-glasses

At the end of the eighties Fisher and Huse introduced a so called droplet picture [33] to describe the equilibrium phenomena for short-range spin glasses. For a clear example of this picture we take a model which has the energy function (2.17) of the Edward-Anderson model. The couplings (the spin-interactions) we choose symmetrically and continuously distributed. We set the field h to zero.

For finite dimensional Ising spin glasses there are two possibilities for the equilibrium behavior for small T. There is a critical dimension d_l such that:

$d < d_l$: System is paramagnetic at all $T > 0$ so $T_c = \infty$.
$d \geq d_l$: There exists exactly one pair of (flip-related) ground states. For $0 < T < T_c <$

∞ the behavior of the system is described by a small non-zero density of excitations with a volume which is non-zero relatively to the (infinite-sized) system.

Now we take a particular ground state $\mathcal{G}$ and look at its excitations. Because $T > 0$ there are excited regions where the spins have opposite values compared to $\mathcal{G}$. As in the Ising model we can define contours by the boundaries of these regions. The contours do exist on various scales. For a large enough system the probability of having at least one large contour is of order 1, although the probability of having a particular large contour is small. For low enough T we assume that the contours with the lowest energies dominate the physics. These contours we call droplets. More precisely

Definition 2.2. *A droplet $D_L(j)$ of length scale L is a contour Γ enclosing site j and has the minimum of energy of all possible contours Γ enclosing j and containing between L^d and $(2L)^d$ spins.*

The energy $F_L(j)$ of a droplet $D_L(j)$ equals

$$F_L(j) = \min_{\substack{\Gamma \text{ encl. } j, \\ L^d \leq |\Gamma| < (2L)^d}} E_{\mathcal{G}}(\Gamma) \tag{2.30}$$

where $E_{\mathcal{G}}(\Gamma)$ is the energy of configuration $\{\Gamma\}$ relatively to the ground state energy $E_{\mathcal{G}}(\emptyset)$.

In case of an Ising ferromagnet (i.e. (2.17) with $J_{ij} \equiv 1$ and $h = 0$) $F_L(j) = O(L^{d-1})$. For the current Ising spin glass with the random symmetric couplings it is expected that the droplet energy is much lower. This because there is a big amount of frustration and also there are many configurations which are almost like the ground states. However for a generic contour the energy scales still like L^{d-1}. Given this we make the scaling ansatz:

$$F_L(j) = O(L^\theta),\ \theta < d - 1 \tag{2.31}$$

In [33] it is argued that

$$\theta \leq \frac{d-1}{2} \tag{2.32}$$

However the arguments in favor of (2.32) use some assumptions which need not hold in general [24].

For $\theta > 0$ we expect the following picture. Because of the almost degenerate ground state the Gibbs weight of the event $F_L \approx 0$ is bigger than zero even for zero energy. As we see from the Hamiltonian only the droplets at length scale L with energy $F_L \leq O(T)$ do contribute significantly to the Gibbs measure. When $T \ll O(L^\theta)$, only a small fraction of these droplets does appear. Because of the positive weight of F_L near zero some of the droplets will be excited at any positive temperature. These properties make that $\theta > 0$ implies $d \geq d_l$.

When $\theta < 0$, the energy cost is so low that the entropy will dominate and the droplet-picture breaks down. Because every spin can be flipped with arbitrarily small energy cost (by taking the system size large enough) the system is to be expected to behave like a paramagnet. Therefore $\theta < 0$ implies $d < d_l$.

2.4.2 Parisi's Replica Symmetry breaking picture

Parisi cleverly conjectured in the eighties an expression for the free energy function of the SK-model and also an expression of the (Parisi) overlap distribution [56]. The idea of this solution is also known as replica symmetry breaking (RSB). Recently the conjectured free energy expression was mathematically rigorously proven to be the correct expression by Talagrand [74] who used in his proof results of Guerra and co-workers. However, some of the aspects of Parisi's RSB-picture are still open.

This RSB-picture predicts that in the infinite-volume limit states do appear which are composed out of infinitely many pure states. It is not clear what a pure state means for the infinite range SK-model. Assuming we still can define overlaps between different 'pure states', the overlap between 'pure state' α and α' is

$$q_{\alpha\alpha'} = \frac{1}{N}\sum_{i=1}^{N} < \sigma_i >_\alpha < \sigma_i >_{\alpha'} \tag{2.33}$$

By $< . >_\alpha$ we mean the Gibbs measure over the pure Gibbs state μ_α. From this quantity we can read how much state μ_α looks like state $\mu_{\alpha'}$. For every pure state μ_α it holds

$$q_\alpha = < \sigma_\alpha >^2 = q_{EA} \tag{2.34}$$

where q_{EA} is the same parameter as in (2.18). Furthermore we see

$$-q_{EA} \leq q_{\alpha\alpha'} \leq q_{EA} \tag{2.35}$$

To explicit construct the pure states is impossible. However, still some things can be said about the distribution of the overlaps. We choose at random two pure states from the Gibbs measures appearing in the limit of the SK-model. Then we denote by $P(q)dq$ the probability that the overlap of these two states lays in between q and $q + dq$. This distribution is also called the Parisi overlap distribution. It looks like

$$P_J(q) = \sum_{\alpha,\alpha'} w_J(\alpha) w_J(\alpha') \delta(q - q_{\alpha\alpha'}) \tag{2.36}$$

For high temperature the SK-model becomes a paramagnet and $P(q) = \delta(q = 0)$. However the symmetric overlap function $P(q)$ is highly non-trivial when the temperature T is low enough and consist of many δ-functions of non-zero weight. Furthermore

it is coupling dependent, i.e. a non self-averaging object. When we average over the couplings the resulting distribution shows to be continuous non zero between two δ spikes at $\pm q_{EA}$. Furthermore there is chaotic size dependence. When we look at two different infinite volumes which has a large difference in volume sizes then in general $P(q)$ also looks very different.

Another interesting concept which holds according to Parisi's theory is ultrametricity. Recall that for two equal pure states the overlap equals q_{EA}. With this we create a distance function between two pure states

$$d_{\alpha\alpha'} = q_{EA} - q_{\alpha\alpha'} \tag{2.37}$$

Then we take at random three states 1, 2, 3. With these states we can make three pairs. Ultrametricity then claims that either

$$\begin{aligned} &d_{12} = d_{13} = d_{23} \text{ or } d_{12} = d_{13} < d_{23} \text{ or} \\ &d_{12} = d_{23} < d_{21} \text{ or } d_{21} = d_{23} < d_{12} \end{aligned} \tag{2.38}$$

So the three overlaps of the state pairs are intimately related. A mathematical rigorous proof of this ultrametric structure is still an open problem.

2.4.3 Chaotic Pairs

The remaining possibility is [65, 66] that for large L the Gibbs measure looks like (with dependence on J)

$$\mu_L \approx \frac{1}{2}\mu_{\alpha_L,J} + \frac{1}{2}\mu_{-\alpha_L,J} \tag{2.39}$$

When we put $L \to \infty$ and take the union of all possible states emerging then we obtain a set of uncountably many states. The Gibbs measure is approximately a combination of two pure Gibbs states α_L and $-\alpha_L$ out of the infinitely many. These two states are each other's global spin-flip. The state-labels α_L are chaotically dependent on L.

We encounter in Chapter 3 an example of an infinite-range system which has infinitely many ground states. For fixed size L only two pairs of ground states do appear (or triples of pairs in case of 3-Potts spins) in the way of the Chaotic Pairs scenario.

2.5 Metastates

In spin glasses to get a grip on the quenched disorder we consider the following. Look at a sequence of finite volume Gibbs measures μ_Λ^η. The disorder of the spin glasses is

prescribed by the parameter η. It is treated as quenched disorder so we consider it as fixed. Then we take the empirical average of these measures

$$\mathcal{K}_N^\eta = \frac{1}{N}\sum_{n=1}^{N} \delta_{\mu^\eta_{\Lambda_n}} \tag{2.40}$$

We try to take the limit $N \to \infty$. The result provides the so called (empirical) metastate. This metastate is a probability measure on the Gibbs measure and is dependent on the quenched disorder η [66]. The metastate gives the relative weight of the event that a quenched disordered system of a very large volume behaves like a particular Gibbs measure.

In general however (2.40) does not converge for almost every configuration η unless we take a sparse enough subsequence. It does converge however in distribution. The resulting distribution over the infinite Gibbs measures does not depend on η anymore. The limiting process of the whole path $t \to \mu^\eta_{\Lambda_{[tN]}}$ is described by the so called super-state. The value $[tN]$ is equal to the largest integer smaller equal tN [54]. In [51] and [53] there are two examples for which this behavior has been examined thoroughly.

For the $d = 2$ random boundary field Ising model, which we consider in Chapter 4, the metastate does concentrate on two extremal Gibbs measures μ^+ and μ^-. We conjecture that for $d = 2, 3$ every mixture of μ^+ and μ^- can appear as a limit point along the regular sequence of cubes. These mixtures are null-recurrent. So in the metastate they do not appear and for this particular model the metastate is a.s. convergent.

For $d > 3$ for the random weak boundary field Ising model, the limit points along the regular sequences are only μ^+ and μ^- almost surely. Each extremal Gibbs measure appears with probability $1/2$ [28].

As example of an a.s. non-converging metastate we take the Curie-Weiss random field Ising model. It has as Hamiltonian

$$H_N = -\frac{\beta}{N}\sum_{i<j}\sigma_i\sigma_j - \beta\epsilon\sum_{i=1}^{N}\eta_i\sigma_i \tag{2.41}$$

The random variables η_i are i.i.d. and have $\mathbb{P}(\eta_i = \pm 1) = 1/2$. For β large enough and ϵ small the model behaves like a ferromagnet with one $+$-phase $\mu_\infty^{+,\eta}$ and one $-$-phase $\mu_\infty^{-,\eta}$. When one takes the sequence $n = 1, 2, \cdots$ the corresponding metastate converges in distribution to

$$\lim_{N\to\infty} K_N^\eta = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\delta_{\mu_n^\eta} \overset{\text{law}}{=} n_\infty \delta_{\mu_\infty^{+,\eta}} + (1-n_\infty)\delta_{\mu_\infty^{-,\eta}} \tag{2.42}$$

The variable n_∞ is a random variable independent of η. It is distributed as $\mathbb{P}(n_\infty < x) = \frac{2}{\pi}\arcsin(\sqrt{x})$ [51, 53].

Chapter 3

Gaussian Potts-Hopfield model

In this chapter we study a Gaussian Potts-Hopfield model. Whereas for Ising spins and two disorder variables per site the chaotic pair scenario is realized, we find that for q-state Potts spins $q(q-1)$-tuples occur. Beyond the breaking of a continuous stochastic symmetry, we study the fluctuations and obtain the Newman-Stein metastate description for our model.

3.1 Introduction

The Gaussian Potts-Hopfield model is equal to the Potts-Hopfield model but with Gaussian noise as patterns. What happens for two patterns with Ising or Potts-like neurons is, surprisingly, that there are infinitely many ground-states. We study the mean-field Potts model with Hopfield-Mattis disorder, and more in particular with Gaussianly distributed disorder. This model is a generalization of the Ising version of the model studied in [11]. It provides yet another example of a disordered model with infinitely many low-temperature pure states, such as is sometimes believed to be typical for spin-glasses [33]. In our model, however, in contrast to [11], instead of chaotic pairs we find that the chaotic size dependence is realized by chaotic $q(q-1)$-tuples.

A somewhat different generalization of the Hopfield model to Potts spins was introduced by Kanter in [47] and was mathematically rigorously analysed in [37]. However, whereas the version we treat here (in which the form of the disorder is the Mattis-Hopfield one) displays the phenomenon of stochastic symmetry breaking, in which a finite-spin, "finite pattern" model can end up with chaotic size dependence, and a realization of chaotic n-tuples out of infinitely many "pure states", we do not see how to obtain such results in a version of Kanter's form of the disorder distribution.

We are concerned in particular with the infinite-volume limit behaviour of the Gibbs

and ground state measures. The possible limit points are labelled as the minima of an appropriate mean-field (free) energy functional. These minima can be obtained as solutions of a suitable mean-field equation. These minima lie on the minimal-free-energy surface, which is a $m(q-1)$-sphere in the $(\mathbf{e}^1, \cdots, \mathbf{e}^q)^{\otimes m}$ space. This space for q-state Potts spins and m patterns is formed by the m-fold product of the hyperplane spanned by the end points of the unit vectors $\mathbf{e}_q$, which are the possible values of the spins. But only a limited area of the minimal-free-energy surface is accessible. Only those values for which certain mean-field equations hold, are allowed. These equations have the structure of fixed point equations. We derive them in Chapter 3.4. To obtain the Gibbs states we need to find the solutions of these equations on the minimal free energy surface.

The structure of the ground or Gibbs states for Ising spins, where $q = 2$, and 2 standard-Gaussian patterns $\vec{\xi}$, $\vec{\eta}$ is known since a few years [11]. Due to the Gaussian distribution we have a nice symmetric structure: the extremal ground (and Gibbs) states form a circle. The first time this degeneracy of the ground states due to the rotational symmetry of the Gaussian's is mentioned is in [2].

For a fixed configuration and a large finite volume the possible order-para-meter values become close to two diametrical points (which ones depend on the volume of the system) on this circle. This chapter treats the generalization of this structure to q-state Potts spins with $q > 2$. To have a concrete example, we concentrate on the case $q = 3$. It turns out that we again obtain a circle symmetry but also a discrete symmetry, which generalizes the one for Ising spins. One gets instead of a single pair a triple of pairs (living on 3 separate circles), where for each pair one has a similar structure as for the single pair for $q = 2$. For $q > 3$ we get $\frac{q(q-1)}{2}$ pairs and a similar higher-dimensional structure.

Our model contains quenched disorder. It turns out that there is some kind of self-averaging. The thermodynamic behaviour of the Hamiltonian is the same for almost every realization. This is the case for the free energy and the associated fixed point equations, as is familiar from many quenched disordered models. However, this is not precisely true for the order parameters. We will see that they show a form of chaotic size dependence, i.e. the behaviour strongly depends both on the chosen configuration and on the way one takes the infinite-volume limit $N \to \infty$ (that is, along which subsequence).

3.2 Notations and definitions

We start with some definitions. Consider the set $\Lambda_N = \{1, \cdots, N\} \subset \mathbb{N}^+$. Let the single-spin space χ be a finite set and the N-spin configuration space be $\chi^{\otimes N}$. We

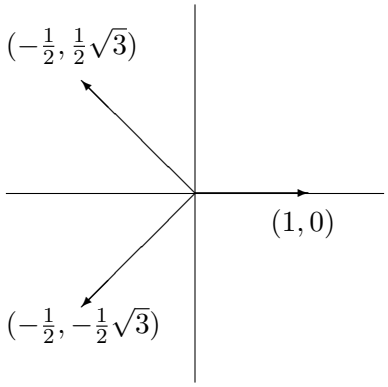


Figure 3.1: *Wu representation spin values for* $q = 3$

denote a spin configuration by σ and its value at site i by σ_i. We will consider Potts spins, in the Wu representation [76]. Each of the possible q values provides a spin-vector $\mathbf{e}_i$. The i-th coordinates are given by $\mathbf{e}_{i,j} = \delta_{i,j}$. Then the set $\chi^{\otimes N}$ is the N-fold tensor product of the single-spin space $\chi = \{\mathbf{e}^1, \cdots, \mathbf{e}^q\}$. The $\mathbf{e}^{\sigma_i}$ are the projections of the spin-vectors $\mathbf{e}_{\sigma_i}$ on the hypertetrahedron in $\mathbb{R}^{q-1}$ spanned by the end points of $\mathbf{e}_{\sigma_i}$. So every spinvalue σ_i is represented by the projection vector $\mathbf{e}^{\sigma_i}$.

For $q = 3$ we get for example for $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{e}^3$ the vectors of Figure 3.1. We have set the projection of the origin $(0, 0, 0)$ to $(0, 0)$ and rescaled the projection of $\mathbf{e}_1$ to $(1, 0)$.

The Hamiltonian of our model is defined as follows:

$$-\beta H_N = \frac{\beta}{N} \sum_{k=1}^{m} \sum_{i,j=1}^{N} \xi_i^k \xi_j^k \delta(\sigma_i, \sigma_j) \tag{3.1}$$

with

$$\delta(\sigma_i, \sigma_j) = \frac{1}{q} \left[1 + (q-1)\mathbf{e}^{\sigma_i} \cdot \mathbf{e}^{\sigma_j}\right] \tag{3.2}$$

where ξ_i^k is the i-th component of the random N-component vector ξ^k. For the ξ_i^k's we choose i.i.d. $N(0, 1)$ distributions. The vectors $\xi^k = (\xi_1^k, \cdots, \xi_N^k)$, by analogy with the standard Hopfield model, are called patterns. If we combine the above, we can rewrite the Hamiltonian H_N as:

$$-\beta H_N = \beta \frac{q-1}{q} N \sum_{k=1}^{m} \left[\left(\frac{\sum_{i=1}^N \xi_i^k \mathbf{e}^{\sigma_i}}{N} \right)^2 + \frac{1}{q-1} \left(\frac{\sum_{i=1}^N \xi_i^k}{N} \right)^2 \right] \tag{3.3}$$

So asymptotically

$$
\begin{aligned}
-\beta H_N &= N\frac{K}{2}\sum_{k=1}^{m} q_{kN}^2 \\
\text{with } K = 2\beta\left(\frac{q-1}{q}\right) &\text{ and order parameters } q_{kN} = \frac{1}{N}\sum_{i=1}^{N}\xi_i^k \mathbf{e}^{\sigma_i}
\end{aligned}
\tag{3.4}
$$

The last term in (3.3) inside the brackets is an irrelevant constant; in fact it approaches zero, due to the strong law of large numbers. Note that for the infinite pattern-limit $m \to \infty$ the Hamiltonian is still of the same form asymptotically. (The ξ_i^k's are i.i.d. $N(0,1)$ distributed so $\mathbb{E}\xi_i^k = 0$.) Note that any i.i.d. distribution with zero mean, finite variance and symmetrically distributed around zero will give an analogous form of H_N, but we plan to consider only Gaussian distributions, for which we will find that a continuous symmetry can be stochastically broken, just as in [11]. From now on we drop the subscript N to simplify the notation, when no confusion can arise.

3.3 Ground states

Now it is time to reveal the characteristics of the ground states for the Potts model. First we discuss the simple behaviour for 1 pattern. Then the more interesting part: $q > 2$ and 2 patterns.

3.3.1 Ground states for 1 pattern

For one pattern $\vec{\xi}$ the Hamiltonian is of the following form:

$$
-\beta H_N = N\frac{K}{2}\vec{q}_1^{\,2} = \frac{\beta}{N}\sum_{i,j=1}^{N}\xi_i\xi_j\delta(\sigma_i,\sigma_j) \tag{3.5}
$$

We easily see that the ground states are obtained by directing the spins with $\xi_i > 0$ in one direction and the spins with $\xi_i \leq 0$ in a different direction. If we have as the distribution for the ξ_i's $P(\xi_i = \pm 1) = \frac{1}{2}$ **,** then the order parameter is of the form: $\vec{q}_1 = \frac{1}{2}(\mathbf{e}^{\sigma_i} - \mathbf{e}^{\sigma_j})$, with $1 \leq i,\ j \leq q$ and $i \neq j$, see also [27]. So for $q = 3$ we have only 6 ground states. They form a regular hexagon:

$$
\left(\pm 3/4, \mp\sqrt{3}/4\right),\ \pm\left(3/4, \sqrt{3}/4\right),\ \left(0, \pm\sqrt{3}/2\right) \tag{3.6}
$$

This regular hexagon with its interior is the convex set of possible order parameter values. It is easy to see that for ξ_i $N(0,1)$-distributed we get the same ground states except for a scaling factor $\sqrt{2/\pi}$ multiplying the values of the order parameter values.

3.3.2 Ground states for 2 patterns

The Hamiltonian for 2 patterns (Gaussian i.i.d.) is:

$$-\beta H_N = \frac{\beta}{N}\sum_{i,j=1}^{N}(\xi_i^1\xi_j^1 + \xi_i^2\xi_j^2)\delta(\sigma_i,\sigma_j) = N\frac{K}{2}(\vec{q}_1^{\,2} + \vec{q}_2^{\,2}) \tag{3.7}$$

Similarly as in [11], we make use of the fact that the distribution of 2 independent identically distributed Gaussians has a continuous rotation symmetry. This symmetry shows also up in the order parameters.

Ising spins

First we consider Ising-spins (i.e. we take $q = 2$). In [11] it is proven that the ground states are as follows. The order parameters become $\pm(r^\star\cos\theta, r^\star\sin\theta)$, with $\theta \in [0,\pi)$ and $r^\star = \sqrt{2/\pi}$. Note that there are uncountably many ground-states.

This can be made plausible by the following observations. Note that the random fields $\{\mathrm{sign}\,(\xi_i^\mu)\}$ are equally distributed as standard Hopfield-patterns: $P(\mathrm{sign}\,(\xi_i^\mu) = \pm 1) = 1/2$. So if we choose σ such that for each i: $\sigma_i = \mathrm{sign}(\xi_i^1)$, then we obtain the state with corresponding order parameters $(r^\star, 0)$ for the limit $N \to \infty$, which is the ground state configuration corresponding to $\theta = 0$. The spin-configuration $\sigma_i = \mathrm{sign}(\xi_i^2)$ for all i corresponds to $\theta = \pi/2$. By the global spin-flip symmetry of the Hamiltonian we obtain the ground-states corresponding to $\theta = \pi$ and $\theta = 3\pi/2$.

But what about the θ values in between? The set of Gaussian patterns has a continuous rotation symmetry. We obtain two new patterns for which we multiply the patterns with a rotation matrix, i.e. rotating the patterns over an angle θ (with $0 \le \theta < \pi/2$):

$$\begin{pmatrix} \eta_i^1(\theta) \\ \eta_i^2(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \xi_i^1 \\ \xi_i^2 \end{pmatrix} \tag{3.8}$$

The corresponding order-parameters we define as $\vec{q}(\theta)$. To obtain the original patterns from $\vec{\eta}^1(\theta)$ and $\vec{\eta}^2(\theta)$ simply perform the rotation again:

$$\begin{pmatrix} \xi_i^1 \\ \xi_i^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \eta_i^1(\theta) \\ \eta_i^2(\theta) \end{pmatrix} \tag{3.9}$$

By the rotation (3.8) of the standard Gaussian patterns $\vec{\xi}^1$ and $\vec{\xi}^2$ we obtain two new patterns $\vec{\eta}^1$ and $\vec{\eta}^2$ which again are Gaussian distributed. Note that $\mathbb{E}\eta_i^1(\theta) = \mathbb{E}\eta_i^2(\theta) = \mathbb{E}\xi_i^1 = \mathbb{E}\xi_i^2 = 0$. Furthermore the variance of $\eta_i^1(\theta)$ and $\eta_i^2(\theta)$ is the same as for ξ_i^1 and ξ_i^2, i.e. 1. Therefore the distribution of the rotated patterns $\vec{\eta}^1(\theta)$ and $\vec{\eta}^2(\theta)$ is the same as for the old ones, namely standard N-multivariate Gaussian. It is easily checked that

each $\eta_i^1(\theta)$ and $\eta_i^2(\theta)$ are uncorrelated and because they are both Gaussian they are also independent.

For any θ it holds

$$\xi_i^1\xi_j^1 + \xi_i^2\xi_j^2 = \eta_i^1(\theta)\eta_j^1(\theta) + \eta_i^2(\theta)\eta_j^2(\theta) \tag{3.10}$$

By this it follows that the energy of the configurations

$$\sigma(\theta) = \{\mathrm{sign}(\eta_i(\theta))\} \tag{3.11}$$

are the same in the limit $N \to \infty$ and therefore ground states. This we see by calculating the two corresponding energies:

$$\begin{aligned}
-\beta H_N(\sigma(0)) &= \frac{\beta}{N}\sum_{i=1}^{N} |\xi_i^1\xi_j^1| + \frac{\beta}{N}\sum_{i=1}^{N} \xi_i^2\xi_j^2 \,\mathrm{sign}\,(\xi_i^1\xi_j^1) = \frac{2\beta}{\pi} + O\big(\beta/N\big), \\
-\beta H_N(\sigma(\theta)) &= \frac{\beta}{N}\sum_{i=1}^{N} \big(\xi_i^1\xi_j^1 + \xi_i^2\xi_j^2\big) \,\mathrm{sign}\,(\eta_i^1(\theta)\eta_j^1(\theta)) = \\
&\qquad \frac{\beta}{N}\sum_{i=1}^{N} \big(\eta_i^1(\theta)\eta_j^1(\theta) + \eta_i^2(\theta)\eta_j^2(\theta)\big) \,\mathrm{sign}\,(\eta_i^1(\theta)\eta_j^1(\theta)) = \frac{2\beta}{\pi} + O\big(\beta/N\big)
\end{aligned} \tag{3.12}$$

So in the limit indeed it holds

$$\lim_{N\to\infty} H_N(\sigma(\theta)) = H_N(\sigma(0)) \;\text{ for all } \theta \tag{3.13}$$

This means that we have an uncountable number of ground-state configurations in the limit $N \to \infty$.

Structure of the order parameters

Now we look what this symmetry does mean for the order parameters. We consider the Gibbs measure with the original patterns $\vec{\xi}^\mu$. Take a configuration $\sigma(\theta)$. The corresponding order-parameters we denote by $\vec{q}(\theta)$. Rewriting the patterns $\vec{\xi}^\mu$ into $\vec{\eta}(\theta)$ according to (3.9) gives

$$\begin{aligned}
\begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} &= \begin{pmatrix} \frac{1}{N}\sum_{i=1}^{N} \big(\cos(\theta)\eta_i^1(\theta) + \sin(\theta)\eta_i^2(\theta)\big)\,\mathrm{sign}(\eta_i^1(\theta)) \\ \frac{1}{N}\sum_{i=1}^{N} \big(\sin(\theta)\eta_i^1(\theta) - \cos(\theta)\eta_i^2(\theta)\big)\,\mathrm{sign}(\eta_i^1(\theta)) \end{pmatrix} = \\
&\qquad \sqrt{\frac{2}{\pi}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + O\big(\sqrt{1/N}\big)
\end{aligned} \tag{3.14}$$

because $\mathrm{sign}(\eta_i^1)$ is independent of η_i^2. The $O\big(\sqrt{1/N}\big)$ term is in general different for different θ. However the equality $\vec{q}(\theta) = -\vec{q}(\theta+\pi)$ is exact because $\eta^\mu(\theta) = -\eta^\mu(\pi+\theta)$. Because of this the energy of the configurations $\sigma = \mathrm{sign}(\eta^\mu(\theta))$ and $\sigma = -\mathrm{sign}(\eta^\mu(\theta)) = \mathrm{sign}(\eta^\mu(\pi+\theta))$ are also the same. In [11] it is proven that for finite N only for one pair ($\theta_0(N)$ and $\theta_0(N)+\pi$) the energy is in its global minimum. The value of $\theta_0(N)$ depends on the system size.

When we add to the Hamiltonian the term $-(\epsilon/N)\sum_{i=1}^N \eta_i^1(\theta_1)\sigma_i$, with $\epsilon > 0$ and θ_1 fixed, the degeneracy of the ground-states is broken even when $N \to \infty$. Now only the configuration $\{\mathrm{sign}\,(\eta_i^1(\theta_1))\}$ is a ground-state, i.e. $\vec{q} = \sqrt{2/\pi}(\cos\theta_1$ $, \sin\theta_1)$. For β finite and large enough the same holds in the limit $N \to \infty$ but with $r^\star(\beta)$ instead of $\sqrt{2/\pi}$. This also corresponds to the results proven in [11].

Potts-spins

For obtaining the ground-states in case of Potts-neurons we perform the same strategy as for the Ising-neurons. We consider the distributions $\{\mathrm{sign}\,(\eta_i^1(\theta))\}$. The corresponding ground-states configurations $\sigma(\theta)$ we obtain as follows. If $\mathrm{sign}\,(\eta_i^1(\theta)) = 1$ we set $\sigma_i = k$. When $\mathrm{sign}\,(\eta_i^1(\theta)) = -1$ we set $\sigma_i = k'$, with $k \neq k'$ and $k, k' \in \{1, \cdots, q\}$. This gives us $q(q-1)$ possible values for the order-parameters $\vec{q}(\theta)$ for each θ, the so-called discrete symmetry. If we look carefully at the values of $\vec{q}(\theta)$ we see that when we take the union of $\vec{q}(\theta)$ over all θ the resulting curves consist of $q(q-1)/2$ circles in the order-parameter space. This provides the continuous symmetry of the ground-states which originates from the continuous rotational symmetry between the two Gaussian patterns ξ^1 and ξ^2.

We take one of the $q(q-1)$ values by considering the ground-state configurations

$$\mathrm{sign}\,(\eta_i^1(\theta)) = 1 \;\to\; \sigma_i = \mathbf{e}^1,\ \mathrm{sign}\,(\eta_i^1(\theta)) = -1 \;\to\; \sigma_i = \mathbf{e}^2 \tag{3.15}$$

In the same way as (3.14) we obtain for $\vec{q}_1(\theta)$ by using independence

$$\begin{aligned}\vec{q}_1(\theta) = \frac{\cos(\theta)}{2N}\sum_{i=1}^{N/2}|\eta_i^1(\theta)|\left[\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\sqrt{3}\end{pmatrix}\right] + O\big(\sqrt{1/N}\big)\\ = \cos(\theta)\sqrt{\frac{2}{\pi}}\begin{pmatrix}\frac{3}{4}\\ -\frac{1}{4}\sqrt{3}\end{pmatrix} + O\big(\sqrt{1/N}\big)\end{aligned} \tag{3.16}$$

and

$$\vec{q}_2(\theta) = \sin(\theta)\sqrt{\frac{2}{\pi}}\begin{pmatrix}\frac{3}{4}\\ -\frac{1}{4}\sqrt{3}\end{pmatrix} + O\big(\sqrt{1/N}\big) \tag{3.17}$$

By considering the other possibilities we obtain all the six discrete points. These have the $\vec{q}_1$-coordinates of (3.6) multiplied by the factor $\sqrt{2/\pi}$. By rotating we obtain the circles. Because $\vec{q}(\theta) = -\vec{q}(\theta + \pi)$ we obtain the same structure as for the Ising-spins. The same is true for the order-parameters resulting from the Gibbs-states.

Without much effort this is also seen to be true for infinitely many patterns (as long as their number grows logarithmic compared to the system size). However the precise structure of the Gibbs-states is not proven yet but still being investigated.

This is an example of chaotic size dependence, based on the breaking of a stochastic symmetry, of the same nature as in [11]. Because of weak compactness, different subsequences exist whose $q(q-1)$-tuples of ground states converge to $q(q-1)$-tuples, associated to particular θ-values. These subsequences depend on the random pattern realization. See Section 3.5.

For any finite $m \geq 3$ patterns one has the same discrete structure as before, but instead of a continuous circle symmetry we have a continuous m-sphere symmetry (isomorpic to $O(m)$). The case of an infinite (that is, increasing with the system) m is still open. However the limit meta-state structure of the Gibbs-states when considering infinite sequences in N is more complicated.

3.4 Positive temperatures

In this section we obtain an expression for the free energy which is maximized over the order parameters $\vec{q}_k$. By large deviation arguments we relate this expression and therefore the free energy to the average of the energy over the induced measure of the order parameters $\vec{q}_k$.

3.4.1 Fixed-point mean-field equations

Remember

$$Z_N = \mathrm{Tr}_\sigma \exp\left(N\frac{K}{2}\sum_{k=1}^{m}\vec{q}_k^{\,2}\right) \tag{3.18}$$

Due to the quadratic dependence on $\vec{q}_k$ this is hard to compute. Therefore we like to linearize the terms in the exponential. For this we use the following identity:

$$e^{ax^2/2} = \sqrt{\frac{aN}{2\pi}}\int_{-\infty}^{\infty} dm\, e^{-Nam^2/2+\sqrt{N}amx} \tag{3.19}$$

Note

$$\vec{q}_k^{\,2} = \sum_{i=1}^{q-1} q_{ki}^2 \tag{3.20}$$

So if we set $x = \sqrt{N} q_{ki}$ and $a = K$ we obtain

$$\exp\left(N\frac{K}{2}\vec{q}_k^{\,2}\right) = \prod_{i=1}^{q-1}\sqrt{\frac{K}{2\pi}}\int_{-\infty}^{\infty} dm_{ki}\exp\left(-KNm_{ki}^2/2 + KNm_{ki}q_{ki}\right) \tag{3.21}$$

Applying it for every m order-parameters $\vec{q}_k$ and putting the result into Z_N we obtain

$$Z_N = \mathrm{Tr}\prod_{k=1}^{m}\left(\frac{K}{2\pi}\right)^{\frac{q-1}{2}}\int_{\mathbb{R}^{q-1}} d\vec{m}_k\exp\left(-KN\vec{m}_k^2/2 + KN\vec{m}_k\cdot\vec{q}_k\right) \tag{3.22}$$

This transformation is called the Hubbard-Stratonovich transformation. Notice that the dependence on $\vec{q}_k$ now is linear. Because $N \to \infty$ the integral behaves like its maximal value. Maximizing the exponent in (3.22) gives the saddle point equations for m_{ki}:

$$\begin{aligned}&\frac{\partial}{\partial m_{ki}}\left(-KN\vec{m}_k^2/2 + K\vec{m}_k\cdot N\vec{q}_k\right) = 0 \to \\ &\quad -KNm_{ki} + KNq_{ki} = 0 \;\to\; \vec{m}_k = \vec{q}_k\end{aligned} \tag{3.23}$$

Further rewriting of (3.22) gives that the partition function Z_N is equal to

$$\begin{aligned}Z_N = \left(\frac{K}{2\pi}\right)^{\frac{m(q-1)}{2}}\int_{\mathbb{R}^{m(q-1)}} d\vec{m}_1\cdots d\vec{m}_m\exp\Bigg(-KN\sum_{k=1}^{m}\vec{m}_k^2/2 + \\ N\Big\langle\log\Big\{\mathrm{Tr}_\sigma\exp\Big[\sum_{k=1}^{m}K\vec{m}_k\cdot\xi_1^k\mathbf{e}^\sigma\Big]\Big\}\Big\rangle_{\xi_1^1,\cdots\xi_1^m}\Bigg)\end{aligned} \tag{3.24}$$

Now we maximize this exponent. Using both equations gives the so-called fixed-point mean-field equation. Maximizing and putting $\vec{m}_k = \vec{q}_k$ (the first equation) give the mean field equations for the order parameters which have the structure of a system of fixed point equations $\vec{q} = F(\vec{q})$. When we have only two patterns $\vec{\xi}^1$ and $\vec{\xi}^2$ these are as follows:

$$\begin{cases}\vec{q}_1 = \left\langle\frac{\mathrm{tr}_\sigma\{\xi\mathbf{e}^\sigma\exp\left[K(\xi_1^1\vec{q}_1+\xi_1^2\vec{q}_2)\cdot\mathbf{e}^\sigma\right]\}}{\mathrm{tr}_\sigma\{\exp\left[K(\xi_1^1\vec{q}_1+\xi_1^2\vec{q}_2)\cdot\mathbf{e}^\sigma\right]\}}\right\rangle_{\xi_1^1,\xi_1^2} \\ \vec{q}_2 = \left\langle\frac{\mathrm{tr}_\sigma\{\eta\mathbf{e}^\sigma\{\exp\left[K(\xi_1^1\vec{q}_1+\xi_1^2\vec{q}_2)\cdot\mathbf{e}^\sigma\right]\}}{\mathrm{tr}_\sigma\{\exp\left[K(\xi_1^1\vec{q}_1+\xi_1^2\vec{q}_2)\cdot\mathbf{e}^\sigma\right]\}}\right\rangle_{\xi_1^1,\xi_1^2}\end{cases} \tag{3.25}$$

3.4.2 Induced measure on order parameters

Now we try to find an expression which in the infinite neuron limit equals the induced Gibbs measure $\mathfrak{L}_{\infty,\beta}$ on the order-parameters. For this end we calculate the free energy by using the Laplace method. When we look carefully at the integrand in (3.22) we see

$$-\beta f(\beta) = \lim_{N\to\infty}\frac{1}{N}\log Z_N = \max_{\vec{m}}\left(-Q(\vec{m}) + c(K\vec{m})\right) \tag{3.26}$$

where $c(\vec{m})$ is the generating function of the pattern distributions:

$$c(\vec{m}) = \sum_{k=1}^{m} \left\langle \ln \left\{ \mathbb{E}_\sigma \exp\left(\zeta^k \vec{m}_k \cdot \mathbf{e}^\sigma\right) \right\} \right\rangle_{\zeta^k} \tag{3.27}$$

Because

$$Q = K\vec{m}^2/2,\ \nabla Q(\vec{m}) = K\vec{m} \tag{3.28}$$

From these solutions we can also read out the fixed point equations. When we differentiate (3.28) to $\vec{m}$ componentwise we get

$$\nabla Q(\vec{m}) = K\nabla c(\nabla Q(\vec{m})) \Rightarrow \vec{m} = \nabla c(\nabla Q(\vec{m})) \tag{3.29}$$

This we can relate to the rate-function $c^\star(\vec{t})$, which is the Legendre transform of $c(\vec{m})$:

$$c^\star(\vec{m}) = \sup_{\vec{t}}[\vec{m} \cdot \vec{t} - c(\vec{t})] \tag{3.30}$$

For fixed $\vec{m}$ the vector $\vec{t}$ has to be such that $\vec{m} = \nabla c(\vec{t})$. But because of the fixed point equations $\vec{m} = \nabla c(\nabla Q(\vec{m}))$. Therefore $\vec{t} = \nabla Q(\vec{m})$. So

$$c^\star(\vec{m}) = \vec{m} \cdot \nabla Q(\vec{m}) - c(\nabla Q(\vec{m})) \tag{3.31}$$

Insert this into (3.26) to obtain

$$-\beta f(\beta) = \max_{\vec{m}} \left(Q(\vec{m}) - c^\star(\vec{m})\right) \tag{3.32}$$

For $N \to \infty$ the equation $\vec{m} = \vec{q}$ holds. This gives that

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N = \lim_{N\to\infty} \frac{1}{N} \log \left\{ \int_{\mathbb{R}^{m(q-1)}} d\vec{q} \exp N(Q(\vec{q}) - c^\star(\vec{q})) \right\} \tag{3.33}$$

Also it holds asymptotically $NQ(\vec{q}) = -\beta H_N(\vec{q})$.

Now we define the measure $\tilde{\mathfrak{L}}_{N,\beta}$. For this measure the limit $\lim_{N\to\infty} \tilde{\mathfrak{L}}_{N,\beta}$ approaches the induced infinite-volume Gibbs measure $\mathfrak{L}_{\infty,\beta}$ on the order parameters $\vec{q}$. The density is given by (3.33)

$$\tilde{\mathfrak{L}}_{N,\beta} \equiv \frac{e^{-\beta H_N(\vec{q})} \exp\left(-Nc^\star(\vec{q})\right)}{\int_{\mathbb{R}^{m(q-1)}} e^{-\beta H_N(\vec{q})} \exp\left(-Nc^\star(\vec{q})\right) d\vec{q}} = \frac{e^{-\beta N \phi_N}}{Z_{N,\beta}} \tag{3.34}$$

with

$$\beta\phi_N \equiv -Q(\vec{q}) + c^\star(\vec{q}) = -Q(\vec{q}) + \vec{q} \cdot \nabla Q(\vec{q}) - c(\nabla Q(\vec{q})) \tag{3.35}$$

For this measure $\tilde{\mathfrak{L}}_{N,\beta}$ we have good large deviation estimates, with the rate function given by $c^\star(\vec{q})$, where the rate equals N.

3.4.3 Radius of the circles labeling the Gibbs states

Ising spins

For all Gibbs states the value of the energy is constant, therefore:

$$q_1^2 + q_2^2 = (r^\star(\beta))^2 \tag{3.36}$$

To obtain the radius of the circle of Gibbs states $r^\star(\beta)$ we make use of the rotation symmetry between the two patterns. Now we take the point $(q_1, q_2) = (r^\star(\beta), 0)$. For Ising spins $K = 2\beta(2-1)/2 = \beta$. We insert all this into the fixed point equations (3.25), with $\mathbf{e}^1 = +1$ and $\mathbf{e}^2 = -1$. Then we obtain the following equation for the radius $r^\star(\beta)$:

$$r^\star(\beta) = \frac{1}{\sqrt{2\pi}} \int \xi \tanh\left(\beta\xi r^\star(\beta)\right) \exp\left(-\frac{\xi^2}{2}\right) d\xi \tag{3.37}$$

For $\beta > \beta_0$ this equation has a nontrivial solution for $r^\star(\beta)$. The equation is the same as in [11]. The limit $\beta \to \infty$ gives the radius of the ground-states:

$$r^\star = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\xi| \left(-\frac{\xi^2}{2}\right) d\xi = \sqrt{\frac{2}{\pi}} \tag{3.38}$$

Potts spins

If we take $q = 3$, then $K = \frac{4}{3}\beta$. The set of ground states now can be parametrized by three (in general $\frac{q(q-1)}{2}$) circles, and similarly for the low-temperature Gibbs states. To obtain the radius $\hat{r}$ of such a circle parametrizing the ground or Gibbs states, we follow the same recipe as in the case of Ising spins. Here we take the point $(\vec{q}_1, \vec{q}_2)$ with $\vec{q}_1 = (0, \hat{r})$ and $\vec{q}_2 = (0, 0)$. This we insert into the fixed point equations (3.25) with $\mathbf{e}^1 = (1, 0)$, $\mathbf{e}^2 = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$ and $\mathbf{e}^3 = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$. From which the equation for the radius $\hat{r}(\beta)$ follows:

$$\hat{r}(\beta) = \sqrt{\frac{3}{2\pi}} \int \frac{\xi \sinh\left(2\beta\xi\hat{r}(\beta)/\sqrt{3}\right)}{2\cosh\left(2\beta\xi\hat{r}(\beta)/\sqrt{3}\right) + 1} \exp\left(-\frac{\xi^2}{2}\right) d\xi \tag{3.39}$$

We can easily check that this expression indeed approaches the one for the radius for the circles through the ground states, by considering the behaviour of the integrand for $K \to \infty$. It behaves like:

$$\hat{r} = \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\xi| \exp\left(-\frac{\xi^2}{2}\right) d\xi = \sqrt{\frac{3}{2\pi}} \tag{3.40}$$

Remark 3.1. *Note that for finite β: $\hat{r} < r^\star\sqrt{3}/2$ and for $\beta \to \infty$: $\hat{r} \nearrow r^\star\sqrt{3}/2$.*

The chaotic size-dependent behaviour of the order parameters is stated in the following theorem which we will prove in the next section:

Theorem 3.2. *Let $\mathfrak{L}_{N,\beta}$ be the induced distribution of the overlap parameters and let $\vec{m}^{ij}(\theta) = (\cos\theta(\boldsymbol{e}^i - \boldsymbol{e}^j)/2, \sin\theta(\boldsymbol{e}^i - \boldsymbol{e}^j)/2)$, where $1 \le i < j \le q$ and $\theta \in [0, \pi)$ is a uniformly distributed random variable. Then there exist $q(q-1)/2$ pairs $\vec{m}^{ij}(\theta)$ such that*

$$\mathfrak{L}_{N,\beta} \xrightarrow{\mathcal{D}} \frac{1}{q(q-1)} \sum_{i<j} \left(\delta_{\vec{m}^{ij}(\theta)} + \delta_{-\vec{m}^{ij}(\theta)}\right) \equiv \mathfrak{L}_{\infty,\beta}[\{\vec{m}^{ij}(\theta)\}]$$

Furthermore, the (induced) AW-metastate is the image of the uniform distribution of θ under the measure-valued map $\theta \to \mathcal{L}_{\infty,\beta}[\{\vec{m}^{ij}(\theta)\}]$.

The case of m an arbitrary finite number of patterns is a straightforward extension.

3.5 Stochastic symmetry breaking for $q = 3$

In this section we adapt the fluctuation analysis of [11] to include Potts spins. We essentially follow the same line of argument, and find that the fluctuations, properly scaled, after dividing out the discrete symmetry, approach again a Gaussian process on the circle.

For notational simplicity we treat the case $q = 3$ only. For $q > 3$ a similar analysis applies. We denote the two patterns $\vec{\xi}^1$ and $\vec{\xi}^2$ by $\vec{\xi}$ and $\vec{\eta}$ respectively.

Define the function ϕ_N as (3.35):

$$\beta\phi_N(\vec{z}) = -Q(\vec{z}) + \vec{z}\cdot\nabla Q(\vec{z}) - c(\nabla Q(\vec{z})) \tag{3.41}$$

where $c(\vec{t})$ equals (3.27):

$$c(\vec{t}) = \frac{1}{N}\ln\left\{\mathbb{E}_\sigma \exp \vec{t}_1\cdot N\vec{q}_1 + \vec{t}_2\cdot N\vec{q}_2\right\} = \frac{1}{N}\sum_{i=1}^{N}\ln\left\{\mathbb{E}_{\sigma_i}\exp\vec{t}_1\cdot\xi_i\mathbf{e}^{\sigma_i} + \vec{t}_2\cdot\eta_i\mathbf{e}^{\sigma_i}\right\} \tag{3.42}$$

Then for $N \to \infty$ the measure

$$\mathfrak{L}_{N,\beta} = \frac{e^{-\beta N\phi_N}}{Z_N} \to \mathfrak{L}_{\infty,\beta} \tag{3.43}$$

where $\mathfrak{L}_{\infty,\beta}$ is the induced distribution of the overlap parameters.

For $q = 3$ it holds:

$$Q(\vec{z}) = \frac{K}{2}\|\vec{z}\|_2^2 = \frac{2}{3}\beta\|\vec{z}\|^2 \tag{3.44}$$

Thus:

$$\phi_N(\vec{z}) = \frac{2}{3}\|\vec{z}\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N} \ln\left\{ \mathbb{E}_\sigma \exp \frac{4}{3}\beta(\xi_i \vec{z}_1 \cdot \mathbf{e}^{\sigma_i} + \eta_i \vec{z}_2 \cdot \mathbf{e}^{\sigma_i}) \right\} \equiv \frac{2}{3}\|\vec{z}\|_2^2 - \frac{1}{\beta N}\Xi_N \tag{3.45}$$

Using $\mathbf{e}^1 = (1,0)$, $\mathbf{e}^2 = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $\mathbf{e}^3 = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$ and $K = \frac{4}{3}\beta$

$$\Xi_N = \sum_{i=1}^{N} \ln\left\{ \frac{\exp K(\xi_i z_{11} + \eta_i z_{21})}{3} + \frac{2}{3}\exp\frac{-K}{2}(\xi_i z_{11} + \eta_i z_{21}) \cosh\frac{K\sqrt{3}}{2}(\xi_i z_{12} + \eta_i z_{22}) \right\} \equiv$$

$$\sum_{i=1}^{N} \ln\left\{ \frac{1}{3}\phi_1(z_{11}, z_{21})_{\xi,\eta} + \frac{2}{3\sqrt{\phi_1(z_{11}, z_{21})_{\xi,\eta}}}\phi_2(z_{12}, z_{22})_{\xi,\eta} \right\} \tag{3.46}$$

Because for finite N the set of 6 Gibbs states has a discrete symmetry, as mentioned before, we choose out of these 6 states one preferred one, namely the one of the form $\vec{q} = (0, \hat{r}(\beta)\sin\theta, 0, \hat{r}(\beta)\cos\theta)$, see (3.6) and the remark below (3.16). Note that the θ depends both on N and on the realization of the random disorder variable. Then because of this particular choice $z_{11} = z_{21} = \phi_1 = 0$. Inserting this and defining $z_{12} = \tilde{z}_1$ and $z_{22} = \tilde{z}_2$ we get for ϕ_N:

$$\phi_N(\tilde{z}_1, \tilde{z}_2) = \frac{2}{3}\|(\tilde{z}_1, \tilde{z}_2)\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N} \ln\left\{ \frac{1}{3} + \frac{2}{3}\cosh\frac{2}{\sqrt{3}}\beta(\xi_i \tilde{z}_1 + \eta_i \tilde{z}_2) \right\} \tag{3.47}$$

Now we re-scale by putting $(z_1, z_2) = \frac{2}{\sqrt{3}}(\tilde{z}_1, \tilde{z}_2)$. Note that $\frac{2}{\sqrt{3}}\hat{r}(\beta) \to r^\star$ for $\beta \to \infty$, so after re-scaling to $r^\star$. We obtain:

$$\phi_N(z_1, z_2) = \frac{1}{2}\|(z_1, z_2)\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N} \ln\left\{ \frac{1}{2} + \cosh\beta(\xi_i z_1 + \eta_i z_2) \right\} + \frac{1}{\beta}\ln\frac{3}{2} \tag{3.48}$$

So it is enough to prove now that *with* the $\frac{1}{2}$ term we get the desired chaotic pairs structure between the patterns due to the quenched disorder for this class of ground states, once we divide out the appropriate discrete Potts permutation symmetry. Thus the original model displays chaotic 6-tuples.

First we show that the induced measure is exponentially concentrated about these circles around the origins. We denote by $\mathbb{P}$ the probability given by the Gibbs measure.

Lemma 3.3 (Concentration near the circle). *Let $\delta_N = N^{-1/10}$ and $\tilde{r}(\beta) = \frac{2}{\sqrt{3}}\hat{r}(\beta)$. Then for K and l strictly positive constants it holds:*

$$\frac{\int_{\|\vec{z}\| - \tilde{r}(\beta)| \geq \delta_N} \exp\left(-\beta N \phi_N(\vec{z})\right) d\vec{z}}{\int_{\|\vec{z}\| - \tilde{r}(\beta)| < \delta_N} \exp\left(-\beta N \phi_N(\vec{z})\right) d\vec{z}} \leq K \exp\left(-K N^l\right) \tag{3.49}$$

on a set of $\mathbb{P}$-measure which converges exponentially fast to one as $N \to \infty$.
(Equivalent to lemma 2.1 in [11])

Because of the circle concentration of $\mathfrak{L}_{N,\beta}$ it is convenient to transform ϕ_N to polar coordinates. Define $\vec{z}(r,\theta) = (r\cos\theta, r\sin\theta)$. Then transforming (3.48) leads to:

$$|(\phi_N - \mathbb{E}\phi_N)(r,\theta)| = \left| \frac{1}{\beta}\mathbb{E}_\psi \mathbb{E}_\zeta \ln\left[\frac{1}{2} + \cosh\left\{\beta\zeta r\cos\psi\right\}\right] - \frac{1}{\beta N}\sum_{i=1}^{N} \ln\left[\frac{1}{2} + \cosh\left\{\beta r\zeta_i \cos\left(\theta - \psi_i\right)\right\}\right]\right| \tag{3.50}$$

Here ζ, ψ denote the polar decomposition of the two-dimensional vector (ξ, η), i.e. ζ is distributed with density $x\exp -x^2/2$ on $\mathbb{R}^+$ and ψ uniformly on the circle $[0, 2\pi)$. See [11, page 188]. This we see easily because:

$$\begin{aligned}\xi z_1 + \eta z_2 = (\zeta\cos\psi)(r\cos\theta) + (\zeta\sin\psi)(r\sin\theta) = \zeta r(\cos\theta\cos\psi + \sin\theta\sin\psi) \\ = \zeta r\cos\left(\theta - \psi\right) \text{ and } \mathbb{E}_\psi \cos\left(\theta - \psi\right) = \mathbb{E}_\psi \cos\psi \end{aligned} \tag{3.51}$$

Proof of lemma 3.3: Denote

$$\exp\left(-\beta N\phi_N(\vec{z})\right) = \exp\left(-\beta N\mathbb{E}\phi_N(\vec{z})\right)\exp\left(-\beta N(\phi_N - \mathbb{E}\phi_N(\vec{z}))\right) \tag{3.52}$$

The function $\mathbb{E}\phi_N(\vec{z})$ has its minimum for a $\tilde{r}(\beta)$ which follows from the fixed point equations. Because $\mathbb{E}\phi_N(\vec{z})$ is sufficiently smooth it can estimated from above and below by a quadratic function $C(\|\vec{z}\| - \tilde{r}(\beta))^2$.

Now we consider the fluctuations of $\phi_N(\vec{z})$ from its mean. The proof is equal to the proof of lemma 2.1 of [11], which we do not reproduce here. Important in this proof is the random matrix $A = (\vec{\zeta}, \vec{\eta})^T(\vec{\zeta}, \vec{\eta})/N$. The eigenvalues of this matrix are very close to one. Therefore it follows (see [11]):

$$\mathbb{P}[\|A\| - 1 \geq x] \leq Ce^{-Nx^2/C} \tag{3.53}$$

Now define

$$f(\cdot) = \frac{1}{\beta}\ln\left(\frac{1}{2} + \cosh\left(\beta\,\cdot\right)\right) \tag{3.54}$$

Note $|f'(\cdot)| < |\tanh(.)|$ and $|f'(\cdot)| < 1$. Then with this function f we have by [11] an upper bound of the form $\|\vec{z} - \vec{z}'\|\|A\|^{1/2}$. The lemma is proven by a coarse graining procedure. □

Only a small part of the circles does contribute to almost all the mass of the distribution. The lemma below makes this precise:

Lemma 3.4 (Concentration around minima). *Assume the hypotheses of lemma 3.3. Let $a_N = N^{-1/25}$. Then there exist strictly positive constants K_1, K_2, C_1, C_2 such that on a set of $\mathbb{P}$-measure at least $1 - K_1 e^{-N^{1/25}}$ the following bound holds,*

$$\frac{\int_{A'_N} e^{-\beta N \phi_N(\vec{z})} d\vec{z}}{\int_{A_N} e^{-\beta N \phi_N(\vec{z})} d\vec{z}} \leq C_1 e^{-N^{1/5}} \tag{3.55}$$

where

$$\begin{aligned} A_N &= \{(r,\theta) \in I\!R_0^+ \times [0,2\pi) || r - \tilde{r}| < \delta_N,\ g_N(\theta) - \min_\theta g_N(\theta) < a_N\} \\ A'_N &= \{(r,\theta) \in I\!R_0^+ \times [0,2\pi) || r - \tilde{r}| < \delta_N,\ g_N(\theta) - \min_\theta g_N(\theta) \geq a_N\} \end{aligned} \tag{3.56}$$

(Equivalent to lemma 2.2 in [11])

For above lemma we look at the following decomposition of the fluctuations of $\phi_N(\vec{z})$:

$$(\phi_N - I\!E\phi_N)(\vec{z}) = \beta\sqrt{N}(g_N(\vec{z}') + h_N(\vec{z})) \tag{3.57}$$

where

$$h_N(\vec{z}) = g_N(\vec{z}) - g_N(\vec{z}') \tag{3.58}$$

The variable $\vec{z}'(\vec{z})$ is the projection of $\vec{z}$ onto the circle $S^1(\tilde{r})$. In above lemma

$$g_N(\theta) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\ln \left\{ \frac{1}{2} + \cosh \left\{ \beta \tilde{r} \zeta_i \cos(\theta - \psi_i) \right\} \right\} - I\!E_\psi I\!E_\zeta \ln \left\{ \frac{1}{2} + \cosh \beta \zeta \tilde{r} \cos \psi \right\} \right] \tag{3.59}$$

which is the polar coordinate form of the function $g_N(\vec{z})$, but for the projected $\vec{z}'(\vec{z})$ instead of $\vec{z}$, where

$$g_N(\vec{z}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\ln \left\{ \frac{1}{2} + \cosh \beta \vec{z} \cdot (\xi, \eta) \right\} - I\!E \ln \left\{ \frac{1}{2} + \cosh \beta \vec{z} \cdot (\xi, \eta) \right\} \right] \tag{3.60}$$

If we look at (3.57) then we see that $h_N(\vec{z})$ represents the error in the fluctuations when we replace $\vec{z}$ by its projection on the circle. The function $g_N(\vec{z}')$ represents the fluctuations for the projected $\vec{z}$ on the circle.

Proof of Lemma 3.4: Again its proof is equal to the proof in [11]. First it is proven there that the projectional error $h_N(\vec{z})$ is vanishing exponentially in N. A important tool for this is the mean value theorem. This because $|f'(\cdot)| < 1$ of function $f(.)$ in (3.54). Then after some calculations a lemma similar to above lemma is proven. □

Proof of theorem 3.2: In the preceding paragraphs we have seen that the measures $\tilde{\mathcal{L}}$ concentrate on a circle at the places where the random function $g_N(\theta)$ takes its minimum. Now it only remains to show that these sets degenerate to a single point, a.s. in the limit $N \to \infty$. In [11] this is already done for a class of functions which include the aperiodic even function

$$g(\cdot) = \ln \left\{ \cosh \beta \cdot + \frac{1}{2} \right\} \tag{3.61}$$

This means that the process $\eta_N = g_N(\theta) - \mathbb{E} g_N(\theta)$ converges to a strictly stationary Gaussian process, having a.s. continuously differentiable sample paths. And on any interval $[s, s+t],\ t < \pi$ the function η_N has only one global minimum. Furthermore, if we define the sets:

$$L_N = \{\theta \in [0, \pi) :\ \eta_N(\theta) - \min_{\theta'} \eta_N(\theta') \leq \epsilon_N\} \tag{3.62}$$

with ϵ_N some sequence converging to zero, $L_N \xrightarrow{\mathcal{D}} \theta^\star$. Then some remaining considerations are needed from [11] to conclude the proof. □

Remark 3.5. *We can easily generalize the above two lemmas to the infinite pattern case where the number of patterns $m = \log N$. The corrections to the stated probabilities are only of order $\exp(P(\log N))$, where P is a polynomial of finite degree. However these trivial considerations do break down when we choose $m = \alpha N$, where $\alpha > 0$ even if it is small. In these considerations we make use of the fact that for a C^3-function f from $\mathbb{R}^m \to \mathbb{R}$ it holds that the Hessian of f at its minima is positive definite. Therefore there exists positive constants c, C such that for every vector $\vec{h}$ with $\|\vec{h}\| \geq 0$*

$$c\|\vec{h}\|^2 \leq f(\min + \vec{h}) - f(\min) \leq C\|\vec{h}\|^2 \tag{3.63}$$

When $m \to \infty$ there is sphere-concentration of the order-parameters itself, so the minima of ϕ_N might not be unique (except for spin-symmetry) anymore.

Chapter 4

The 2d Ising model with random boundary conditions

In this chapter we study the infinite-volume limit behavior of the 2d Ising model under possibly strong random boundary conditions. The model exhibits chaotic size-dependence at low temperatures and we prove that the '+' and '-' phases are the only almost sure limit Gibbs measures, assuming that the limit is taken along a sparse enough sequence of squares. In particular, we provide an argument to show that in a sufficiently large volume a typical spin configuration under a typical boundary condition contains no interfaces. In order to exclude mixtures as possible limit points, a detailed multi-scale contour analysis is performed. Finally we show that the 2d Ising model with a high random boundary field coincide with 2d Ising model with random boundary conditions.

4.1 Introduction

A fundamental problem in equilibrium statistical mechanics is to determine the set of physically accessible thermodynamic states for models defined via a family of local interactions. Usually [26, 38] one interprets the extremal elements of the set of translationally invariant Gibbs measures as the pure thermodynamic phases of the model. In particular this means that one gathers all periodic or quasiperiodic extremal Gibbs measures into symmetry-equivalent classes and identifies the latter with the pure phases. Examples are the ferromagnetic, the antiferromagnetic, crystalline or quasicrystalline phases exhibited by various models. In this approach one does not consider either interface states or mixtures as pure phases. The mixtures allow for a unique decomposition into the extremal measures and are traditionally interpreted in terms of a lack of the knowledge about the thermodynamic state of the system. They can also be classified as

less stable than the extremal measures [40, 59]. It is thought that interface states which are extremal Gibbs measures are more stable than mixed states, but less so than pure phases. However, such an "intrinsic" characterization has not been developed. Note, moreover, that in disordered systems such as spin glasses, the stability of pure phases is a priori not clear and characterizing them remains an open question.

An efficient strategy for models with a simple enough structure of low-temperature phases is to associate these with suitable *coherent boundary conditions*. The latter are usually chosen as *ground states* of the model. As an example, the '+' and '-' Ising phases can be obtained by fixing the constant '+', respectively the constant '-' configurations at the boundaries and by letting the volume tend to infinity. This idea has been generalized to a wide class of models with both a finite and a 'mildly' infinite number of ground states, and is usually referred to as the *Pirogov-Sinai theory* [8, 15, 77, 70, 71]. The main assumption is that the different ground states are separated by high enough energy barriers, which can be described in terms of domain walls, referred to as contours. A useful criterion to check this so-called Peierls condition is within the formalism of m-potentials due to Holzstynski and Slawny [43].

An alternative strategy is to employ a boundary condition that does not favor any of the phases. Examples are the free and periodic boundary conditions for the zero-field Ising model, or the periodic boundary conditions for the Potts model at the critical temperature. In all these cases, an infinite-volume Gibbs measure is obtained that is a homogenous mixture of all phases.

Another scenario has been expected to occur for spin glasses. Namely, Newman and Stein have conjectured [61, 62, 63, 65, 66] that some spin glass models under symmetric boundary conditions exhibit non-convergence to a single thermodynamic limit measure, a phenomenon called *chaotic size dependence* (see also [32, 55, 24]). In this case, both the set of limit points of the sequence of the finite-volume Gibbs measures and their empirical frequency along the sequence of increasing volumes are of interest, and the formalism of *metastates* has been developed [63, 65, 64] to deal with these phenomena. These arguments have been made rigorous for a class of mean-field models [12, 51, 53, 11, 29, 67, 52], whereas no such results are available for short-range spin glasses. For some general background on spin glasses and disordered models we refer to [10, 34, 54, 73].

A natural toy-problem where the usual contour methods can be used in the regime of chaotic size-dependence is the zero field Ising model with the boundary condition sampled from a random distribution which is symmetric under the spin flip. In dimension 2 or more and at any subcritical temperature (including $T = 0$) the finite-volume Gibbs measures are expected to oscillate randomly between the '+' and the '-' phases, demonstrating the chaotic size dependence with exactly two limit points coinciding with the thermodynamic phases of the model [62]. In particular, one does not expect either any

interface (e.g. Dobrushin) Gibbs states or any non-trivial statistical mixtures to occur as the limit points. This problem was addressed in [28] where the conjecture was rigorously proven as the almost sure picture in the regime of the weak boundary coupling. In this regime, the boundary bonds are made sufficiently weaker w.r.t. the bulk bonds so that the interface configurations become damped exponentially with the size of the system, uniformly for *all* boundary conditions. Hence, all translationally non-invariant Gibbs measures are forbidden as possible limit points and one only needs to prove that the mixtures do not appear with probability 1.

In this chapter we continue this study by removing the weakness assumption on the boundary bonds. To be specific, we consider the 2d Ising model with the random boundary condition sampled from the symmetric i.i.d. field $\{-1,1\}^{\mathbb{Z}^2}$ and coupled to the system via the *bulk* coupling constant. The conjecture remains true in this case and the crucial novelty of our approach is a detailed multi-scale analysis of contour models in the regime where realizations of the boundary condition are allowed that violate the 'diluteness' (Peierls) condition, possibly making interfaces likely configurations. To be precise, these interfaces can have large Gibbs probabilities for certain boundary conditions, but we will show that such boundary conditions are sufficiently unlikely to occur for large volumes. An important side-result is the almost sure absence of interface configurations. This means that for a typical boundary condition, the probability of the set of configurations containing an interface tends to zero in the infinite-volume limit. Note that this excludes interfaces in a stronger way than the familiar result about the absence of translationally non-invariant Gibbs measures in the 2d Ising model [36, 1, 42]. Indeed, the absence of fluctuating interfaces basically means that not only the expectations of local functions but also their space averages (e.g. the volume-averaged magnetization) have only two limit points, corresponding to the two Ising phases. Hence, we believe that our techniques allow for a natural generalization to any dimension $d \geq 2$. However, as already argued in [28], in dimensions $d \geq 4$, the set $\{\mu^+, \mu^-\}$ is expected (and partially proven) to be the *almost sure* set of limit measures, the limit being taken along the regular sequence of cubes. On the other hand, for $d = 2, 3$ the same result can only be obtained if the limit is taken along a sparse enough sequence of cubes. In the latter case it remains an open problem to analyze the set of limit points along the regular sequence of cubes. Our conjecture is that the almost sure set of limit points coincides then with the set of all translationally invariant Gibbs measures, i.e. including the mixtures.

The structure of this chapter is as follows. We will first introduce our notation in Section 4.2, and describe our results in Section 4.3. Then in Sections 4.4 and 4.5 we will introduce a contour representation of the model and set up our cluster expansion formalism. In Section 4.6 we first exclude the occurrence of interfaces. In the rest of the chapter we develop a multiscale argument, providing a weak version of the local limit

theorem to show that no mixed states can occur as limit points in the infinite-volume limit. In Section 4.12 we discuss the high field case. Two general results, the first one on a variant of the cluster expansion convergence criteria for polymer models and the second one on local limit upper bounds, are collected in two final sections.

4.2 Set-up

We consider the two-dimensional square lattice $\mathbb{Z}^2$ and use the symbols $\sigma, \eta, \ldots$ for the maps $\mathbb{Z}^2 \mapsto \{-1, 1\}$. They are called *spin configurations* and the set of all spin configurations is $\Omega = \{-1, 1\}^{\mathbb{Z}^2}$. Furthermore, the symbol σ_A is used for the restriction of a spin configuration $\sigma \in \Omega$ to the set $A \subset \mathbb{Z}^2$. If $A = \{x\}$, we write σ_x instead. The set of all restrictions of Ω to the set A is Ω_A.
A function $f : \Omega \mapsto \mathbb{R}$ is called *local* whenever there is a finite set $D \subset \mathbb{Z}^2$ such that $\sigma_D = \sigma'_D$ implies $f(\sigma) = f(\sigma')$. The smallest set with this property is called the *dependence set* of the function f and we use the symbol $\mathcal{D}_f$ for it. To every local function f we assign the supremum norm $\|f\| = \sup_{\sigma\in\Omega} |f(\sigma)|$.
The spin configuration space Ω comes equipped with the product topology, which is followed by the weak topology on the space $M(\Omega)$ of all probability measures on Ω. The latter is introduced via the collection of seminorms

$$\|\mu\|_X = \sup_{\substack{\|f\|=1 \\ \mathcal{D}_f \subset X}} |\mu(f)| \tag{4.1}$$

upon all finite $X \subset \mathbb{Z}^2$. Then, the weak topology is generated by the collection of open balls $B^\epsilon_X(\mu) = \{\nu;\ \|\nu - \mu\|_X < \epsilon\}$, $\epsilon > 0$, X finite, and a sequence $\mu_n \in M(\Omega)$ weakly converges to μ if and only if $\|\mu_n - \mu\|_X \to 0$ for all finite $X \subset \mathbb{Z}^2$. Under the weak topology, $M(\Omega)$ is compact.

We consider a collection of the Hamiltonians $H^\eta_\Lambda : \Omega_\Lambda \mapsto \mathbb{R}$ for all square volumes $\Lambda = \Lambda(N)$, $N = 1, 2, \ldots$,

$$\Lambda(N) = \{x \in \mathbb{Z}^2;\ \|x\| \le N\} \qquad \|x\| = \max\{|x_1|, |x_2|\} \tag{4.2}$$

and *boundary conditions* $\eta \in \Omega$. The Hamiltonians are given by

$$H^\eta_\Lambda(\sigma_\Lambda) = -\beta \sum_{\langle x,y\rangle \subset \Lambda} (\sigma_x\sigma_y - 1) - \beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\ y\in\Lambda^c}} \sigma_x\eta_y \tag{4.3}$$

where $\langle x, y\rangle$ stands for pairs of nearest neighboring sites, i.e. such that $\|x - y\|_1 := |x_1 - y_1| + |x_2 - y_2| = 1$, and $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$. We consider the ferromagnetic case,

$\beta > 0$. Following a familiar framework, we introduce the *finite-volume Gibbs measure* $\mu^\eta_\Lambda \in M(\Omega)$ by

$$\mu^\eta_\Lambda(\sigma) = \frac{1}{Z^\eta_\Lambda} \exp[-H^\eta_\Lambda(\sigma_\Lambda)] \, \mathbb{1}_{\{\sigma_{\Lambda^c} = \eta_{\Lambda^c}\}} \tag{4.4}$$

and define the set $\mathcal{G}_\beta$ of (infinite-volume) *Gibbs measures*, $\mathcal{G}_\beta$, as the weak closure of the convex hull over the set of all weak limit points of the sequences $(\mu^\eta_{\Lambda(N)})_{N\to\infty}$, $\eta \in \Omega$. A standard result reads that there exists β_c such that for any $\beta > \beta_c$ the set of Gibbs measures $\mathcal{G}_\beta = \{\alpha\mu^+ + (1-\alpha)\mu^-;\, 0 \le \alpha \le 1\}$. Here, the extremal measures $\mu^\pm$ are translation-invariant, they satisfy the symmetry relation $\int \mathrm{d}\mu^+(\sigma)\, f(\sigma) = \int \mathrm{d}\mu^-(\sigma)\, f(-\sigma)$, and can be obtained as the weak limits $\lim_{N\to\infty} \mu^\eta_{\Lambda(N)}$ for $\eta \equiv \pm 1$.

4.3 Results

We consider the limit behavior of the sequence of finite-volume Gibbs measures $(\mu^\eta_{\Lambda(N)})_{N\in\mathbb{N}}$ under boundary conditions η sampled from the i.i.d. symmetric random field

$$\boldsymbol{P}\{\eta_x = 1\} = \boldsymbol{P}\{\eta_x = -1\} = \frac{1}{2} \tag{4.5}$$

Our first result concerns the almost sure structure of the set of all limit points of the sequence of the finite-volume Gibbs measures, the limit being taken along a sparse enough sequence of squares.

Theorem 4.1. *For arbitrary $\omega > 0$ there is a $\beta_1 = \beta_1(\omega)$ such that for any $\beta \ge \beta_1$ the set of all weak limit points of any sequence $(\mu_{\Lambda(k_N)})_{N=1,2,\dots}$, $k_N \ge N^{2+\omega}$, is $\{\mu^+, \mu^-\}$, $\boldsymbol{P}$-a.s.*

Remark 4.2. *The above theorem does not exclude other measures as the almost sure limit points, provided that other (non-sparse) sequences of squares are taken instead. Actually, our conjecture is that, for β large enough, the set of all weak limit points of $(\mu_{\Lambda(N)})_{N=1,2,\dots}$ coincides $\boldsymbol{P}$-a.s. with $\mathcal{G}_\beta$. On the other hand, in dimension 3, it is rather expected to coincide with the set of all translation-invariant Gibbs measures, and, in any dimension higher than 3, with the set $\{\mu^+, \mu^-\}$.*

Remark 4.3. *A modification of the Hamiltonian* (4.3) *is obtained by re-scaling the boundary coupling by a factor λ to get*

$$H^{\lambda,\eta}_\Lambda(\sigma_\Lambda) = -\beta \sum_{\langle x,y\rangle \subset \Lambda} (\sigma_x\sigma_y - 1) - \lambda\beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\, y\in\Lambda^c}} \sigma_x\eta_y \tag{4.6}$$

In this case, the claim of Theorem 4.1 for the sequence of the finite-volume Gibbs measures

$$\mu_\Lambda^{\lambda,\eta}(\sigma) = \frac{1}{\mathcal{Z}_\Lambda^{\lambda,\eta}} \exp[-H_\Lambda^{\lambda,\eta}(\sigma_\Lambda)]\, \mathbb{1}_{\{\sigma_{\Lambda^c}=\eta_{\Lambda^c}\}} \tag{4.7}$$

was proven in [28] under the condition that $|\lambda|$ is small enough (= the boundary coupling is sufficiently weak w.r.t. the bulk one). It was also shown that $\{\mu^+, \mu^-\}$ is the almost sure set of limit points of the sequence $(\mu^\eta_{\Lambda(N)})_{N\in\mathbb{N}}$, provided that the space dimension is at least 4.

To reveal the nature of all possible limit points that can appear along the sequence of squares $\Lambda(N)$, $N = 1, 2, \ldots$, we study the empirical frequency for the finite-volume Gibbs states from the sequence $(\mu^\eta_{\Lambda(N)})_{N\in\mathbb{N}}$ to occur in a fixed set of measures. More precisely, for any set $B \subset M(\Omega)$, boundary condition $\eta \in \Omega$, and $N = 1, 2, \ldots$, we define

$$Q_N^{B,\eta} = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\{\mu^\eta_{\Lambda(k)} \in B\}} \tag{4.8}$$

The next theorem shows the *null-recurrent* character of all measures different from both μ^+ and μ^-. We use the notation $\bar{B}$ and B^0 for the weak closure and the weak interior of B, respectively.

Theorem 4.4. *There is β_2 such that for any $\beta \geq \beta_2$ and any set $B \subset M(\Omega)$, one has*

$$\lim_{N\uparrow\infty} Q_N^{B,\eta} = \begin{cases} 0 & \text{if } \mu^+, \mu^- \notin \bar{B} \\ \frac{1}{2} & \text{if } \mu^\pm \in B^0 \text{ and } \mu^\mp \notin \bar{B} \\ 1 & \text{if } \mu^+, \mu^- \in B^0 \end{cases} \tag{4.9}$$

with $\boldsymbol{P}$-probability 1.

Both theorems follow in a straightforward way from the following key estimate that will be proven in the sequel of this chapter.

Proposition 4.5. *Given $\alpha > 0$, there is a $\beta_0 = \beta_0(\alpha)$ such that for any $\beta \geq \beta_0$, $\epsilon > 0$ and $X \subset \mathbb{Z}^d$ finite,*

$$\overline{\lim_{N\to\infty}}\, N^{\frac{1}{2}-\alpha}\, \boldsymbol{P}\{(\|\mu^\eta_{\Lambda(N)} - \mu^+\|_X \wedge \|\mu^\eta_{\Lambda(N)} - \mu^-\|_X) \geq \epsilon\} < \infty \tag{4.10}$$

Remark 4.6. *The proposition claims that, for a* typical *$\eta \in \Omega$, the finite-volume Gibbs measures are expected to be near the extremal Gibbs measures $\mu^\pm$. The above probability upper-bound of the form $\mathcal{O}\big(N^{-\frac{1}{2}+\alpha}\big)$ will be proven by means of a variant of the*

local limit theorem for the sum of weakly dependent random variables. Although we conjecture the correct asymptotics to be of order $N^{-\frac{1}{2}}$, the proof of any *lower bound goes beyond the presented technique. This is why the detailed structure of the almost sure set of the limit Gibbs measures is not available, except for the limits taken along sparse enough sequences of squares.*

Proof of Theorem 4.1. Given $\omega > 0$, we choose an $\alpha < \omega/(2(2+\omega))$ and define $\beta_1(\omega) = \beta_0(\alpha)$. Let $\beta \geq \beta_1(\omega)$ and $k_N \geq N^{2+\omega}$.

First let $\mu \notin \{\mu^+, \mu^-\}$. There exists a weakly open set $B \subset M(\Omega)$ such that $\mu \in B$ and $\mu^+, \mu^- \notin \bar{B}$. Choosing a finite set $X \subset \mathbb{Z}^2$ and $\epsilon > 0$ such that $B^\epsilon_X(\mu^\pm) \cap B = \emptyset$, Proposition 4.5 gives the bound

$$\begin{aligned} &\boldsymbol{P}\{\mu^\eta_{\Lambda(k_N)} \in B\} \leq \boldsymbol{P}\{\mu^\eta_{\Lambda(k_N)} \notin B^\epsilon_X(\mu^+) \cup B^\epsilon_X(\mu^-)\} \\ &= \mathcal{O}(k(N)^{-\frac{1}{2}+\alpha}) = \mathcal{O}(N^{-1+\alpha(2+\omega)-\frac{\omega}{2}}) \end{aligned} \tag{4.11}$$

Since $\sum_N \boldsymbol{P}\{\mu^\eta_{\Lambda(k_N)} \in B\} < \infty$, the set B contains $\boldsymbol{P}$-a.s. no limit points of the sequence $\mu^\eta_{\Lambda(k_N)}$ due to the Borel-Cantelli argument. Hence, with $\boldsymbol{P}$-probability 1, μ is not a limit point.

To prove that both μ^+ and μ^- are $\boldsymbol{P}$-a.s. limit points, take any finite set of sites X and $\epsilon > 0$ such that $B^\epsilon_X(\mu^+) \cap B^\epsilon_X(\mu^-) = \emptyset$. By the symmetry of the distribution, $\boldsymbol{P}\{\mu_{\Lambda(k_N)} \in B^\epsilon_X(\mu^+)\} = \boldsymbol{P}\{\mu_{\Lambda(k_N)} \in B^\epsilon_X(\mu^-)\}$ and, employing Proposition 4.5 again, $\lim_N \boldsymbol{P}\{\mu_{\Lambda(k_N)} \in B^\epsilon_X(\mu^\pm)\} = \frac{1}{2}$. By the Borel-Cantelli and the compactness arguments, the weak closure $\bar{B}^\epsilon_X(\mu^\pm)$ contains a limit point, $\boldsymbol{P}$-a.s. As $\mu^\pm = \cap_{X,\epsilon} \bar{B}^\epsilon_X(\mu^\pm)$, the statement is proven. □

Proof of Theorem 4.4. Choose $\beta_2 = \beta_0(\alpha)$ for an arbitrary $\alpha \in (0, \frac{1}{2})$ and assume $\beta \geq \beta_2$, $B \in M(\Omega)$. Using the notation $q^{B,\eta}_N = \boldsymbol{P}\{\mu^\eta_{\Lambda(N)} \in B\}$ and repeating the reasoning in the proof of Theorem 4.1, one gets

$$\begin{aligned} \boldsymbol{E}\, \mathbb{1}_{\{\mu^\eta_{\Lambda(N)} \in B\}} &= q^{B,\eta}_N \\ &= \begin{cases} \mathcal{O}(N^{-\frac{1}{2}+\alpha}) \to 0 & \text{if } \mu^+, \mu^- \notin \bar{B} \\ \frac{1}{2} - \mathcal{O}(N^{-\frac{1}{2}+\alpha}) \to \frac{1}{2} & \text{if } \mu^\pm \in B^0 \text{ and } \mu^\mp \notin \bar{B} \\ 1 - \mathcal{O}(N^{-\frac{1}{2}+\alpha}) \to 1 & \text{if } \mu^\pm \in B^0 \end{cases} \end{aligned} \tag{4.12}$$

and

$$\operatorname{Var}\, \mathbb{1}_{\{\mu^\eta_{\Lambda(N)} \in B\}} = q^{B,\eta}_N (1 - q^{B,\eta}_N) \leq \frac{1}{4} \tag{4.13}$$

Hence, $\sum_N \frac{1}{N^2} \operatorname{Var}\, \mathbb{1}_{\{\mu^\eta_{\Lambda(N)} \in B\}} < \infty$ and since the functions $\mathbb{1}_{\{\mu^\eta_{\Lambda(N)} \in B\}}$, $N = 1, 2, \ldots$ are independent, the result immediately follows from the strong law of large numbers [23]. □

4.4 Geometrical representation of the model

We define the *dual lattice* $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (1/2, 1/2)$. The (unordered) pairs of nearest neighboring sites $\langle x, y\rangle \subset \mathbb{Z}^2$ are called *bonds* and to every bond we assign a unique *dual bond* $\langle x^*, y^*\rangle \equiv \langle x, y\rangle^* \subset (\mathbb{Z}^2)^*$. Given a set of dual bonds A^*, we use the symbol $|A^*|$ to denote the number of all dual bonds in A^*. Further, with a slight abuse of notation, we also write $x^* \in A^*$ whenever there exists a dual bond $\langle x^*, y^*\rangle \in A^*$, i.e. A^* also stands for the corresponding set of dual sites.

Any set A^* of dual bonds is called *connected* whenever for any dual sites $x^*, y^* \in A^*$ there exists a sequence of dual bonds $\langle x^*, x_1^*\rangle, \langle x_1^*, x_2^*\rangle, \ldots, \langle x_{k-1}^*, y^*\rangle \in A^*$. The distance $d[A^*, B^*]$ of the sets of dual bonds A^*, B^* is defined as the smallest integer k such that there exist $x^* \in A^*$, $y^* \in B^*$, and a sequence of dual bonds $\langle x^*, x_1^*\rangle, \langle x_1^*, x_2^*\rangle$, $\ldots, \langle x_{k-1}^*, y^*\rangle \subset (\mathbb{Z}^2)^*$. Similarly, a set of sites $A \subset \mathbb{Z}^2$ is called *connected* whenever for all $x, y \in A$ there exists a sequence of bonds $\langle x, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{k-1}, y\rangle \subset A$. Correspondingly, the distance $d[A, B]$ of the sets $A, B \subset \mathbb{Z}^2$ is understood in the sense of the $\|.\|_1$-norm.

In the sequel we assume that a volume $\Lambda = \Lambda(N)$ is fixed and we define the *boundary* $\partial\Lambda$ as the set of all dual bonds $\langle x, y\rangle^*$ such that $x \in \Lambda$ and $y \in \Lambda^c$. In general, ∂A, $A \subset \Lambda$ is the set of all dual bonds $\langle x, y\rangle^*$, $x \in A$, $y \in \Lambda^c$. For any subset $P \subset \partial\Lambda$ we use the symbol $\underline{P}$ to denote the set of all sites $y \in \Lambda^c$ such that there is a (unique) bond $\langle x, y\rangle^* \in P$, $x \in \Lambda$. If $\underline{P}$ is a connected set of sites, then P is called a *boundary interval*. Obviously, any boundary interval is a connected set of dual bonds, however, the opposite is not true. However, any set $P \subset \partial\Lambda$ has a unique decomposition into a family of (maximal) boundary intervals. Furthermore, consider all connected sets P_i of dual bonds satisfying $P \subset P_i \subset \partial\Lambda$ which are minimal in the sense of inclusion. The smallest of these sets is called $\mathrm{Con}(P)$ (in the case of an equal size take the first one in the lexicographic order) and we use the shorthand $|P|_{\mathrm{con}} = |\,\mathrm{Con}(P)|$. Finally, we define the *corners* of $\Lambda(N)$ as the dual sites $x_{C,1}^* = (-N - 1/2, -N - 1/2)$, $x_{C,2}^* = (N + 1/2, -N - 1/2)$, $x_{C,3}^* = (N + 1/2, N + 1/2)$, and $x_{C,4}^* = (-N - 1/2, N + 1/2)$.

Pre-contours

Given a configuration $\sigma \in \Omega_\Lambda = \{-1, +1\}^\Lambda$, the dual bond $\langle x, y\rangle^*$ to a bond $\langle x, y\rangle \subset \Lambda$ is called *broken* whenever $\sigma_x \neq \sigma_y$, and the set of all the broken dual bonds is denoted by $\Delta_\Lambda(\sigma)$. In order to define a suitable decomposition of the set $\Delta_\Lambda(\sigma)$ into components, we take advantage of a certain freedom in such a construction to obtain the components with suitable geometrical properties. In this first step, we define the *pre-contours* as follows. Consider all maximal connected components of the set of dual bonds $\Delta_\Lambda(\sigma)$. By the standard 'rounding-corner' procedure, see Figure 4.1, we further

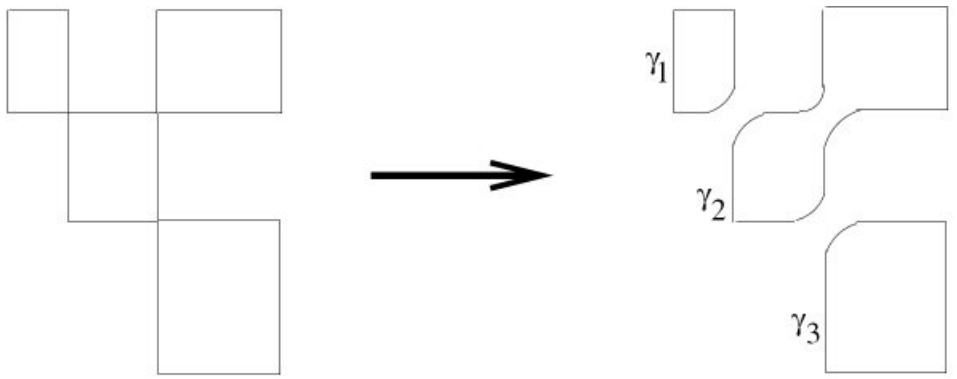


Figure 4.1: *Pre-contours constructed via the rounding corner procedure.*

split them into connected (not necessarily disjoint) subsets, γ, which can be identified with (open or closed) simple curves. Namely,

$$\gamma = \{\langle x_0^*, x_1^*\rangle, \langle x_1^*, x_2^*\rangle, \ldots, \langle x_{k-1}^*, x_k^*\rangle\} \qquad k \in \mathbb{N}$$

such that if $x_i^* = x_j^*$, $i \neq j$, then $\{i, j\} = \{0, k\}$ and γ is closed. Otherwise, $x_i^* \neq x_j^*$ for all $i \neq j$ and γ is open with $x_0^*, x_k^* \in \partial\Lambda$.

These γ are called *pre-contours* and we use the symbol $\tilde{\mathcal{D}}_\Lambda(\sigma)$ for the set of all pre-contours corresponding to σ; write also $\tilde{\mathscr{D}}_\Lambda = \{\tilde{\mathcal{D}}_\Lambda(\sigma),\, \sigma \in \Omega_\Lambda\}$ and use the symbol $\tilde{\mathcal{K}}_\Lambda$ for the set of all pre-contours in Λ. Any pair of pre-contours $\gamma_1, \gamma_2 \in \tilde{\mathcal{K}}_\Lambda$ is called *compatible* whenever there is a configuration $\sigma \in \Omega_\Lambda$ such that $\gamma_1, \gamma_2 \in \tilde{\mathcal{D}}_\Lambda(\sigma)$. A set of pairwise compatible pre-contours is called a *compatible set*. Obviously, $\tilde{\mathscr{D}}_\Lambda$ is simply the collection of all compatible sets of pre-contours from $\tilde{\mathcal{K}}_\Lambda$. Intuitively, the pre-contours that are closed curves coincide with the familiar Ising contours, whereas the pre-contours touching the boundary become open curves.

Obviously, $\Omega_\Lambda \mapsto \tilde{\mathscr{D}}_\Lambda$ is a two-to-one map with the images of the configurations σ and $-\sigma$ being identical. In order to further analyze this map, we introduce the concept of interior and exterior of the pre-contours briefly as follows (the details can be found in [9, 28]). If $\sigma \in \Omega_\Lambda$ is a configuration such that $\tilde{\mathcal{D}}_\Lambda(\sigma) = \{\gamma\}$, then there is a unique decomposition of the set Λ into a pair of disjoint connected subsets, $\Lambda = \Lambda_1 \cup \Lambda_2$, such that for any bond $\langle x, y\rangle$, $x \in \Lambda_1$, $y \in \Lambda_2$, one has $\langle x, y\rangle^* \in \gamma$. These are called the *exterior*, $\mathrm{Ext}(\gamma)$, and the *interior*, $\mathrm{Int}(\gamma)$, where the assignment is given by the following procedure. We distinguish three mutually exclusive classes of pre-contours:

i) *Bulk pre-contours.*
$\partial\Lambda = \partial\Lambda_1$. Then, $\mathrm{Ext}(\gamma) := \Lambda_1$ and $\mathrm{Int}(\gamma) := \Lambda_2$, see Figure 4.1.

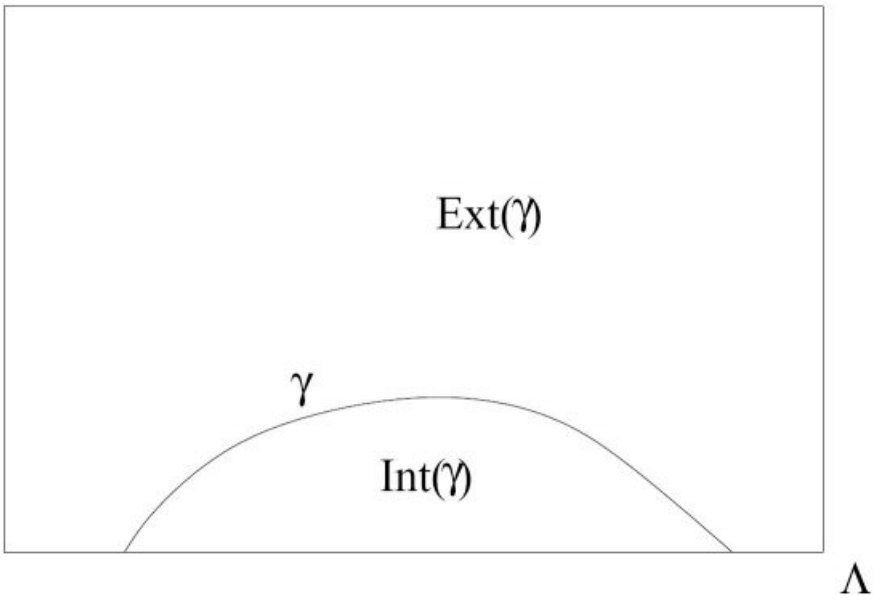


Figure 4.2: *Small boundary pre-contour.*

ii) *Small boundary pre-contours.*
Λ_1 contains at least three corners of Λ and $\partial\Lambda_2 \neq \emptyset$. Then, $\mathrm{Ext}(\gamma) := \Lambda_1$ and $\mathrm{Int}(\gamma) := \Lambda_2$, see Figure 4.2.

iii) *Interfaces.*
Both Λ_1 and Λ_2 contain exactly two corners of Λ and a) $|\Lambda_1| > |\Lambda_2|$, or b) $|\Lambda_1| = |\Lambda_2|$ and $x_{C,1} \in \Lambda_1$. Then, $\mathrm{Ext}(\gamma) := \Lambda_1$ and $\mathrm{Int}(\gamma) := \Lambda_2$, see Figure 4.3.

The set $\partial\gamma := \partial\,\mathrm{Int}(\gamma)$ is called the *boundary* of the pre-contour γ.

Contours

Next, we define contours by gluing some boundary pre-contours together via the following procedure. Any compatible pair of pre-contours $\gamma_1, \gamma_2 \in \tilde{\mathcal{K}}_\Lambda$ is called *boundary-matching* iff $\partial\gamma_1 \cap \partial\gamma_2 \neq \emptyset$. Any compatible set of pre-contours such that the graph on this set obtained by connecting the pairs of boundary-matching pre-contours becomes connected is called a *contour*. In particular, every bulk pre-contour is boundary-matching with no other compatible pre-contour. Therefore, every bulk pre-contour is

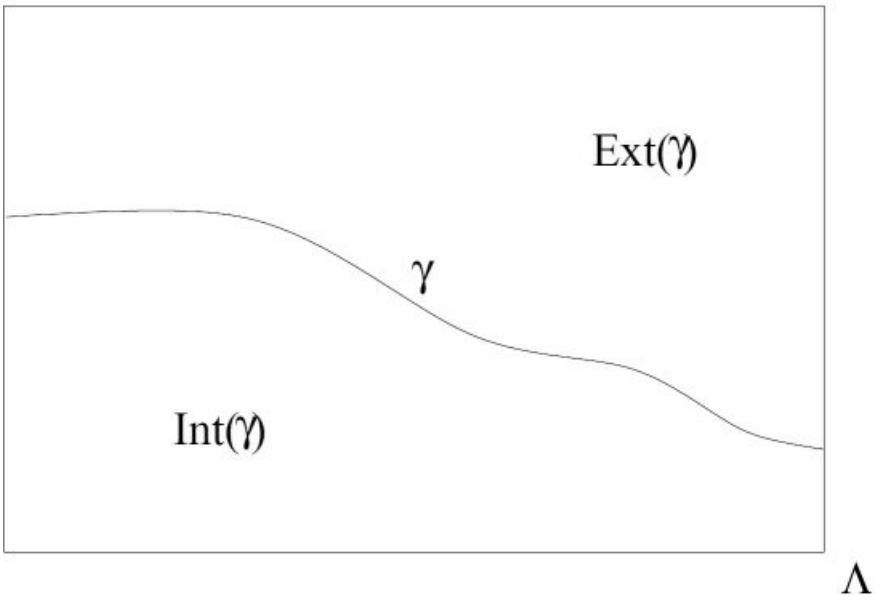


Figure 4.3: *Interface.*

trivially a contour. We use the symbol $\mathcal{D}_\Lambda(\sigma)$ for the set of all contours corresponding to $\sigma \in \Omega_\Lambda$ and $\mathcal{K}_\Lambda$ for the set of all contours in Λ. Any pair of contours Γ_1, Γ_2 is compatible, $\Gamma_1 \sim \Gamma_2$, whenever all pairs of pre-contours $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ are compatible, and we write $\mathscr{D}_\Lambda$ for the set of all families of pairwise compatible contours in Λ. All the above geometrical notions naturally carry over to contours and we define the exterior, $\mathrm{Ext}(\Gamma) := \cap_{\gamma\in\Gamma} \mathrm{Ext}(\gamma)$, the interior, $\mathrm{Int}(\Gamma) := \Lambda \setminus \mathrm{Ext}(\Gamma)$ (in general, not a connected set anymore), the boundary $\partial\Gamma := \cup_{\gamma\in\Gamma}\partial\gamma$, and the length $|\Gamma| := \sum_{\gamma\in\Gamma} |\gamma|$. Similarly, if $\partial \in \mathscr{D}_\Lambda$ is a configuration of contours, let $\mathrm{Ext}(\partial) := \cap_{\Gamma\in\partial} \mathrm{Ext}(\Gamma)$, $\mathrm{Int}(\partial) := \Lambda \setminus \mathrm{Ext}(\partial)$, and $|\partial| := \sum_{\Gamma\in\partial} |\Gamma|$.

Eventually we arrive at the following picture. The set $\mathcal{K}_\Lambda$ of contours is a union of three disjoint sets of contours, namely of the sets of all

i) *bulk (pre-)contours*.

ii) *small boundary contours* Γ defined by 1) $\partial\Gamma \neq \emptyset$, and 2) no pre-contour $\gamma \in \Gamma$ is an interface.

   a) *simple* small boundary contours: the boundary $\partial\Gamma$ contains no corner, i.e. $\partial\Gamma$ is a boundary interval.

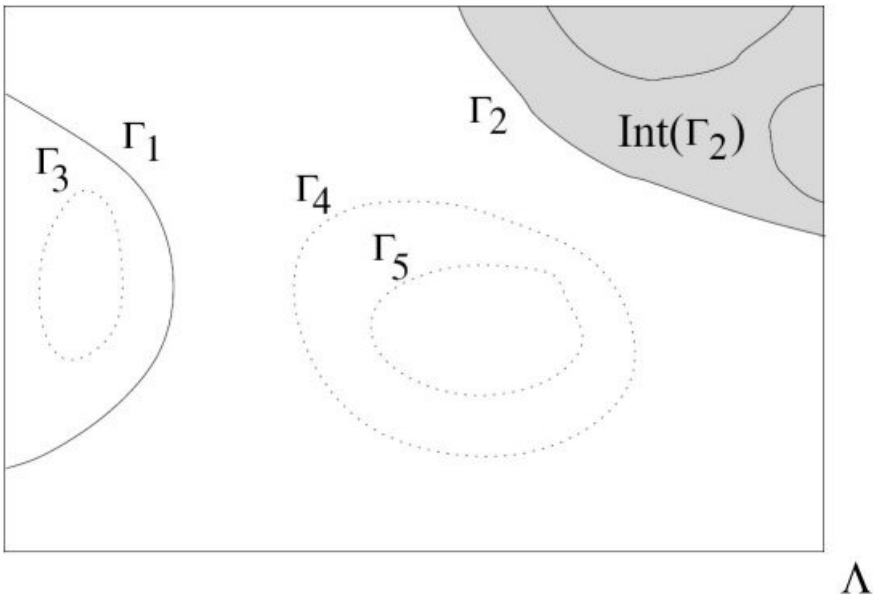


Figure 4.4: *Bulk and small boundary contours.*

b) *corner* small boundary contours: there is exactly one corner $x^*_{C,i} \in \partial\Gamma$.

iii) *large boundary contours* Γ, i.e. containing at least one interface $\gamma \in \Gamma$.

Examples of the bulk, small boundary, and large boundary contours are given in Figures 4.4 and 4.5.

Furthermore, $\mathcal{D}_\Lambda(\sigma)$ is a two-to-one map $\Omega_\Lambda \mapsto \mathscr{D}_\Lambda$ satisfying the spin-flip symmetry $\mathcal{D}_\Lambda(\sigma) = \mathcal{D}_\Lambda(-\sigma)$. Since σ takes a unique spin value in the set $\mathrm{Ext}(\mathcal{D}_\Lambda(\sigma))$, there is a natural decomposition $\Omega_\Lambda = \Omega^+_\Lambda \cup \Omega^-_\Lambda$ according to this value, i.e.

$$\Omega^\pm_\Lambda := \{\sigma \in \Omega_\Lambda;\, \sigma|_{\mathrm{Ext}(\mathcal{D}_\Lambda(\sigma))} = \pm 1\} = -\Omega^\mp_\Lambda \tag{4.14}$$

As a consequence, $\mathcal{D}_\Lambda$ splits into a conjugated (by spin-flip symmetry) pair of one-to-one maps $\Omega^\pm_\Lambda \mapsto \mathscr{D}_\Lambda$. This enables us to represent the finite-volume Gibbs measure (4.4) in the form of a convex combination of two conjugated constrained Gibbs measures as follows:

$$\mu^\eta_\Lambda(\sigma) = \Big[1 + \frac{\mathcal{Z}^{-,\eta}_\Lambda}{\mathcal{Z}^{+,\eta}_\Lambda}\Big]^{-1} \nu^{+,\eta}_\Lambda(\sigma) + \Big[1 + \frac{\mathcal{Z}^{+,\eta}_\Lambda}{\mathcal{Z}^{-,\eta}_\Lambda}\Big]^{-1} \nu^{-,\eta}_\Lambda(\sigma) \tag{4.15}$$

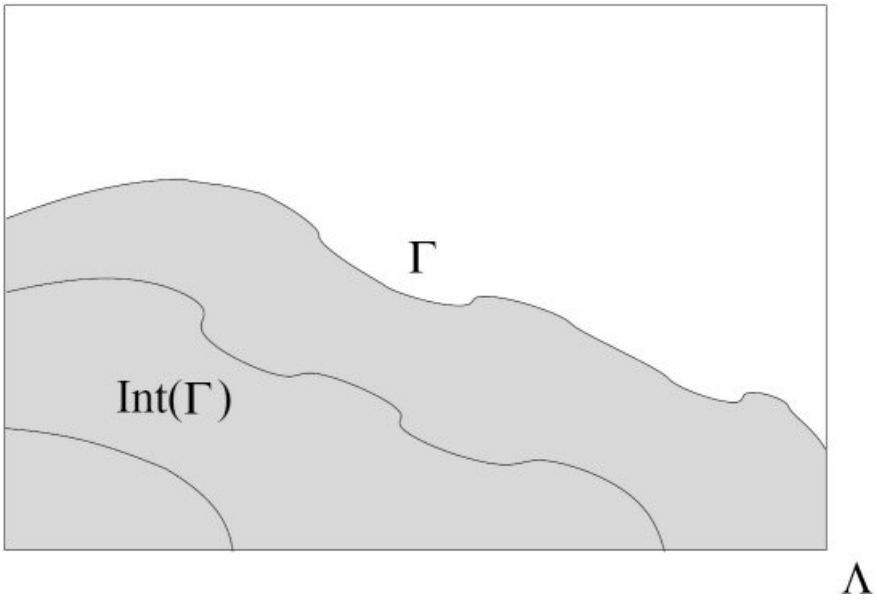


Figure 4.5: *Large boundary contour.*

where we have introduced the Gibbs measure constrained to $\Omega_\Lambda^\pm$ by

$$\nu_\Lambda^{\pm,\eta}(\sigma) = \frac{1}{\mathcal{Z}_\Lambda^{\pm,\eta}} \exp[-H_\Lambda^\eta(\sigma)]\, \mathbf{1}_{\{\sigma\in\Omega_\Lambda^\pm\}} \tag{4.16}$$

Moreover, for any $\sigma \in \Omega_\Lambda^\pm$, the Hamiltonian can be written as

$$H_\Lambda^\eta(\sigma) = E_\Lambda^{\pm,\eta}(\partial) + 2\beta \sum_{\Gamma\in\partial} |\Gamma| \tag{4.17}$$

with $\partial = \mathcal{D}_\Lambda(\sigma)$, and we have introduced

$$E_\Lambda^{\pm,\eta}(\partial) = -\beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\, y\in\Lambda^c}} \sigma_x \eta_y \tag{4.18}$$

Finally, $\mathcal{Z}_\Lambda^{\pm,\eta}$ is essentially the partition function of a polymer model [50], see also Section 4.14,

$$\mathcal{Z}_\Lambda^{\pm,\eta} = \exp\left(-E_\Lambda^{\pm,\eta}(\emptyset)\right) \sum_{\partial\in\mathscr{D}_\Lambda} \prod_{\Gamma\in\partial} \rho^{\pm,\eta}(\Gamma) \tag{4.19}$$

where the polymers coincide with the contours and the polymer weights are defined by

$$\rho^{\pm,\eta}(\Gamma) = \exp\left(-2\beta|\Gamma|\right) \exp\left(-E^{\pm,\eta}(\Gamma) + E^{\pm,\eta}(\emptyset)\right) \tag{4.20}$$

By the spin-flip symmetry, we can confine ourselves to the '+' case and use the shorthand notations $\rho^{\eta}(\Gamma) := \rho^{+,\eta}(\Gamma) = \rho^{-,-\eta}(\Gamma)$, $\mathcal{Z}^{\eta}_{\Lambda} := \mathcal{Z}^{+,\eta}_{\Lambda} = \mathcal{Z}^{-,-\eta}_{\Lambda}$, $E^{\eta}_{\Lambda} := E^{+,\eta}_{\Lambda} = E^{-,-\eta}_{\Lambda}$, and $\nu^{\eta}_{\Lambda}(\sigma) := \nu^{+,\eta}_{\Lambda}(\sigma) = \nu^{-,-\eta}_{\Lambda}(-\sigma)$. Moreover, the boundary $\partial\Gamma$ of a contour Γ has a natural decomposition into components as follows. Let $\sigma \in \Omega^{+}_{\Lambda}$ be such that $\mathcal{D}_{\Lambda}(\sigma) = \{\Gamma\}$. Then the '$\pm$' boundary component $\partial\Gamma^{\pm}$ is defined as the set of all dual bonds $\langle x, y\rangle^{*}$ such that $x \in \Lambda$, $y \in \Lambda^{c}$, $\sigma_x = \pm 1$. With this definition, the contour weight (4.20) is

$$\rho^{\eta}(\Gamma) = \exp\Big[-2\beta\big(|\Gamma| + \sum_{x \in \underline{\partial\Gamma^{-}}} \eta_x\big)\Big] \tag{4.21}$$

Using the representation (4.15) of the finite-volume Gibbs measure μ^{η}_{Λ}, the strategy of our proof consists of two main parts:

1. To prove that the constrained (random) Gibbs measure ν^{η}_{Λ} asymptotically coincides with the Ising '+' phase, for almost all η.

2. To show that a sufficiently sparse subsequence of the sequence of random free energy differences $\log \mathcal{Z}^{\eta}_{\Lambda} - \log \mathcal{Z}^{-\eta}_{\Lambda}$ has $+\infty$ and $-\infty$ as the only limit points, for almost all η.

Then, Proposition 4.5 follows almost immediately. Moreover, we will show that for a $\boldsymbol{P}$-typical boundary condition η and a μ^{η}_{Λ}-typical configuration $\sigma \in \Omega_{\Lambda}$, the corresponding set of pre-contours $\tilde{\mathcal{D}}_{\Lambda}(\sigma)$ contains no interfaces.

Theorem 4.7. *There is β_3 such that for any $\beta \geq \beta_3$ one has*

$$\lim_{N\uparrow\infty} \mu^{\eta}_{\Lambda(N)}\{\tilde{\mathcal{D}}_{\Lambda(N)}(\sigma) \textit{ contains an interface}\} = 0 \tag{4.22}$$

for $\boldsymbol{P}$-a.e. $\eta \in \Omega$.

Remark 4.8. *Note that the low-temperature result by Gallavotti [36], extended to all subcritical temperatures in [1, 42], about the absence of translationally non-invariant Gibbs measures in the 2d Ising model does not exclude fluctuating interfaces under a suitably arranged ('Dobrushin-like') boundary condition. On the other hand, the above theorem claims that a* typical *boundary condition gives rise to a Gibbs measure in which interfaces* anywhere *are suppressed. We mention this side-result to demonstrate the robustness of the presented multi-scale approach and to argue that it is essentially dimension-independent, the $d = 2$ case being chosen only for simplicity.*

It is easy to realize that, for a typical η, the polymer model (4.19) fails the 'diluteness' condition on the sufficient exponential decay of the polymer weights, which means one cannot directly apply the familiar formalism of cluster expansions. These violations of the diluteness condition occur locally along the boundary with low probability, and hence have typically low densities. Nevertheless, their presence on all scales forces a sequential, multi-scale, treatment. Multi-scale methods have been employed at various occasions, such as for one-phase models in the presence of Griffiths singularities or for the random field Ising model [13, 14, 22, 31, 35, 45]. In contrast to the usual case of cluster expansions one does not obtain analyticity (which may not even be valid). In our approach, we loosely follow the ideas of Fröhlich and Imbrie [35]. For other recent work developing their ideas, see [6, 7].

4.5 Cluster expansion of balanced contours

In this section we perform the zeroth step of the multi-scale analysis for the polymer model (4.19), and set up the cluster expansion for a class of contours the weight of which is sufficiently damped. As a result, an interacting polymer model is obtained that will be dealt with in the next section.

Let an integer l_0 be fixed. It is supposed to be large enough and the precise conditions will be specified throughout the sequel. It plays the role of an η-independent 'cut-off scale'. Given any boundary condition η (fixed throughout this section), we start by defining the set of contours that allow for the cluster expansion. Obviously, every bulk contour Γ has the weight $\rho^\eta(\Gamma) = \exp(-2\beta|\Gamma|)$. For boundary contours, there is no such exponential bound with a strictly positive rate, uniformly in η. Instead, we segregate an η-dependent subset of sufficiently damped boundary contours as follows.

Definition 4.9. *Given $\eta \in \Omega$, a boundary contour Γ is called balanced (or η-balanced) whenever*

$$\sum_{x \in \underline{\partial^- \Gamma}} \eta_x \geq -\big(1 - \frac{1}{l_0}\big)|\Gamma| \tag{4.23}$$

Otherwise Γ is called unbalanced.
A set $B \subset \partial\Lambda$ is called unbalanced if there exists an unbalanced contour Γ, $\partial^-\Gamma = B$.

While the case of large boundary contours will be discussed separately in the next section, some basic properties of unbalanced small boundary contours are collected in the following lemma. We define the *height* of any simple boundary contour Γ as

$$h(\Gamma) = \max_{y^* \in \Gamma} d[y^*, \partial\Gamma] \tag{4.24}$$

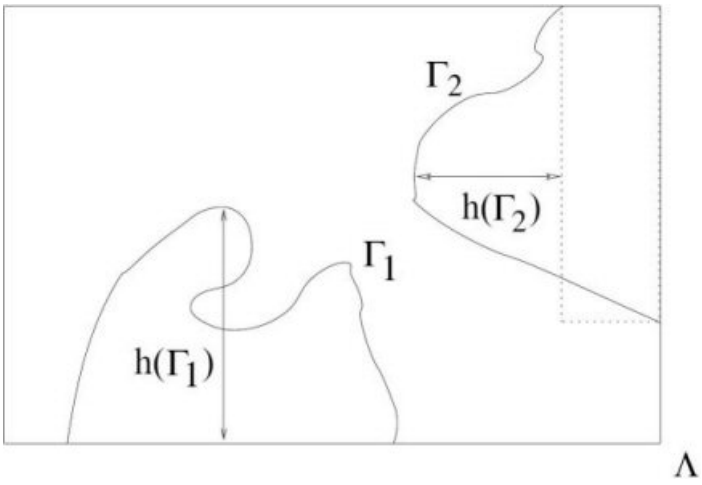


Figure 4.6: *Height of small boundary contours.*

In order to extend this definition to small boundary contours Γ such that $\partial\Gamma$ contains an (exactly one) corner, we make the following construction. If $\partial\Gamma$ is a connected subset of the boundary with the endpoints $[\pm(N+1/2), a]$ and $[b, \pm(N+1/2)]$, then we define the set $R(\Gamma) \subset (\mathbb{Z}^2)^*$ as the (unique) rectangle such that $[\pm(N+1/2), a]$, $[b, \pm(N+1/2)]$, and $[\pm(N+1/2), \pm(N+1/2)]$ are three of its corners. Now the height is the maximal distance of a point in the contour to this rectangle,

$$h(\Gamma) = \max_{y^* \in \Gamma} d[y^*, R(\Gamma)] \tag{4.25}$$

The situation is illustrated in Figure 4.6.

Lemma 4.10. *Let Γ be an unbalanced small boundary contour. Then,*

i) $\sum_{x \in \underline{\partial\Gamma}} \eta_x \leq -\big(1 - \frac{2}{l_0}\big)|\partial\Gamma|$.

ii) $|\partial\Gamma| \geq l_0 h(\Gamma)$. *In particular, if Γ is simple then $|\partial\Gamma| \geq l_0$.*

Proof. For any unbalanced contour Γ, Definition 4.9 together with the bound $|\Gamma| \geq |\partial\Gamma|$ valid for any small boundary contour implies the inequalities

$$-|\partial^-\Gamma| \leq \sum_{x \in \underline{\partial^-\Gamma}} \eta_x < -\big(1 - \frac{1}{l_0}\big)|\Gamma| \leq -\big(1 - \frac{1}{l_0}\big)|\partial\Gamma| \tag{4.26}$$

Hence, $|\partial^+\Gamma| \leq \frac{1}{l_0}|\partial\Gamma|$ and we obtain

$$\sum_{x\in\underline{\partial\Gamma}} \eta_x \leq -\big(1-\frac{1}{l_0}\big)|\partial\Gamma| + |\partial^+\Gamma| \leq -\big(1-\frac{2}{l_0}\big)|\partial\Gamma| \tag{4.27}$$

proving i).

If Γ is simple, then we use (4.26) again together with the refined relation $|\Gamma| \geq |\partial\Gamma| + 2h(\Gamma)$ to get

$$|\partial\Gamma| \geq \big(1-\frac{1}{l_0}\big)|\Gamma| \geq \big(1-\frac{1}{l_0}\big)(|\partial\Gamma| + 2h(\Gamma)) \tag{4.28}$$

which implies $|\partial\Gamma| \geq l_0 h(\Gamma) \geq l_0$, assuming $l_0 \geq 2$ and using that $h(\Gamma) \geq 1$ for any simple small boundary contour.
Since the definition of the height is such that the inequality $|\Gamma| \geq |\partial\Gamma| + 2h(\Gamma)$ remains true as well for any small boundary contour Γ such that $\partial\Gamma$ contains a corner, the lemma is proven. □

The union of the set of all bulk contours and of the set of all balanced boundary contours is denoted by $\mathcal{K}_0^\eta$. We also write $\mathscr{D}_0^\eta$ for the set of all compatible families of contours from $\mathcal{K}_0^\eta$, and $\mathscr{D}_{>0}^\eta$ for the set of all compatible families of contours from $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$. Later we will show that, for almost every η, all large boundary contours (i.e. those containing at least one interface) are balanced for all but finitely many squares $\Lambda(N)$.

Formally, the partition function (4.19) can be partially computed by summing over all contours from the set $\mathcal{K}_0^\eta$. We start by rewriting partition function (4.19) as

$$\mathcal{Z}_\Lambda^\eta = \exp\left(-E^\eta(\emptyset)\right) \sum_{\partial\in\mathscr{D}_{>0}^\eta} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \sum_{\substack{\partial^0\in\mathscr{D}_0^\eta \\ \partial^0\sim\partial}} \prod_{\Gamma^0\in\partial^0} \rho^\eta(\Gamma^0) \tag{4.29}$$

Here, the first sum runs over all compatible families ∂ of contours not belonging to $\mathcal{K}_0^\eta$, while the second one is over all collections of contours from $\mathcal{K}_0^\eta$, compatible with ∂. Let $\mathfrak{C}_\Lambda^0$ denote the set of all clusters of contours from $\mathcal{K}_0^\eta$. Then, the cluster expansion reads, see Section 4.14,

$$\mathcal{Z}_\Lambda^\eta = \exp\left(-E^\eta(\emptyset)\right) \sum_{\partial\in\mathscr{D}_{>0}^\eta} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \exp\Big(\sum_{\substack{C\in\mathfrak{C}_0^\eta \\ C\sim\partial}} \phi_0^\eta(C)\Big) \tag{4.30}$$

where the sum runs over all clusters of contours from $\mathcal{K}_0^\eta$ that are compatible with ∂, and we have denoted the weight of a cluster C by $\phi_0^\eta(C)$. Note that the cluster expansion

was applied only formally here and it needs to be justified by providing bounds on the cluster weights. This is done in Proposition 4.12 below.

Hence, we rewrite the model with the partition function $\mathcal{Z}_\Lambda^\eta$ as an effective model upon the contour ensemble $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$, with a contour interaction mediated by the clusters:

$$\mathcal{Z}_\Lambda^\eta = \mathcal{Z}_1^\eta \, \exp\Big(-E^\eta(\emptyset) + \sum_{C \in \mathfrak{C}_0^\eta} \phi_0^\eta(C)\Big) \tag{4.31}$$

where

$$\mathcal{Z}_1^\eta = \sum_{\partial \in \mathscr{D}_{>0}^\eta} \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_0^\eta \\ C \not\sim \partial}} \phi_0^\eta(C)\Big) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{4.32}$$

After establishing an exponential upper bound on the number of incompatible contours in the next lemma, a bound on the cluster weights immediately follows by recalling the basic result on the convergence of the cluster expansions [50].

Lemma 4.11. *There exists a constant $c_1 > 0$ (independent of l_0) such that the number of all contours $\Gamma' \in \mathcal{K}_\Lambda$, $|\Gamma'| = n$, $\Gamma' \not\sim \Gamma$ is upper-bounded by $|\Gamma| \, e^{c_1 n}$, for any $\Gamma \in \mathcal{K}_\Lambda$ and $n = 1, 2, \dots$*

Proof. Note that Γ is not necessarily a connected set. However, the relation $\Gamma' \nsim \Gamma$ implies $(\Gamma' \cup \partial\Gamma') \cap (\Gamma \cup \partial\Gamma) \neq \emptyset$, and using that $\Gamma \cup \partial\Gamma$ is connected, we get:

$$\begin{aligned}
\#\{\Gamma' : \Gamma' \nsim \Gamma, \, |\Gamma'| = n\} &\leq |\Gamma \cup \partial\Gamma| \sup_{x^*} \#\{\Gamma' : \, x^* \in \Gamma' \cup \partial\Gamma', \, |\Gamma'| = n\} \\
&\leq 3|\Gamma| \sup_{x^*} \#\{A \subset (\mathbb{Z}^2)^* \text{ connected}, \, x^* \in A, \, |A| \leq 3n\} \\
&\leq |\Gamma| \cdot 4^{6n+1} \leq |\Gamma| \, e^{c_1 n}
\end{aligned}$$

by choosing c_1 large enough. □

Assigning to any cluster $C \in \mathfrak{C}_0^\eta$ the *domain* $\mathrm{Dom}(C) = \underline{\partial C}$ where $\partial C = \cup_{\Gamma \in C} \partial\Gamma$ is the boundary of C, and the length $|C| = \sum_{\Gamma \in C} |\Gamma|$, we have the following result.

Proposition 4.12. *There are constants $\beta_4, c_2 > 0$ (independent of l_0) such that for any $\beta \geq l_0 \beta_4$, one has the upper bound*

$$\sup_{x^*} \sum_{\substack{C \in \mathfrak{C}_0^\eta \\ x^* \in C}} |\phi_0^\eta| \, \exp\Big[\big(\frac{2\beta}{l_0} - c_2\big)|C|\Big] \leq 1 \tag{4.33}$$

uniformly in Λ.
Moreover, $\phi_0^\eta(C)$ only depends on the restriction of η to the set $\mathrm{Dom}(C)$.

Proof. Using Definition 4.9 and equation (4.21), we have $\rho^\eta(\Gamma) \le \exp(-\frac{2\beta}{l_0}|\Gamma|)$ for any balanced contour Γ. In combination with Lemma 4.11, we get

$$\sum_{\substack{\Gamma\in\mathcal{K}_0^\eta \\ x^*\in\Gamma}} |\rho^\eta(\Gamma)| \, \exp\big[(\frac{2\beta}{l_0} - c_2 + 1)|\Gamma|\big] \le \sum_{n=1}^{\infty} \exp[-(c_2 - c_1 - 1)n] \le 1 \tag{4.34}$$

provided that c_2 is chosen large enough. The proposition now follows by applying Proposition 4.50, with $\beta_4 = \frac{c_2}{2}$. □

4.6 Absence of large boundary contours

By the construction, all unbalanced contours are boundary contours, either small or large. In this section we show that unbalanced large boundary contours actually do not exist under a typical realization of the boundary condition. This observation will allow us to restrict our multi-scale analysis entirely to the class of small boundary contours.

Lemma 4.13. *There is a constant $c_3 > 0$ such that for any $N \in \mathbb{N}$ and any unbalanced large boundary contour $\Gamma \in \mathcal{K}_{\Lambda(N)}$, the inequality*

$$\sum_{x\in\underline{\partial\Gamma}} \eta_x \le -c_3 N \tag{4.35}$$

holds true.

Proof. Using the geometrical inequality $|\Gamma| \ge 2N + |\partial^+\Gamma|$ and Definition 4.9, we have

$$\begin{aligned} \sum_{x\in\underline{\partial\Gamma}} \eta_x &\le -\big(1 - \frac{1}{l_0}\big)|\Gamma| + \sum_{x\in\underline{\partial^+\Gamma}} \eta_x \\ &\le -\big(1 - \frac{1}{l_0}\big)(2N + |\partial^+\Gamma|) + |\partial^+\Gamma| \\ &\le -2N\big(1 - \frac{3}{l_0}\big) \end{aligned} \tag{4.36}$$

where in the last inequality we used that $|\partial^+\Gamma| \le |\partial\Gamma| \le 4N$. □

Proposition 4.14. *There is a constant $c_4 > 0$ such that for any $N \in \mathbb{N}$,*

$$\boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \textit{ large unbalanced}\} \le \exp(-c_4 N) \tag{4.37}$$

Proof. If $B \subset \partial\Lambda(N)$ is a connected set containing exactly two corners, then, using Lemma 4.13,

$$\begin{aligned} \boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced} : \partial\Gamma = B\} &\leq \boldsymbol{P}\big\{\sum_{x\in\underline{B}} \eta_x \leq -c_3 N\big\} \\ &\leq \boldsymbol{P}\big\{\sum_{x\in\underline{B}} \eta_x \leq -\frac{c_3}{2}|B|\big\} \leq \exp\big(-\frac{c_3^2}{8}|B|\big) \end{aligned} \tag{4.38}$$

Hence,

$$\begin{aligned} &\boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} \\ &\quad\leq \sum_{B\subset\partial\Lambda} \boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced} : \partial\Gamma = B\} \\ &\quad\leq \sum_{l\geq 2N} 8N \exp\big(-\frac{c_3^2}{8} l\big) \leq \frac{128N}{c_3^2} \exp\big(-\frac{c_3^2 N}{4}\big) \leq \exp(-c_4 N) \end{aligned} \tag{4.39}$$

for N large enough and an appropriate c_4. Choosing c_4 small enough gives (4.37) for all N. □

Corollary 4.15. *There exist a set $\Omega^* \subset \Omega$, $\boldsymbol{P}\{\Omega^*\} = 1$ and a function $N^* : \Omega^* \mapsto \mathbb{N}$ such that for any b.c. $\eta \in \Omega^*$ and any volume $\Lambda = \Lambda(N)$, $N \geq N^*(\eta)$, all large boundary contours are balanced.*

Proof. Since

$$\sum_N \boldsymbol{P}\{\exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} < \infty \tag{4.40}$$

the Borel-Cantelli lemma implies

$$\boldsymbol{P}\{\forall N_0 \in \mathbb{N} : \exists N \geq N_0 : \exists \Gamma \in \mathcal{K}_{\Lambda(N)} \text{ large unbalanced}\} = 0 \tag{4.41}$$

proving the statement. □

We are now ready to prove the almost sure absence of interfaces in the large-volume limit.

Proof of Theorem 4.7. Let $\eta \in \Omega^* \cap (-\Omega^*)$ and $N \geq N^*(\eta)$. Then, any large boundary contour Γ is both η- and $(-\eta)$-balanced and, using the Peierls inequality and (4.15), the

Gibbs probability of any collection of (possibly large boundary) contours $\Gamma_1, \dots, \Gamma_m$, $m = 1, 2, \dots$ has the upper bound

$$\begin{aligned} \mu^{\eta}_{\Lambda(N)}(\Gamma_1, \dots, \Gamma_m) &\leq \max_{a \in \{-1,1\}} \nu^{a\eta}_{\Lambda(N)}(\Gamma_1, \dots, \Gamma_m) \\ &\leq \max_{a \in \{-1,1\}} \prod_{i=1}^{m} \rho^{a\eta}(\Gamma_i) \leq \exp\bigl(-\frac{2\beta}{l_0} \sum_{i=1}^{m} |\Gamma_i|\bigr) \end{aligned} \tag{4.42}$$

Hence, using Lemma 4.11 and the bound $|\Gamma| \geq 2N$ for any large boundary contour Γ, we get

$$\begin{aligned} \mu^{\eta}_{\Lambda(N)}(\exists \text{ a large boundary contour}) &\leq \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{\Gamma_1, \dots, \Gamma_m \text{ large} \\ \forall i : \Gamma_i \cap \partial\Lambda \neq \emptyset}} \mu^{\eta}_{\Lambda(N)}(\Gamma_1, \dots, \Gamma_m) \\ &\leq \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{x_1, \dots, x_m \in \partial\Lambda} \sum_{\Gamma_1 \ni x_1, \dots, \Gamma_m \ni x_m} \exp\bigl(-\frac{2\beta}{l_0} \sum_{i=1}^{m} |\Gamma_i|\bigr) \\ &\leq \exp\bigl(-\frac{2\beta}{l_0} N\bigr) \sum_{m=1}^{\infty} \frac{1}{m!} \bigl(4N \sum_{\substack{\Gamma \ni x \\ |\Gamma| \geq 2N}} e^{-\frac{\beta}{l_0}|\Gamma|}\bigr)^m \\ &\leq \exp\bigl[-\bigl(\frac{2\beta}{l_0} - 8e^{-2(\frac{\beta}{l_0} - c_1)N}\bigr) N\bigr] \longrightarrow 0 \end{aligned} \tag{4.43}$$

provided that β is large enough. Since $\boldsymbol{P}\{\Omega^* \cap (-\Omega^*)\} = 1$, the theorem is proven. □

As a consequence, all interfaces get $\boldsymbol{P}$-a.s. and for all but finitely many volumes uniformly exponentially damped weights. Hence, their Gibbs probabilities become exponentially small as functions of the size of the system and, therefore, no interfacial infinite-volume Gibbs measure occurs as a limit point, with $\boldsymbol{P}$-probability 1. While such a result is not sensational in $d = 2$ (in this case, no translationally non-invariant Gibbs measure exists by [1, 42]), similar arguments are expected to apply in higher dimensions.

In the next sections, a perturbation technique is developed that allows us to address the question whether non-trivial mixtures of μ^+ and μ^- can occur as limit measures.

4.7 Classification of unbalanced contours

We now consider the interacting contour model introduced by the partition function (4.32), defined on the set of unbalanced contours $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$. As a consequence of Corollary 4.15, we can restrict our analysis to the set Ω^* of boundary conditions under which

the set $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ of unbalanced contours contains only small boundary contours, both simple and corner ones.

Our multi-scale analysis consists of a sequential expansion of groups of unbalanced contours that are far enough from each other. The groups are supposed to be typically sufficiently rarely distributed, so that the partition function (4.31) can be expanded around the product over the partition functions computed within these groups only. Under the condition that the density of the groups decays fast enough with their space extension, one can arrive at an expansion that essentially shares the locality features of the usual cluster expansion, at least for $\boldsymbol{P}$-typical boundary conditions η. To make this strategy work, we define a suitable decomposition of the set $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ into disjoint groups associated with a hierarchy of length scales. Also, the unbalanced contours close enough to any of the four corners will be dealt with differently and expanded in the end.

Definition 4.16. *Assuming l_0 to be fixed, we define the two sequences $(l_n)_{n=1,2,\dots}$ and $(L_n)_{n=1,2,\dots}$ by the following recurrence relations:*

$$L_n = \frac{l_{n-1}}{5^n}\,, \qquad l_n = \exp\big(\frac{L_n}{2^n}\big) \qquad\qquad n = 1, 2, \dots \tag{4.44}$$

For any $n = 1, 2, \dots$, any pair of contours Γ, Γ' is called L_n-connected, if $d[\Gamma, \Gamma'] \leq L_n$. Furthermore, fixing a positive constant $\epsilon > 0$, we introduce the N-dependent length scale

$$l_\infty = (\log N)^{1+\epsilon} \tag{4.45}$$

Introducing the boundary $\partial\Delta$ for any set of contours $\Delta \subset \mathcal{K}_\Lambda$ by $\partial\Delta = \cup_{\Gamma\in\Delta}\partial\Gamma$, we consider the η-dependent decomposition of the set of contours $\mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ defined by induction as follows.

Definition 4.17. *1) A maximal L_1-connected subset $\Delta \subset \mathcal{K}_\Lambda \setminus \mathcal{K}_0^\eta$ is called a* 1-aggregate *whenever i) $|\partial\Delta|_{con} \leq l_1$, ii) there is no corner $x^*_{C,i}$ such that $\max_{y^*\in\partial\Delta} d[y^*, x^*_{C,i}] \leq l_\infty$. We use the notation $(\mathcal{K}^\eta_{1,\alpha})$ for the collection of all 1-aggregates, and write $\mathcal{K}_1^\eta = \cup_\alpha \mathcal{K}^\eta_{1,\alpha}$.*

$\vdots$

n) Assume the sets $(\mathcal{K}^\eta_{j,\alpha})_{j=1,\dots,n-1}$ have been defined. Then, the n-aggregates are defined as maximal L_n-connected subsets $\Delta \subset \mathcal{K}_\Lambda \setminus \cup_{j<n}\mathcal{K}_j^\eta$ satisfying i) $|\partial\Delta|_{con} \leq l_n$, ii) there is no corner $x^*_{C,i}$ such that $\max_{y^*\in\partial\Delta} d[y^*, x^*_{C,i}] \leq l_\infty$. The set of all n-aggregates is denoted by $(\mathcal{K}^\eta_{n,\alpha})$, and $\mathcal{K}_n^\eta = \cup_\alpha \mathcal{K}^\eta_{n,\alpha}$.

To each n-aggregate $\mathcal{K}^\eta_{n,\alpha}$ we assign the domain

$$\mathrm{Dom}(\mathcal{K}^\eta_{n,\alpha}) := \underline{\{x^* \in \partial\Lambda;\, d[x^*, \partial\mathcal{K}^\eta_{n,\alpha}] \leq L_n\}} \tag{4.46}$$

Obviously, the set $\mathcal{K}_\infty^\eta := \mathcal{K}_\Lambda \setminus (\mathcal{K}_1^\eta \cup \mathcal{K}_2^\eta \cup \ldots)$ need not be empty, and since all large boundary contours are balanced, for every contour $\Gamma \in \mathcal{K}_\infty^\eta$ there is exactly one corner $x_{C,i}^*$ such that $\max_{y^* \in \partial\Gamma} d[y^*, x_{C,i}^*] \leq l_\infty$. Hence, there is a natural decomposition of the set $\mathcal{K}_\infty^\eta$ into at most four *corner aggregates*, $\mathcal{K}_\infty^\eta = \cup_i \mathcal{K}_{\infty,i}^\eta$, each of them consisting of contours within the logarithmic neighborhood of one of the corners. In general, any corner aggregate contains both simple and corner boundary contours. Later we will show that with $\boldsymbol{P}$-probability 1, every unbalanced corner boundary contour belongs to a corner aggregate. In other words, every n-aggregate, $n = 1, 2, \ldots$ contains only simple boundary contours.

Remark 4.18. *By Definition 4.17, any n-aggregate has a distance at least L_n from all m-aggregates, $m \geq n$. In this way, in the n-th step of our expansion, after having removed all lower-order aggregates, we will be able to use the 'essential independence' of all n-aggregates. Namely, on the assumption that L_n is big enough, depending on the aggregate size l_n, both the interaction among the n-aggregates and the interaction between n-aggregates and m-aggregates, $m \geq n$ will be controlled by a cluster expansion.*

Our first observation is a local property of the above construction, which will be crucial to keep the dependence of expansion terms to be defined later depending only on a sufficiently small set of boundary spins.

Lemma 4.19. *Let a set of small boundary contours Δ be fixed and assume that $\eta, \eta' \in \Omega$ are such that $\eta_{\mathrm{Dom}(\Delta)} = \eta'_{\mathrm{Dom}(\Delta)}$. Then, Δ is an n-aggregate w.r.t. the boundary condition η if and only if it is an n-aggregate w.r.t. η'.*

The super-exponential growth of the scales l_n will imply an exponential decay of the probability for an n-aggregate to occur. An upper bound on this probability is stated in the following proposition, the proof of which is given in Section 4.11.1.

Proposition 4.20. *There is a constant $c_5 > 0$ (independent of l_0) such that for any $n = 1, 2, \ldots$ and any connected set $B \subset \partial\Lambda$,*

$$\boldsymbol{P}\{\exists \mathcal{K}_{n,\alpha}^\eta : \mathrm{Con}(\partial\mathcal{K}_{n,\alpha}^\eta) = B\} \leq e^{-c_5|B|} \tag{4.47}$$

uniformly in Λ.

Note that, given a connected set $B \subset \partial\Lambda$, there is at most one aggregate $\mathcal{K}_{n,\alpha}^\eta$, $n = 1, 2, \ldots$ such that $\mathrm{Con}(\partial\mathcal{K}_{n,\alpha}^\eta) = B$.

Corollary 4.21. *There exists $\Omega^{**} \subset \Omega^*$, $\boldsymbol{P}\{\Omega^{**}\} = 1$ and $N^{**} : \omega^{**} \mapsto \mathbb{N}$, $N^{**}(\omega) \geq N^*(\omega)$ such that for any $\omega \in \Omega^{**}$ and any $\Lambda = \Lambda(N)$, $N \geq N^{**}(\omega)$ every aggregate $\mathcal{K}_{n,\alpha}^\eta$, $n = 1, 2, \ldots$ satisfies the inequality $|\partial\mathcal{K}_{n,\alpha}^\eta|_{con} \leq l_\infty$. In particular:*

i) The set $\mathrm{Con}(\mathcal{K}^\eta_{n,\alpha})$ *is a boundary interval and there is at most one corner* $x^*_{C,i}$ *such that* $d[x^*_{C,i}, \partial\mathcal{K}^\eta_{n,\alpha}] \le l_\infty$.

ii) All contours $\Gamma \in \mathcal{K}^\eta_{n,\alpha}$ *are simple boundary contours.*

Proof. Using Proposition 4.20, the probability for any aggregate to occur can be estimated as

$$\begin{aligned}\boldsymbol{P}\{\exists \mathcal{K}^\eta_{n,\alpha},\, n=1,2,\dots:\, |\partial\mathcal{K}^\eta_{n,\alpha}|_{\mathrm{con}} > l_\infty\} &\le \sum_{\substack{B\subset\partial\Lambda \text{ conn.}\\ |B|>(\log N)^{1+\epsilon}}} \boldsymbol{P}\{\exists \mathcal{K}^\eta_{n,\alpha} :\, \mathrm{Con}(\mathcal{K}^\eta_{n,\alpha}) = B\} \\ &\le |\partial\Lambda| \sum_{l>(\log N)^{1+\epsilon}} e^{-c_5 l} \le \frac{16}{c_5} N^{1-c_5(\log N)^\epsilon} = o\big(N^{-\delta}\big)\end{aligned} \tag{4.48}$$

for any (arbitrarily large) $\delta > 0$. Hence,

$$\sum_{N=1}^{\infty} \boldsymbol{P}\{\exists \mathcal{K}^\eta_{n,\alpha},\, n=1,2,\dots:\, |\partial\mathcal{K}^\eta_{n,\alpha}|_{\mathrm{con}} > l_\infty\} < \infty \tag{4.49}$$

and the statement follows by a Borel-Cantelli argument. □

For convenience, let us summarize the results of the last three sections by reviewing all types of contours again together with their balancedness properties. For any $\eta \in \Omega^{**}$ and $\Lambda = \Lambda(N)$, $N \ge N^{**}(\omega)$, any configuration of contours $\partial \in \mathscr{D}_\Lambda$ possibly contains

i) *Bulk contours* (trivially balanced).

ii) *Large boundary contours* that are balanced.

iii) *Corner boundary contours* that are either balanced or elements of corner aggregates.

iv) *Simple boundary contours* which are balanced or elements of either n-aggregates, $n = 1, 2, \dots$, or of corner aggregates.

4.8 Sequential expansion of unbalanced contours

The next step in our strategy is to proceed by induction in the order of aggregates, rewriting at each step the interacting polymer model (4.32) as an effective model over the contour ensembles $\mathcal{K}_\Lambda \setminus (\mathcal{K}^\eta_0 \cup \mathcal{K}^\eta_1)$, $\mathcal{K}_\Lambda \setminus (\mathcal{K}^\eta_0 \cup \mathcal{K}^\eta_1 \cup \mathcal{K}^\eta_2)$, etc. At the n-th step, a compatible set of contours inside all corner and all normal m-aggregates, $m > n$, is

fixed, and we perform the summation over contours in all normal n-aggregates. This is a constrained partition function which is approximately a product over the normal n-aggregates. By the construction, the latter are sufficiently isolated on the scale L_n, which will allow for the control of the remaining interaction by means of a cluster expansion. At the end, we arrive at an effective model over the contour ensemble $\mathcal{K}^\eta_\infty$, which is the union of (at most four) corner aggregates. In large volumes, the corner aggregates become essentially independent, the error being exponentially small in the size of the volume. The reason we distinguish between the n-aggregates and the corner aggregates is that the partition function within the former allows for a much better control, which will be essential in our analysis of the characteristic function of the random free energy difference $\log \mathcal{Z}^\eta_\Lambda - \log \mathcal{Z}^{-\eta}_\Lambda$ in Section 4.10. Note that the lack of detailed control around the corners is to be expected as there may more easily occur some low-energy (unbalanced) boundary contours, but at most of logarithmic size in N.

The n-th step of the expansion, $n \geq 1$, starts from the partition function,

$$\mathcal{Z}^\eta_n = \sum_{\partial \in \mathscr{D}^\eta_{>n-1}} \exp\Big(- \sum_{\substack{C \in \mathfrak{C}^\eta_{n-1} \\ C \nsim \partial}} \phi^\eta_{n-1}(C) \Big) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{4.50}$$

which in the case $n = 1$ coincides with (4.32). Here, $\mathscr{D}^\eta_{>n-1}$ is the set of all compatible families of contours from $\mathcal{K}^\eta_{>n-1} := \mathcal{K}^\eta_\Lambda \setminus (\mathcal{K}^\eta_0 \cup \mathcal{K}^\eta_1 \cup \ldots \cup \mathcal{K}^\eta_{n-1})$, i.e. with all normal m-aggregates, $m \leq n-1$, being removed. Furthermore, we use the notation $\mathfrak{C}^\eta_{n-1}$ for the set of all $(n-1)$-clusters. Here, the 0-clusters have been introduced in Section 4.5, and the clusters of higher order will be defined inductively in the sequel.

In order to analyze partition function (4.50), we follow the ideas of Fröhlich and Imbrie [35], however, we choose to present them in a slightly different way. Observing that, by construction, the family of aggregates compose a 'sparse set', one is tempted to approximate the partition function by a product over the aggregates and to control the error by means of a cluster expansion. However, to make this strategy work, we need to 'renormalize' suitably the contour weights. Namely, only the clusters that intersect at least two distinct aggregates generate an interaction between them, and are sufficiently damped by using the sparsity of the set of aggregates. On the other hand, the (sufficiently short) clusters intersecting a single aggregate cannot be expanded, and they modify the weights of contour configurations within the aggregate. An important feature of this procedure is that the weight of these contour configurations is kept positive. In some sense, it is this very renormalization of the weights within each aggregate that can hardly be done via a single expansion and requires an inductive approach. In what follows, we present this strategy in detail, via a number of steps.

4.8.1 Renormalization of contour weights

For any compatible set of contours $\partial \subset \mathcal{K}_n^\eta$, define the renormalized weight

$$\hat{\rho}^\eta(\partial) = \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \nsim \partial;\, |C| < L_n}} \phi_{n-1}^\eta(C)\Big) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{4.51}$$

Note that the above sum only includes the clusters of length smaller than L_n. By construction, any such cluster is incompatible with at most one n-aggregate. Hence, the renormalized weight $\hat{\rho}^\eta(\Gamma)$ factorizes over the n-aggregates and we have $\hat{\rho}^\eta(\partial) = \prod_\alpha \hat{\rho}^\eta(\partial^\alpha)$ where $\partial^\alpha = \partial \cap \mathcal{K}_{n,\alpha}^\eta$. Therefore, formula (4.50) gets the form

$$\begin{aligned} \mathcal{Z}_n^\eta &= \sum_{\partial \in \mathscr{D}_{>n}^\eta} \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \sum_{\partial^n \in \mathscr{D}_n^\eta} \hat{\rho}^\eta(\partial^n) \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ (C \nsim \partial) \vee (C \nsim \partial^n;\, |C| \geq L_n)}} \phi_{n-1}^\eta(C)\Big) \\ &= \sum_{\partial \in \mathscr{D}_{>n}} \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \nsim \partial}} \phi_{n-1}^\eta(C)\Big) \\ &\quad \times \sum_{\partial^n \in \mathscr{D}_n^\eta} \hat{\rho}^\eta(\partial^n) \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta;\, |C| \geq L_n \\ C \sim \partial;\, C \nsim \partial^n}} \phi_{n-1}^\eta(C)\Big) \end{aligned} \tag{4.52}$$

Defining the renormalized partition function $\hat{\mathcal{Z}}_{n,\alpha}^\eta$ of the contour ensemble $\mathcal{K}_{n,\alpha}^\eta$ as

$$\hat{\mathcal{Z}}_{n,\alpha}^\eta = \sum_{\partial^n \in \mathscr{D}_{n,\alpha}^\eta} \hat{\rho}^\eta(\partial^n) \tag{4.53}$$

and using the shorthand

$$\tilde{\phi}_{n-1}^\eta(C, \partial^n) = \phi_{n-1}^\eta(C) \mathbf{1}_{\{C \nsim \partial^n;\, |C| \geq L_n\}} \tag{4.54}$$

we obtain

$$\begin{aligned} \mathcal{Z}_n^\eta = &\prod_\alpha \hat{\mathcal{Z}}_{n,\alpha}^\eta \sum_{\partial \in \mathscr{D}_{>n}^\eta} \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \nsim \partial}} \phi_{n-1}^\eta(C)\Big) \\ &\times \sum_{\partial^n \in \mathscr{D}_n^\eta} \prod_\alpha \frac{\hat{\rho}^\eta(\partial^{n,\alpha})}{\hat{\mathcal{Z}}_{n,\alpha}^\eta} \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \sim \partial}} \tilde{\phi}_{n-1}^\eta(C, \partial^n)\Big) \end{aligned} \tag{4.55}$$

where $\partial^{n,\alpha} = \partial^n \cap \mathcal{K}_{n,\alpha}^\eta$ is the restriction of ∂^n to the n-aggregate $\mathcal{K}_{n,\alpha}^\eta$. In the last expression, the second sum contains the interaction between n-aggregates, to make a correction to the product over the renormalized partition functions $\hat{\mathcal{Z}}_{n,\alpha}^\eta$.

4.8.2 Cluster expansion of the interaction between n-aggregates

Now we employ a trick familiar from the theory of high-temperature (Mayer) expansions, and assign to any family $\mathcal{C} \subset \mathfrak{C}^\eta_{n-1}$ of $(n-1)$-clusters the weight

$$w^\eta_n(\mathcal{C}) = \frac{1}{\prod_\alpha \hat{\mathcal{Z}}^{n,\alpha}_\Lambda} \sum_{\partial^n \in \mathscr{D}^\eta_n} \hat{\rho}^\eta(\partial^n) \prod_{C\in\mathcal{C}} \big(e^{-\tilde{\phi}^\eta_{n-1}(C,\partial^n)} - 1\big) \tag{4.56}$$

See Figure 4.7 for an example of a family of 1-clusters that generically gets a nontrivial weight according to this construction.

Definition 4.22. *Any pair of $(n-1)$-clusters $C_1, C_2 \in \mathfrak{C}^\eta_{n-1}$ is called* n-incompatible, *$C_1 \overset{n}{\not\leftrightarrow} C_2$, whenever there exists an n-aggregate $\mathcal{K}^\eta_{n,\alpha}$ such that $C_1 \nsim \mathcal{K}^\eta_{n,\alpha}$ and $C_2 \nsim \mathcal{K}^\eta_{n,\alpha}$.*
In general, the sets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathfrak{C}^\eta_{n-1}$ are n-incompatible *if there are $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$, $C_1 \overset{n}{\not\leftrightarrow} C_2$.*

One easily checks the following properties of the weight $w^\eta_n(\mathcal{C})$.

Lemma 4.23. *For any set of $(n-1)$-clusters $\mathcal{C} \in \mathfrak{C}^\eta_{n-1}$,*

i) $\sup_\eta |w^\eta_n(\mathcal{C})| \le \prod_{C\in\mathcal{C}} (e^{|\phi^\eta_{n-1}(C)|} - 1)$.

ii) If $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ such that $\mathcal{C}_1 \overset{n}{\leftrightarrow} \mathcal{C}_2$, then $w^\eta_n(\mathcal{C}) = w^\eta_n(\mathcal{C}_1)\, w^\eta_n(\mathcal{C}_2)$.

iii) The weight $w^\eta_n(\mathcal{C})$ depends only on the restriction of η to the set $(\cup_{C\in\mathcal{C}} \mathrm{Dom}(C)) \cup (\cup'_\alpha \mathrm{Dom}(\mathcal{K}^\eta_{n,\alpha})$ where the second union is over all n-aggregates $\mathcal{K}^\eta_{n,\alpha}$ such that $\mathcal{C} \nsim \mathcal{K}^\eta_{n,\alpha}$.

In the second sum in (4.55) we recognize the partition function of a polymer model with the polymers being defined as the n-connected subsets of $\mathfrak{C}^\eta_{n-1}$, which are incompatible if and only if they are n-incompatible. Treating this polymer model by the cluster expansion, and using the symbols $\mathfrak{D}^\eta_n$ for the set of all clusters in this polymer model and $\psi^\eta_n(D)$ for the weight of a cluster $D \in \mathfrak{D}^\eta_n$, we get

$$\begin{aligned}
\mathcal{Z}^\eta_n &= \prod_\alpha \hat{\mathcal{Z}}^\eta_{n,\alpha} \sum_{\partial\in\mathscr{D}^\eta_{>n}} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \exp\Big(-\sum_{\substack{C\in\mathfrak{C}^\eta_{n-1}\\ C\nsim\partial}} \phi^\eta_{n-1}(C)\Big) \sum_{\substack{\mathcal{C}\subset\mathfrak{C}^\eta_{n-1}\\ \mathcal{C}\sim\partial}} w^\eta_n(\mathcal{C}) \\
&= \exp\Big(\sum_{D\in\mathfrak{D}^\eta_n} \psi^\eta_n(D)\Big) \prod_\alpha \hat{\mathcal{Z}}^\eta_{n,\alpha} \sum_{\partial\in\mathscr{D}^\eta_{>n}} \prod_{\Gamma\in\partial} \rho^\eta(\Gamma) \\
&\quad \times \exp\Big(-\sum_{\substack{C\in\mathfrak{C}^\eta_{n-1}\\ C\nsim\partial}} \phi^\eta_{n-1}(C) - \sum_{\substack{D\in\mathfrak{D}^\eta_n\\ D\nsim\partial}} \psi^\eta_n(D)\Big)
\end{aligned} \tag{4.57}$$

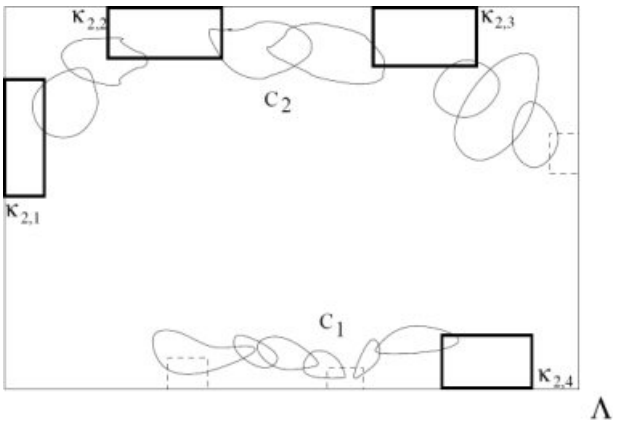


Figure 4.7: *A pair of 2-compatible families of 1-clusters $\mathcal{C}_1, \mathcal{C}_2 \subset \mathfrak{C}_1^\eta$ intersecting 2-aggregates $K_{2,\alpha}^\eta$, $\alpha = 1, 2, 3, 4$. The dashed rectangles illustrate 1-aggregates which have become parts of the 1-clusters. By construction, $w_2^\eta(\mathcal{C}_1 \cup \mathcal{C}_2) = w_2^\eta(\mathcal{C}_1)\, w_2^\eta(\mathcal{C}_2)$*

Defining the set of all n-clusters $\mathfrak{C}_n^\eta = \mathfrak{C}_{n-1}^\eta \cup \mathfrak{D}_n^\eta$ and the weight of any n-cluster $C \in \mathfrak{C}_n^\eta$ as

$$\phi_n^\eta(C) = \begin{cases} \phi_{n-1}^\eta(C) & \text{if } C \in \mathfrak{C}_{n-1}^\eta \\ \psi_n^\eta(C) & \text{if } C \in \mathfrak{D}_n^\eta \end{cases} \tag{4.58}$$

we finish the inductive step by obtaining the final expression

$$\mathcal{Z}_n^\eta = \mathcal{Z}_{n+1}^\eta \prod_\alpha \hat{\mathcal{Z}}_{n,\alpha}^\eta \, \exp\Big(\sum_{D \in \mathfrak{D}_n^\eta} \psi_n^\eta(D) \Big) \tag{4.59}$$

with the partition function of a new interacting polymer model

$$\mathcal{Z}_{n+1}^\eta = \sum_{\partial \in \mathscr{D}_{>n}^\eta} \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_n^\eta \\ C \sim \partial}} \phi_n^\eta(C) \Big) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{4.60}$$

We need to extend the notion of domain from the set of $(n-1)$-clusters $\mathfrak{C}_{n-1}^\eta$ to the set of n-clusters $\mathfrak{C}_n^\eta$. Realizing that any n-cluster $D \in \mathfrak{D}_n^\eta$ is a collection $(\mathcal{C}_i)$ of L_n-connected families of $(n-1)$-clusters, $\mathcal{C}_i = (C_i^s)$, we first introduce the domain of any such family $\mathcal{C}_i$ as $\mathrm{Dom}(\mathcal{C}_i) = \cup_s \mathrm{Dom}(C_i^s)$. Next, we define

$$\mathrm{Dom}(D) := \bigcup_i \mathrm{Dom}(\mathcal{C}_i) \cup \bigcup_{\alpha \colon \mathcal{K}_{n,\alpha}^\eta \nsim D} \mathrm{Dom}(\mathcal{K}_{n,\alpha}^\eta) \tag{4.61}$$

Furthermore, the length $|D|$ of the cluster is defined as

$$|D| := \sum_i |\mathcal{C}_i| = \sum_i \sum_s |C_i^s| \tag{4.62}$$

Note that this is possibly much smaller than the diameter of the cluster, since the sizes of the n-aggregates in the domain of D are not counted in the length of the cluster. The reason for this definition is that the cluster weights are not expected to be exponentially damped with the cluster diameter. Note, however, that the *probability* of a cluster to occur is exponentially damped with the size of the n-aggregates in its domain.

In the next proposition, we provide uniform bounds on the n-cluster weights. For the proof, see Section 4.11.

Proposition 4.24. *There is $\beta_5 > 0$ such that for any $\beta \geq l_0\beta_5$, $\eta \in \Omega^{**}$, $\Lambda = \Lambda(N)$, $N \geq N^{**}(\eta)$, the inequalities*

$$\sup_{x^*} \sum_{\substack{D \in \mathfrak{D}_n^\eta \\ x^* \in D}} \exp\big(\frac{\beta}{l_0}|D|\big)\, |\psi_n^\eta(D)| \leq 2^{-n} \qquad n = 1, 2, \dots \tag{4.63}$$

and

$$\sup_n \sup_{x^*} \sum_{\substack{C \in \mathfrak{C}_n^\eta \\ x^* \in C}} \exp\big(\frac{\beta}{l_0}|C|\big)\, |\phi_n^\eta(C)| \leq 1 \tag{4.64}$$

hold true.
Moreover, if $C \in \mathfrak{C}_n^\eta$ and $\eta'|_{\mathrm{Dom}(C)} = \eta|_{\mathrm{Dom}(C)}$, then also $C \in \mathfrak{C}_n^{\eta'}$ and $\phi_n^{\eta'}(C) = \phi_n^\eta(C)$. Similarly, $D \in \mathfrak{D}_n^\eta$ and $\eta'|_{\mathrm{Dom}(D)} = \eta|_{\mathrm{Dom}(D)}$ implies both $D \in \mathfrak{D}_n^{\eta'}$ and $\psi_n^{\eta'}(D) = \psi_n^\eta(D)$.

4.8.3 Expansion of corner aggregates

For any finite square $\Lambda = \Lambda(N)$ and $\eta \in \Omega$, all aggregates from the set $\cup_n \mathcal{K}_n^\eta$ are expanded in a finite number of steps. Afterwards, all corner aggregates are treated by a similar procedure. Throughout this section, we use the notation n_0 for the highest order in the collection of all normal aggregates. The expansion goes similarly as in the case of normal aggregates, so we only sketch it.

The renormalized weight of any compatible family of contours $\partial \subset \mathcal{K}_\infty^\eta$ is defined by the formula

$$\hat{\rho}^\eta(\partial) = \exp\Big(- \sum_{\substack{C \in \mathfrak{C}_{n_0}^\eta \\ C \nsim \partial;\, |C| < 2l_\infty}} \phi_{n_0-1}^\eta(C)\Big) \prod_{\Gamma \in \partial} \rho^\eta(\Gamma) \tag{4.65}$$

which factorizes over the corners, $\hat{\rho}^\eta(\partial) = \prod_i \hat{\rho}^\eta(\partial \cap \mathcal{K}^\eta_{\infty,i})$, assuming $\Lambda(N)$ to be large enough. Clusters $C_1, C_2 \subset \mathfrak{C}^\eta_{n_0}$ are called ∞-incompatible whenever there is a corner aggregate $\mathcal{K}^\eta_{\infty,i}$ such that $C_1 \nsim \mathcal{K}_{\infty,i}$ and $C_2 \nsim \mathcal{K}_{\infty,i}$. Defining the weight $w^\eta(C)$ as

$$w^\eta(\mathcal{C}) = \frac{1}{\prod_i \hat{\mathcal{Z}}^\eta_{\infty,i}} \sum_{\partial \in \mathscr{D}^\eta_\infty} \hat{\rho}^\eta(\partial) \prod_{C \in \mathcal{C}^\eta_{n_0}} \big(e^{-\tilde{\phi}^\eta_{n_0}(C,\partial)} - 1\big) \tag{4.66}$$

where

$$\hat{\mathcal{Z}}^\eta_{\infty,i} = \sum_{\partial \in \mathscr{D}^\eta_{\infty,i}} \hat{\rho}^\eta(\partial) \tag{4.67}$$

and

$$\tilde{\phi}^\eta_\infty(C,\partial) = \phi^\eta_\infty(C) \mathbf{1}_{\{C \nsim \partial;\, |C| \geq 2l_\infty\}} \tag{4.68}$$

an obvious variant of Lemma 4.23 holds true and $w^\eta(\mathcal{C})$ factorizes into a product over maximal connected components of $\mathcal{C}$ w.r.t. ∞-incompatibility. Treating these as polymers in a new polymer model with ∞-incompatibility used as the incompatibility relation, and using the notation $\mathfrak{D}^\eta_\infty$ for the set of all clusters in this polymer model and $\psi^\eta_\infty(D)$ for the cluster weights, we obtain as the final step of the sequential expansion,

$$\mathcal{Z}^\eta_{n_0+1} = \exp\Big(\sum_{D \in \mathfrak{D}^\eta_\infty} \psi^\eta_\infty(D) \Big) \prod_i \hat{\mathcal{Z}}^\eta_{\infty,i} \tag{4.69}$$

Proposition 4.25. *There exist constants $\beta_6 \geq \beta_5, c_6 > 0$ such that for any $\beta \geq l_0\beta_6$, $\eta \in \Omega^{**}$, and volume $\Lambda(N)$, $N \geq N^{**}(\eta)$, one has the bound*

$$\sum_{D \in \mathfrak{D}^\eta_\infty} |\psi^\eta_\infty(D)| \leq e^{-c_6 l_\infty} \tag{4.70}$$

Gathering all expansion steps, we arrive at the final expression for the partition function $\mathcal{Z}^\eta_\Lambda$ in the form

$$\begin{aligned} \log \mathcal{Z}^\eta_\Lambda = &- E^\eta(\emptyset) + \sum_{C \in \mathfrak{C}^\eta_0} \phi^\eta_0(C) + \sum_{n \geq 1} \sum_{D \in \mathfrak{D}^\eta_n} \psi^\eta_n(D) + \sum_{D \in \mathfrak{D}^\eta_\infty} \psi^\eta_\infty(D) \\ &+ \sum_i \log \hat{\mathcal{Z}}^\eta_{\infty,i} + \sum_{n \geq 1} \sum_\alpha \log \hat{\mathcal{Z}}^\eta_{n,\alpha} \end{aligned} \tag{4.71}$$

The terms collected on the first line contain the 'vacuum' energy under the boundary condition η, together with the contributions of clusters of all orders. Recall that the latter

allow for a uniform exponential upper bound. On the second line there are the partition functions of all n- and all corner aggregates. Although we can provide only rough upper bounds for these terms, a crucial property to be used is that the probability of an aggregate to occur is exponentially small in the size of its boundary, see Section 4.7. In this sense, the above expansion is a natural generalization of the familiar 'uniform' cluster expansion [50].

4.8.4 Estimates on the aggregate partition functions

In expression (4.71) we do not attempt to perform any detailed expansion of the aggregate's (log-)partition functions $\hat{\mathcal{Z}}^{\eta}_{n,\alpha}$ and $\hat{\mathcal{Z}}^{\eta}_{\infty,i}$ via a series of local and exponentially damped terms. Instead, we follow the idea that a locally ill-behaving boundary condition forces a partial coarse-graining represented above via the framework of aggregates of different orders. Although the detailed (cluster expansion-type) control within the aggregates is lost, we still can provide generic upper bounds on these partition functions. Notice a basic difference between n-aggregates and corner aggregates: The former contain only simple boundary contours the weights of which exponentially decay with the height of the contours. In some sense, the partition functions $\hat{\mathcal{Z}}^{\eta}_{n,\alpha}$ can be compared with the partition function of a 1d interface to get an upper bound. On the other hand, the corner aggregates are ensembles of contours the weight of which obey no uniform exponential bound with the space extension of the contours, and allow possibly for a non-trivial 'degeneracy of vacuum'. As a consequence, only rough (counting-type) estimates can be provided for the partition functions $\hat{\mathcal{Z}}^{\eta}_{\infty,i}$.

Lemma 4.26. *There are constants $c_7, c_7' > 0$ ($c_7 \downarrow 0$ if $\beta \to \infty$) such that for any n-aggregate $\mathcal{K}^{\eta}_{n,\alpha}$, one has the bound*

$$\log \hat{\mathcal{Z}}^{\eta}_{n,\alpha} \leq c_7 |\partial \mathcal{K}^{\eta}_{n,\alpha}| \tag{4.72}$$

For any corner aggregate $\mathcal{K}^{\eta}_{\infty,i}$,

$$\log \hat{\mathcal{Z}}^{\eta}_{\infty,i} \leq c_7' l_{\infty}^2 \tag{4.73}$$

4.9 Asymptotic triviality of the constrained Gibbs measure ν^{η}_{Λ}

As the first application of expansion (4.71) we prove that the weak limit of the constrained measure ν^{η}_{Λ} coincides with the '+' phase Gibbs measure μ^{+}, finishing the first part of our program.

Proposition 4.27. *There exists a constant $c > 0$ such that for any $\beta \geq l_0\beta_6$ (with the β_6 the same as in Proposition 4.25), any $\eta \in \Omega^{**}$, and $X \subset \mathbb{Z}^2$ finite,*

$$\|\nu^{\eta}_{\Lambda(N)} - \mu^{+}\|_X = O(e^{-cN}) \tag{4.74}$$

In particular, $\lim_{N\to\infty} \nu^{\eta}_{\Lambda(N)} = \mu^{+}$*, $\boldsymbol{P}$-a.s.*

Proof. The idea of the proof is to express the expectation $\nu^{\eta}_{\Lambda(N)}(f)$ of any local function f as the sum of a convergent series by using the multi-scale scheme developed in the last section, and to compare the series with a standard cluster expansion for $\nu^{\eta\equiv+1}_{\Lambda(N)}$. The difference between both series is given in terms of clusters both touching the boundary and the dependence set of f. Restricting only to the boundary conditions $\eta \in \Omega^{**}$ and volumes $\Lambda(N)$, $N \geq N^{**}(\eta)$ and using the exponential decay of the cluster weights, we prove the exponential convergence $\nu^{\eta}_{\Lambda(N)}(f) \to \nu^{+}(f)$.

For notational simplicity, we only restrict to a special case and give a proof of the equality

$$\lim_{\Lambda} \nu^{\eta}_{\Lambda}(\sigma_0 = -1) = \mu^{+}(\sigma_0 = -1) \tag{4.75}$$

The general case goes along the same lines.

Assuming $\sigma \in \Omega^{+}_{\Lambda}$, observe that $\sigma_0 = -1$ if and only if the set $\mathcal{D}_{\Lambda}(\sigma)$ contains an odd number of contours Γ such that $0 \in \mathrm{Int}(\Gamma)$. In an analogy with (4.19), we write the ν^{η}_{Λ}-probability that $\sigma_0 = -1$ in the form

$$\nu^{\eta}_{\Lambda}(\sigma_0 = -1) = \frac{1}{\mathcal{Z}^{\eta}_{\Lambda}} \sum_{\Delta \sqsubset \Lambda} \mathcal{Z}^{\eta}_{\Lambda}(\backslash\Delta) \prod_{\Gamma\in\Delta} \rho^{\eta}(\Gamma) \tag{4.76}$$

where we have used the shorthand $\Delta \sqsubset \Lambda$ for any compatible family of contours in Λ such that $\mathrm{card}(\Delta)$ is an odd integer and $0 \in \mathrm{Int}\,\Gamma$ for every $\Gamma \in \Delta$. Furthermore, $\mathcal{Z}^{\eta}_{\Lambda}(\backslash\Delta)$ is the partition function

$$\mathcal{Z}^{\eta}_{\Lambda}(\backslash\Delta) = \exp\left(-E^{\eta}_{\Lambda}(\emptyset)\right) \sum_{\partial\in\mathscr{D}_{\Lambda}(\backslash\Delta)} \prod_{\Gamma\in\partial} \rho^{\eta}(\Gamma) \tag{4.77}$$

of a polymer model over the restricted ensemble $\mathcal{K}_{\Lambda}(\backslash\Delta) \subset \mathcal{K}_{\Lambda}$ of all contours Γ such that i) $\Gamma \sim \Delta$, and ii) $0 \notin \mathrm{Int}(\Gamma)$. We can now repeat the same procedure as in the last sections, but with the contour ensemble $\mathcal{K}_{\Lambda}$ being replaced by $\mathcal{K}^{\eta}(\backslash\Delta)$. A crucial observation is that all contours from the set $\mathcal{K}^{\eta} \setminus \mathcal{K}^{\eta}(\backslash\Delta)$ are balanced, at least for all $\eta \in \Omega^{**}$ and provided that the volume $\Lambda(N)$ is large enough. Hence, the sets of unbalanced contours coincide for both contour ensembles $\mathcal{K}^{\eta}$ and $\mathcal{K}^{\eta}(\backslash\Delta)$, hence, the

same is true for the collections of both n- and corner aggregates. Finally, we compare the terms in the expansions for $\mathcal{Z}_\Lambda^\eta$ and $\mathcal{Z}_\Lambda^\eta(\backslash\Delta)$, and arrive at the formula

$$\log \frac{\mathcal{Z}_\Lambda^\eta(\backslash\Delta)}{\mathcal{Z}_\Lambda^\eta} = -\sum_{C\in\mathfrak{C}_0^\eta\setminus\mathfrak{C}_0^\eta(\backslash\Delta)} \phi_0^\eta(C) - \sum_{n\geq 1}\sum_{D\in\mathfrak{D}_n^\eta\setminus\mathfrak{D}_n^\eta(\backslash\Delta)} \psi_n^\eta(D) - \sum_{D\in\mathfrak{D}_\infty^\eta\setminus\mathfrak{D}_\infty^\eta(\backslash\Delta)} \psi_\infty^\eta(D) \tag{4.78}$$

where each of the three sums runs over all (0-, n-, or ∞-)clusters that are either incompatible with Δ or contain a contour Γ, $0 \in \mathrm{Int}(\Gamma)$. By construction, each n-, respectively ∞-cluster is further required to be incompatible with an n-, respectively corner aggregate, and since their weights are uniformly exponentially bounded by Propositions 4.24-4.25, we get the uniform upper bound

$$\sup_\Lambda \Big| \log \frac{\mathcal{Z}_\Lambda^\eta(\backslash\Delta)}{\mathcal{Z}_\Lambda^\eta} \Big| \leq c|\Delta| \tag{4.79}$$

with a constant c large enough, as well as the existence of the limit

$$\lim_\Lambda \log \frac{\mathcal{Z}_\Lambda^\eta(\backslash\Delta)}{\mathcal{Z}_\Lambda^\eta} = -\sum_C{}' \phi_0(C) \tag{4.80}$$

where the sum runs over all finite 0-clusters in $\mathbb{Z}^2$ that are either incompatible with Δ or contain a contour surrounding the origin.
Since every $\Gamma \in \Delta$ surrounds the origin, it is necessarily balanced and satisfies $\rho^\eta(\Gamma) \leq \exp(-\frac{2\beta}{l_0}|\Gamma|)$. Combined with (4.79)-(4.80), one easily checks that

$$\lim_\Lambda \nu_\Lambda^\eta(\sigma_0 = -1) = \sum_{\Delta\sqsubset\mathbb{Z}^2} \exp\big(-\sum_C{}' \phi_0(C)\big) \prod_{\Gamma\in\Delta} \rho(\Gamma) \tag{4.81}$$

and the convergence is exponentially fast. Obviously, the right-hand side coincides with the limit $\lim_\Lambda \mu_\Lambda^{\eta\equiv+1}(\sigma_0 = -1) = \mu^+(\sigma_0 = -1)$, which finishes the proof. □

4.10 Random free energy difference

In this section we analyze the limit behavior of the sequence of the random free energy differences

$$F_\Lambda^\eta = \log \mathcal{Z}_\Lambda^\eta - \log \mathcal{Z}_\Lambda^{-\eta} \tag{4.82}$$

In order to show that the probability that F_Λ^η takes a value in a fixed finite interval is bounded as $\mathcal{O}(N^{-\frac{1}{2}+\alpha})$ with $\alpha > 0$, we can use the local central limit upper bound

proven in Section 4.15, provided that a Gaussian-type upper bound on the characteristic functions of the random variables F_Λ^η can be established. The basic idea is to prove the latter by employing the sequential expansion for $\log \mathcal{Z}_\Lambda^\eta$ developed in section 4.8 and by computing the characteristic functions in a neighborhood of the origin via a Mayer expansion. However, a technical problem arises here due to the high probability of the presence of corner aggregates. That is why we need to split our procedure in two steps that can be described as follows.

In the first step, we fix the boundary condition in the logarithmic neighborhood of the corners and consider the random free energy difference F_Λ^η conditioned on the fixed configurations. For this conditioned quantity a Gaussian upper bound on the characteristic function can be proven, implying a bound on the probability that the conditioned free energy difference $\boldsymbol{P}$-a.s. takes a value in a scaled interval (aN^δ, bN^δ). This can be combined with a Borel-Cantelli argument to exclude all values in such an interval, at least $\boldsymbol{P}$-a.s. and for all but finitely many volumes from a sparse enough sequence of volumes.

In the second step, we consider the contribution to the free energy difference coming from the corner aggregates. However, their contribution to the free energy will be argued to be of a smaller order when compared with the contribution of the non-corner terms.

Note that we also include the ∞-clusters in the first step. Because we have uniform bounds in η for the ∞-cluster weights, we are allowed to do so.

The free energy difference F_Λ^η can be computed by using the sequential expansion (4.71). For convenience, we rearrange the terms in the expansion by introducing

$$\begin{aligned} U^\eta(B) = &\sum_n \sum_\alpha \log \hat{\mathcal{Z}}_{n,\alpha}^\eta \mathbf{1}_{\{\mathrm{Dom}(\mathcal{K}_{n,\alpha}^\eta)=B\}} + \sum_i \log \hat{\mathcal{Z}}_{\infty,i}^\eta \mathbf{1}_{\{\mathrm{Dom}(\mathcal{K}_{\infty,i}^\eta)=B\}} \\ &+ \sum_C \phi_0^\eta(C) \mathbf{1}_{\{\mathrm{Dom}(C)=B\}} + \sum_n \sum_D \psi_n^\eta(D) \mathbf{1}_{\{\mathrm{Dom}(D)=B\}} \\ &+ \sum_D \psi_\infty^\eta(D) \mathbf{1}_{\{\mathrm{Dom}(D)=B\}} \end{aligned} \tag{4.83}$$

for any set $B \subset \partial\Lambda$. Note that any function $U^\eta(B)$ only depends on the restriction of η to the set $\underline{B}$. Using the notation $\bar{U}^\eta(B) = U^\eta(B) - U^{-\eta}(B)$, the expansion for the free energy difference F_Λ^η reads, formally,

$$F_\Lambda^\eta = 2\beta \sum_{x \in \underline{\partial\Lambda}} \eta_x + \sum_{B \subset \partial\Lambda} \bar{U}^\eta(B) \tag{4.84}$$

Obviously, no bulk contours contribute to $\bar{U}^\eta(B)$. Using the notation $\partial\Lambda_{C,i} := \{y^* \in \partial\Lambda \,:\, d[y^*, x_{C,i}^*] \le 2l_\infty\}$ and $\partial\Lambda_C := \cup_{i=1}^4 \partial\Lambda_{C,i}$, we consider the decomposition

$F^\eta_\Lambda = \tilde F^\eta_\Lambda + \hat F^\eta_\Lambda$, where

$$\tilde F^\eta_\Lambda = 2\beta \sum_{x\in \underline{\partial\Lambda\setminus\partial\Lambda_C}} \eta_x + \sum_{\substack{B\subset\partial\Lambda\\ \mathrm{Dom}(B)\not\subset\partial\Lambda_C}} \bar U^\eta(B) \tag{4.85}$$

and

$$\hat F^\eta_\Lambda = 2\beta \sum_{x\in \underline{\partial\Lambda_C}} \eta_x + \sum_{\substack{B\subset\partial\Lambda\\ \mathrm{Dom}(B)\subset\partial\Lambda_C}} \bar U^\eta(B) \tag{4.86}$$

The first term, $\tilde F^\eta(B)$, can be analyzed by means of the Mayer expansion of its characteristic function

$$\begin{aligned}\tilde\Psi^\eta_\Lambda(t) &:= \boldsymbol{E}[\exp(it\tilde F^\eta_\Lambda)\,|\,\eta_{\underline{\partial\Lambda_C}}] = \boldsymbol{E}\big[\exp\big(2it\beta \sum_{x\in \underline{\partial\Lambda\setminus\partial\Lambda_C}} \eta_x\big) \sum_{\mathcal{B}} \prod_{B\in\mathcal{B}} (e^{it\bar U^\eta(B)}-1)\,|\,\eta_{\underline{\partial\Lambda_C}}\big]\\ &= [\Psi_0(t)]^{|\partial\Lambda\setminus\partial\Lambda_C|} \sum_{\mathcal{B}} w_t(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}})\end{aligned} \tag{4.87}$$

where we have assigned to any family $\mathcal{B}$ of subsets of the boundary the weight

$$\begin{aligned}w_t(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}}) = \frac{1}{[\Psi_0(t)]^{|\partial\Lambda\setminus\partial\Lambda_C|}}\, \boldsymbol{E}\big[\exp(2it\beta \sum_{x\in \underline{\partial\Lambda\setminus\partial\Lambda_C}} \eta_x) \prod_{B\in\mathcal{B}} (e^{it\bar U^\eta(B)}-1)\,|\,\eta_{\underline{\partial\Lambda_C}}\big]\\ \times\, \mathbf{1}_{\{\forall B\in\mathcal{B}:\, B\not\subset\partial\Lambda_C\}}\end{aligned} \tag{4.88}$$

and have introduced the notation

$$\Psi_0(t) = \boldsymbol{E}[\exp(2it\beta\eta_0)] = \cos 2t\beta \tag{4.89}$$

Observing that

$$w(\mathcal{B}_1\cup\mathcal{B}_2\,|\,\eta_{\underline{\partial\Lambda_C}}) = w(\mathcal{B}_1\,|\,\eta_{\underline{\partial\Lambda_C}})\, w(\mathcal{B}_2\,|\,\eta_{\underline{\partial\Lambda_C}}) \tag{4.90}$$

whenever $B_1\cap B_2=\emptyset$ for any $B_1\in\mathcal{B}_1$ and $B_2\in\mathcal{B}_2$, the last sum in equation (4.87) is a partition function of another polymer model and using the symbols $\mathfrak{B}, \mathfrak{B}_1, \dots$ for the clusters in this model and w^T_t for the cluster weights, we get

$$\tilde\Psi^\eta_\Lambda(t) = [\Psi_0(t)]^{|\partial\Lambda\setminus\partial\Lambda_C|} \exp\big[\sum_{\mathcal{B}} w^T_t(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}})\big] \tag{4.91}$$

A crucial observation is that for any $\eta\in\Omega^{**}$ and $\Lambda(N)$, $N\ge N^{**}(\eta)$ no corner aggregate contributes to the weight $w_t(\mathcal{B})$ for any $\mathcal{B}$. On the other hand, the partition

function of any n-aggregate is balanced by a small probability of the aggregate to occur. Another observation is that every weight $w_t(\mathcal{B})$ is of order $\mathcal{O}(t^2)$ due to the symmetry of the distribution $\boldsymbol{P}$. To see this explicitly, formula (4.88) can be cast into a more symmetrized form,

$$w_t(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}}) = \frac{1}{[\Psi_0(t)]^{|\,\mathrm{Supp}(\mathcal{B})|}}\,\boldsymbol{E}\Big[T\Big\{t\big[2\beta\sum_{x\in\underline{\mathrm{Supp}(\mathcal{B})}}\eta_x+\frac{1}{2}\sum_{B\in\mathcal{B}}\bar{U}^{\eta}(B)\big]\Big\} \\ \times\prod_{B\in\mathcal{B}}2i\sin\Big(\frac{t\bar{U}^{\eta}(B)}{2}\Big)\,\Big|\,\eta_{\underline{\partial\Lambda_C}}\Big] \tag{4.92}$$

where $\mathrm{Supp}(\mathcal{B}) := \cup_{B\in\mathcal{B}}B$ and

$$T\{Y\} := \begin{cases} i\sin Y & \text{if } \mathrm{card}(\mathcal{B}) = 2k-1 \\ \cos Y & \text{if } \mathrm{card}(\mathcal{B}) = 2k \qquad k\in\mathbb{N} \end{cases} \tag{4.93}$$

In section 4.11.5 we give a proof of the following upper bound on the corresponding cluster weights:

Lemma 4.28. *There exist constants $\beta_8, l_0 > 0$[1] such that for any $\beta \geq \beta_8 l_0$ there is $t_0 = t_0(\beta) > 0$ for which the following is true. For any $\eta \in \Omega^{**}$ and $\Lambda = \Lambda(N)$, $N \geq N^{**}(\eta)$, the inequality*

$$\sup_{x^*\in\partial\Lambda\setminus\partial\Lambda_C}\sum_{\mathcal{B}:\,x^*\in\mathrm{Supp}(\mathcal{B})}|w_t^T(\mathcal{B}\,|\,\eta_{\underline{\partial\Lambda_C}})| \leq \frac{1}{2}\beta^2t^2 \tag{4.94}$$

is satisfied for all $|t| \leq t_0$.

With the help of the last lemma, it is easy to get an upper bound on $\tilde{\Psi}^{\eta}_{\Lambda}(t)$:

Lemma 4.29. *Under the assumptions of Lemma 4.28, we have*

$$\tilde{\Psi}^{\eta}_{\Lambda}(t) \leq \exp\Big(-\frac{1}{2}\beta^2t^2|\partial\Lambda(N)\setminus\partial\Lambda_C(N)|\Big) \tag{4.95}$$

for all $|t| \leq t_0$, $\eta \in \Omega^{**}$, and $N \geq N^{**}(\eta)$.

Proof. It immediately follows by combining Lemma 4.28, equation (4.91), and the bound $\Psi_0(t) \leq \exp[-\beta^2t^2]$. □

For the corner part $\hat{F}^{\eta}_{\Lambda}$ of the free energy difference we use the next immediate upper bound:

[1] Recall that the construction of aggregates depends on the choice of l_0.

Lemma 4.30. *Given $\eta \in \Omega^{**}$ and $\beta \geq \beta_6 l_0$, then $\hat{F}^{\eta}_{\Lambda(N)} = \mathcal{O}(N^{\delta})$ for any $\delta > 0$.*

Proof. Using Proposition 4.25 and Lemma 4.26, we have $\sum_{B \subset \partial\Lambda_C} |\bar{U}^{\eta}(B)| = \mathcal{O}(l_{\infty}^2)$ and the above claim immediately follows. □

Proof of Proposition 4.5. Combining Lemma 4.29 with Proposition 4.53 in the appendix, we get

$$\overline{\lim_{N\to\infty}} N^{\frac{1}{2}-\alpha} \boldsymbol{P}\big\{|\tilde{F}^{\eta}_{\Lambda(N)}| \leq N^{\alpha}\tau \,\big|\, \eta_{\underline{\partial\Lambda_C}}\big\} < \infty \tag{4.96}$$

for any $\alpha, \tau > 0$. By Lemma 4.30, $\tilde{F}$ can be replaced with the full free energy difference F. As a consequence,

$$\overline{\lim_{N\to\infty}} N^{\frac{1}{2}-\alpha} \boldsymbol{P}\{|F^{\eta}_{\Lambda(N)}| \leq \tau\} < \infty \tag{4.97}$$

and the proof is finished by applying Proposition 4.27. □

4.11 Proofs

In this section, we collect the proofs omitted throughout the main text.

4.11.1 Proof of Proposition 4.20

In order to get the claimed exponential upper bound on the probability for an n-aggregate to occur, we need to analyze the way how the aggregates are constructed in more detail. We start with an extension of Definition 4.17. Throughout the section, a finite volume $\Lambda = \Lambda(N)$ is supposed to be fixed.

Definition 4.31. *For every $n = 1, 2, \ldots$, any maximal L_n-connected subset $\Delta \subset \mathcal{K} \setminus (\mathcal{K}_0^{\eta} \cup \mathcal{K}_1^{\eta} \cup \ldots \cup \mathcal{K}_{n-1}^{\eta})$ is called an n-pre-aggregate.*

Obviously, n-aggregates are exactly those n-pre-aggregates Δ that satisfy the condition $|\partial\Delta|_{\text{con}} \leq l_n$. Moreover, every n-pre-aggregate can equivalently be constructed inductively by gluing pre-aggregates of lower orders:

Lemma 4.32. *Every n-pre-aggregate Δ_n is the union of a family of $(n-1)$-pre-aggregates, $\Delta_n = \cup_{\alpha}\Delta_{n-1}^{\alpha}$. Moreover,*

i) Each $(n-1)$-pre-aggregate Δ_{n-1}^{α} satisfies $|\partial\Delta_{n-1}^{\alpha}|_{con} > l_{n-1}$,

ii) The family $(\Delta_{n-1}^{\alpha})_{\alpha}$ is L_n-connected.

Proof. For $n = 1$ the statement is trivial.
Assume that $n \geq 2$, and let Δ_n be an n-pre-aggregate and $\Gamma \in \Delta_n$ be a contour. Then, there exists an $(n-1)$-pre-aggregate Δ^{α}_{n-1} such that $\Gamma \in \Delta^{\alpha}_{n-1}$ (otherwise Γ would be an element of a k-aggregate, $k \leq n-2$). Moreover, since Δ^{α}_{n-1} is not an $(n-1)$-aggregate by assumption, it satisfies $|\partial\Delta^{\alpha}_{n-1}|_{\text{con}} > l_{n-1}$, proving i).

The claim ii) is obvious. □

Lemma 4.33. *Let Δ by any family of unbalanced contours. Then,*

i) There exists a subset $\tilde{\Delta} \subset \Delta$ such that

a) $\partial\tilde{\Delta} = \partial\Delta$,

b) if $\Gamma_1, \Gamma_2, \Gamma_3 \in \tilde{\Delta}$ are any three mutually different contours, then $\partial\Gamma_1 \cap \partial\Gamma_2 \cap \partial\Gamma_3 = \emptyset$.

ii) The inequality

$$\sum_{x \in \underline{\partial\Delta}} \eta_x < -\big(1 - \frac{4}{l_0}\big)|\partial\Delta| \tag{4.98}$$

holds true.

Proof. i) Assume that $\Gamma_1, \Gamma_2, \Gamma_3 \subset \Delta$ is a triple of mutually different contours such that $\partial\Gamma_1 \cap \partial\Gamma_2 \cap \partial\Gamma_3 \neq \emptyset$. Since $\partial\Gamma_i$, $i = 1, 2, 3$ are connected subsets of $\partial\Lambda$, it is easy to realize that, up to a possible permutation of the index set $\{1, 2, 3\}$, one has $\partial\Gamma_1 \subset \partial\Gamma_2 \cup \partial\Gamma_3$. Hence, $\partial(\Delta \setminus \{\Gamma_1\}) = \partial\Delta$. Since the set Δ is finite, a subset $\tilde{\Delta} \subset \Delta$ with the claimed property is constructed by iterating the argument.

ii) Let $\tilde{\Delta} \subset \Delta$ be the same as in i). Then, using Lemma 4.10, the inclusion-exclusion principle implies

$$\begin{aligned}
\sum_{x \in \underline{\partial\Delta}} \eta_x &= \sum_{\Gamma \in \tilde{\Delta}} \sum_{x \in \underline{\partial\Gamma}} \eta_x - \sum_{(\Gamma,\Gamma') \subset \tilde{\Delta}} \sum_{x \in \underline{\partial\Gamma \cap \partial\Gamma'}} \eta_x \\
&< -\big(1 - \frac{2}{l_0}\big) \sum_{\Gamma \in \tilde{\Delta}} |\partial\Gamma| + \sum_{(\Gamma,\Gamma') \subset \tilde{\Delta}} |\partial\Gamma \cap \partial\Gamma'| \\
&\leq -\big(1 - \frac{4}{l_0}\big)|\partial\Delta|
\end{aligned} \tag{4.99}$$

□

It remains to prove that one still gets a large deviation upper bound by replacing the sum over the boundary sites $x \in \underline{\partial\Delta}$ in equation (4.98) with the sum over all $x \in \underline{\mathrm{Con}(\partial\Delta)}$, provided that Δ is a pre-aggregate. Technically, we need to exploit the

basic feature of any pre-aggregate Δ that the set $\mathrm{Con}(\partial\Delta) \setminus \partial\Delta$ is not 'too big'. A minor complication lies in the fact that the boundary distance $d[\partial\gamma, \partial\gamma']$ is allowed to exceed the contour distance $d[\gamma, \gamma']$. To overcome this difficulty, it is useful to define

$$\widetilde{\mathrm{Con}}(\partial\Delta) = \partial\Delta \cup \big\{x \in \mathrm{Con}(\partial\Delta);\, \forall\gamma \in \Delta :\, d[x, \partial\gamma] > \frac{|\partial\gamma|}{l_0}\big\} \tag{4.100}$$

for which the first equation in the proof of Lemma 4.10 implies the upper bound

$$|\partial\Delta|_{\mathrm{con}} \leq (1 + \frac{2}{l_0})|\widetilde{\mathrm{Con}}(\partial\Delta)| \tag{4.101}$$

We are now ready to prove the following key estimate from which Proposition 4.20 immediately follows by using a large deviation upper bound.

Lemma 4.34. *Let Δ be an n-pre-aggregate, $n = 1, 2, \ldots$ Then,*

$$\sum_{x \in \underline{\mathrm{Con}(\partial\Delta)}} \eta_x \leq -\frac{1}{3}|\partial\Delta|_{con} \tag{4.102}$$

uniformly in n.

Proof. We prove by induction in the order of the pre-aggregates the refined bound

$$\sum_{x \in \underline{\widetilde{\mathrm{Con}}(\partial\Delta)}} \eta_x \leq -\big(1 - 3\sum_{i=1}^{n} \frac{L_i}{l_{i-1}}\big)|\widetilde{\mathrm{Con}}(\partial\Delta)| \tag{4.103}$$

for any n-pre-aggregate Δ, from which the statement follows by using the definition 4.16 of length scales l_n and L_n, and equation (4.101). Indeed, one obtains then

$$\begin{aligned} \sum_{x \in \mathrm{Con}(\partial\Delta)} \eta_x &\leq -\big(1 - 3\sum_{i=1}^{\infty} \frac{L_i}{l_{i-1}}\big)|\widetilde{\mathrm{Con}}(\partial\Delta)| + |\,\mathrm{Con}(\partial\Delta) \setminus \widetilde{\mathrm{Con}}(\partial\Delta)| \\ &\leq -\big(\frac{1}{4} - \frac{2}{l_0}\big)\frac{|\partial\Delta|_{\mathrm{con}}}{1 + \frac{2}{l_0}} \leq -\frac{1}{3}|\partial\Delta|_{\mathrm{con}} \end{aligned} \tag{4.104}$$

First, assume that Δ is a 1-pre-aggregate, and let $\Delta = \cup_{i=1}^{m} A_i$ be the (unique) decomposition of Δ into disjoint subsets such that $\partial\Delta = \cup_{i=1}^{m} \partial A_i$ is the decomposition of $\partial\Delta$ into maximal connected components. For convenience, we use the notation $J_i := \partial A_i$. Considering furthermore the decomposition $\widetilde{\mathrm{Con}}(\partial\Delta) \setminus \partial\Delta = \cup_{k=1}^{m-1} G_k$ into maximal connected components, the set $\widetilde{\mathrm{Con}}(\partial\Delta)$ can be finally written as the union

$$\widetilde{\mathrm{Con}}(\partial\Delta) = \big(\cup_{i=1}^{m} J_i\big) \cup \big(\cup_{k=1}^{m-1} G_k\big) \tag{4.105}$$

of disjoint connected subsets, which satisfy the inequalities $|J_i| > l_0$ and $|G_k| \le L_1$, for all $i, k = 1, 2, \dots$ Using Lemma 4.33, we have $\sum_{x \in \underline{J_i}} \eta_x \le -\big(1 - \frac{4}{l_0}\big)|J_i|$, and since $\sum_{k=1}^{m-1} |G_k| \le \frac{L_1}{l_0} \sum_{i=1}^{m} |J_i|$, we finally get

$$\begin{aligned} \sum_{x \in \underline{\widetilde{\mathrm{Con}}(\partial\Delta)}} \eta_x = \sum_{i=1}^{m} \sum_{x \in \underline{J_i}} \eta_x + \sum_{k=1}^{m-1} |G_k| &\le -\big(1 - \frac{L_1 + 4}{l_0}\big) \frac{|\widetilde{\mathrm{Con}}(\partial\Delta)|}{1 + \frac{L_1}{l_0}} \\ &\le -\big(1 - \frac{3L_1}{l_0}\big)|\widetilde{\mathrm{Con}}(\partial\Delta)| \end{aligned} \tag{4.106}$$

provided that, say, $L_1 \ge 4$.
Next, we will prove the statement for an arbitrary n-pre-aggregate Δ. By Lemma 4.32, Δ is the union of a family of $(n-1)$-pre-aggregates, $\Delta = \cup_i \Delta^i_{n-1}$. In order to generalize our strategy used in the $n = 1$ case, we consider the (possibly disconnected) boundary sets $J_i = \widetilde{\mathrm{Con}}(\partial\Delta^i_{n-1})$, and the family of connected sets $(G_i)_{i=1,2,\dots}$ defined as the maximal connected components of the set $\mathrm{Con}(\partial\Delta) \setminus \cup_i \mathrm{Con}(\Delta^i_{n-1})$. Note that $\#\{G_i\} = \#\{J_i\} - 1$ and the identity $\widetilde{\mathrm{Con}}(\Delta) = (\cup_i J_i) \cup (\cup_i G_i)$. Hence, by using the induction hypothesis,

$$\begin{aligned} \sum_{x \in \underline{\widetilde{\mathrm{Con}}(\partial\Delta)}} \eta_x = \sum_{i=1}^{m} \sum_{x \in \underline{J_i}} \eta_x + \sum_{k=1}^{m-1} |G_k| &\le \Big[-\big(1 - 3\sum_{i=1}^{n-1} \frac{L_i}{l_{i-1}}\big) + \frac{L_n}{l_{n-1}}\Big] \frac{|\widetilde{\mathrm{Con}}(\partial\Delta)|}{1 + \frac{L_n}{l_{n-1}}} \\ &\le -\big(1 - 3\sum_{i=1}^{n} \frac{L_i}{l_{i-1}}\big)|\widetilde{\mathrm{Con}}(\partial\Delta)| \end{aligned} \tag{4.107}$$

as required. □

4.11.2 Proof of Proposition 4.24

The proof goes by induction in the order of aggregates.

The case $n = 1$.
As the initial step we bound the sums over 1-clusters in $\mathfrak{D}^{\eta}_1$. Recall that the 1-clusters consist of 0-clusters which connect 1-aggregates $\mathcal{K}^{\eta}_{1,\alpha}$. Throughout this section we use the shorthand $\tilde{\beta} := \frac{\beta}{l_0}$.

From Proposition 4.12 we know that for any integer r_0,

$$\sum_{\substack{C\in\mathfrak{C}_0^\eta:\ |C|\geq r_0\\ C\ni x}} |\phi_0^\eta(C)|\exp\left(\tilde{\beta}(2-(1/8))|C|\right)\leq 1$$

which implies

$$\sum_{\substack{C\in\mathfrak{C}_0^\eta:\ |C|\geq r_0\\ C\ni x}} |\phi_0^\eta(C)|\exp\left(2\tilde{\beta}(1-(1/8))|C|\right)\leq \exp\left(-\tilde{\beta}r_0/8\right) \tag{4.108}$$

We split the procedure into four steps as follows.

Part 1. For any 1-cluster in $\mathfrak{D}_1^\eta$, none of its 0-clusters contributes to the dressed weight of a 1-aggregate. Hence, all these 0-clusters have at least size L_1. Moreover, they are incompatible with a 1-aggregate $\mathcal{K}_{1,\alpha}^\eta$. Using Lemma 4.10 and choosing $r_0 = L_1$ in (4.108), this results in the inequality

$$\sum_{\substack{C\in\mathfrak{C}_0^\eta\\ C\not\sim\mathcal{K}_{1,\alpha}^\eta}} |\phi_0^\eta(C)|\exp\left(2\tilde{\beta}(1-(1/8))|C|\right)\leq l_1^2\exp\left[-(\tilde{\beta}L_1)/8\right]\leq 2^{-2} \tag{4.109}$$

Part 2. In order to prove the convergence of the cluster expansion resulting from the Mayer expansion, we apply Proposition 4.50. As our initial estimate, we get, using (4.109) and since

$$C \overset{1}{\not\leftrightarrow} C' \ \Leftrightarrow\ \exists\alpha \text{ such that } C, C' \not\sim \mathcal{K}_{1,\alpha}^\eta$$

the inequality

$$\begin{aligned}
&\sum_{\substack{C\in\mathfrak{C}_0^\eta\\ C\overset{1}{\leftrightarrow}C_0}} |\phi_0^\eta(C)|\exp\left(2\tilde{\beta}(1-(1/8))|C|\right)\\
&\qquad\leq \sum_{\mathcal{K}_{1,\alpha}^\eta:\ \mathcal{K}_{1,\alpha}^\eta\not\sim C_0}\ \sum_{\substack{C\in\mathfrak{C}_0^\eta\\ C\not\sim\mathcal{K}_{1,\alpha}^\eta}} \exp\left(2\tilde{\beta}(1-(1/8))|C|\right)|\phi_0^\eta(C)| \qquad (4.110)\\
&\qquad\leq 2^{-2}\#\{\mathcal{K}_{1,\alpha}^\eta\not\sim C_0\}
\end{aligned}$$

Part 3. Using Lemma 4.23, the weight of any set of 0-clusters appearing in the Mayer expansion is bounded as

$$|w_1^\eta(\mathcal{C})|\leq\prod_{C\in\mathcal{C}}(e^{|\phi_0^\eta(C)|}-1)\leq\prod_{C\in\mathcal{C}}2|\phi_0^\eta(C)|$$

Hence, by using Proposition 4.50, we obtain the bound

$$\sum_{\substack{C_1 \overset{1}{\not\rightarrow} C_0 \\ C_1 \in \mathfrak{D}_1^\eta}} |\psi_1^\eta(C_1)| \exp\left[(2\tilde{\beta}(1-(1/8))-1/2)|C_1|\right] \leq 2^{-1} \#\{\mathcal{K}_{1,\alpha}^\eta \not\sim C_0\} \tag{4.111}$$

Taking now $C_0 \in \mathfrak{C}_0^\eta$ such that $\mathcal{K}_{1,\alpha}^\eta$ is the only 1-aggregate satisfying $C_0 \nsim \mathcal{K}_1^\alpha$, inequality (4.111) yields

$$\sum_{\substack{C_1 \not\sim \mathcal{K}_{1,\alpha}^\eta \\ C_1 \in \mathfrak{D}_1^\eta}} |\psi_1^\eta(C_1)| \exp\left[(2\tilde{\beta}(1-(1/8))-1/2)|C_1|\right] \leq 2^{-1} \tag{4.112}$$

Part 4. In order to bound the sum over all 1-clusters $C_1 \in \mathfrak{D}_1^\eta$ such that $C_1 \ni x$ and $|C_1| \geq r_1$, we use that $|C_1| \geq L_1$ and write

$$\begin{aligned} &\sum_{\substack{C_1 \ni x,\ |C_1| \geq r_1 \\ C_1 \in \mathfrak{D}_1^\eta}} |\psi_1^\eta(C_1)| \exp\left[(2\tilde{\beta}(1-(1+1/2)/8)-1/2)|C_1|\right] \\ &\qquad \leq \sum_{\mathcal{K}_{1,\alpha}^\eta} \sum_{\substack{C_1 \not\sim \mathcal{K}_{1,\alpha}^\eta :\ C_1 \ni x \\ |C_1| \geq r_1,\ C_1 \in \mathfrak{D}_1^\eta}} |\psi_1^\eta(C_1)| \exp\left[(2\tilde{\beta}(1-(1+1/2)/8)-1/2)|C_1|\right] \end{aligned} \tag{4.113}$$

Substituting (4.112), we obtain

$$\begin{aligned} (4.113) &\leq \sum_{\mathcal{K}_{1,\alpha}^\eta} 2^{-1} \exp\left(-(\tilde{\beta}/8) \cdot \max\left[d(\mathcal{K}_{1,\alpha}^\eta, x), r_1\right]\right) \\ &\leq \sum_{R=0}^{\infty} \sum_{\mathcal{K}_{1,\alpha}^\eta :\ d(\mathcal{K}_{1,\alpha}^\eta, x)=R} 2^{-1} \exp\left(-\epsilon\tilde{\beta} \cdot \max\left[R, r_1\right]\right) \end{aligned} \tag{4.114}$$

The last sum can be estimated by a partial integration and we finally get

$$(4.113) \leq \exp\left(-\tilde{\beta} r_1/8\right) \left[r_1^2 + \frac{16 r_1}{\tilde{\beta}} + 2\right] \leq 2^{-1} \cdot 4 r_1^2 \exp\left(-\tilde{\beta} r_1/8\right)$$

where we have used that $r_1 \geq L_1$ and that L_1 is large enough.

Induction step.
The induction hypothesis reads

$$\sum_{\substack{C_i \ni x:\ |C_i| \geq r_i \\ C_i \in \mathfrak{D}_i^\eta}} |\psi_i^\eta(C_i)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{i+1}(1/2)^j/8) - \sum_{j=1}^{i}(1/2)^j\right)|C_i|\right] \leq 4 \cdot 2^{-i} r_i^2 \exp\left(-\tilde{\beta}(1/2)^{i+1} r_i/8\right) \tag{4.115}$$

for any $1 \leq i \leq n-1$.

Part 1. As in part 1 of the $n = 1$ case, we want to prove first that

$$\sum_{\substack{C \in \mathfrak{C}_{n-1}^\eta \\ C \not\sim \mathcal{K}_{n,\alpha}^\eta}} |\phi_{n-1}^\eta(C)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{n}(1/2)^j/8) - \sum_{j=1}^{n-1}(1/2)^j\right)|C|\right] \leq 2^{-n-1} \tag{4.116}$$

Recalling Definition (4.58) for $\phi_{n-1}^\eta(C)$, we know that $\phi_{n-1}^\eta(C) = \psi_j^\eta(C)$ for any $C \in \mathfrak{D}_j^\eta$. Hence, using (4.115) with $r_i = L_n$, we write

$$\begin{aligned}
(4.116) &\leq l_n^2 \sum_{i=1}^{n-1} \sum_{\substack{C_i \ni x:\ |C_i| \geq L_n \\ C_i \in \mathfrak{D}_i^\eta}} |\psi_i^\eta(C_i)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{n}(1/2)^j/8) - \sum_{j=1}^{n-1}(1/2)^j\right)|C_i|\right] \\
&\leq 4 l_n^2 L_n^2 \sum_{i=0}^{n-1} 2^{-i} \exp\left[\left(-\tilde{\beta} \sum_{j=i+1}^{n}(1/2)^j/4 - \sum_{j=i+1}^{n-1}(1/2)^j\right) L_n\right] \\
&\leq 2^{-n-1} \cdot 32 l_n^2 L_n^2 \exp\left[-\tilde{\beta}(1/2)^n L_n/4\right] \leq 2^{-n-1}
\end{aligned} \tag{4.117}$$

where we have used that $l_n = \exp(L_n/2^n)$ and $\tilde{\beta}$ is large enough. This proves inequality (4.116).

Part 2. Similarly as in the $n = 1$ case, we prove by using (4.116) the inequality

$$\sum_{\substack{C \not\to C_0 \\ C \in \mathfrak{C}_{n-1}^\eta}} |\phi_{n-1}^\eta(C)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{n}(1/2)^j/8) - \sum_{j=1}^{n-1}(1/2)^j\right)|C|\right] \leq 2^{-n-1} \#\{\mathcal{K}_{n,\alpha}^\eta \not\sim C_0\} \tag{4.118}$$

Part 3. By construction, any n-cluster $C_n \in \mathfrak{D}_n^\eta$ consists of a family of 0-clusters $C_0 \in \mathfrak{C}_0^\eta$ and i-clusters $C_1 \in \mathfrak{D}_0^\eta$, $0 \le i \le n-1$, which are all incompatible with $\mathcal{K}_n^\eta$. Using Lemma 4.23 again, we have the upper bound

$$|w_n^\eta(C_n)| \le \prod_{i=0}^{n-1} \prod_{C \in C_n \cap \mathfrak{D}_i^\eta} 2|\psi_i^\eta(C)|$$

where we have identified $\psi_0^\eta(.) \equiv \phi_0^\eta(.)$ and $\mathfrak{D}_0^\eta \equiv \mathfrak{C}_0^\eta$. Applying Proposition 4.50 with $z(C) = 2|\psi_i^\eta(C)|$ then gives

$$\sum_{\substack{C_n \not\sim C_0 \\ C_n \in \mathfrak{D}_n^\eta}} |\psi_n^\eta(C_n)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{n}(1/2)^j/8) - \sum_{j=1}^{n}(1/2)^j\right)|C|\right] \le 2^{-n} \#\{\mathcal{K}_{n,\alpha}^\eta \not\sim C_0\}$$

Taking again $C_0 \in \mathfrak{C}_0^\eta$ such that $\mathcal{K}_{n,\alpha}^\eta \not\sim C_0$ implies the inequality

$$\sum_{\substack{C_n \not\sim \mathcal{K}_{n,\alpha}^\eta \\ C_n \in \mathfrak{D}_n^\eta}} |\psi_n^\eta(C_n)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{n}(1/2)^j/8) - \sum_{j=1}^{n}(1/2)^j\right)|C_n|\right] \le 2^{-n} \quad (4.119)$$

Part 4. Repeating the argument for the $n = 1$ case, we obtain the inequality

$$\sum_{\substack{C_n \ni x,\ |C_n| \ge r_n \\ C_n \in \mathfrak{D}_n^\eta}} |\psi_n^\eta(C_n)| \exp\left[\left(2\tilde{\beta}(1 - \sum_{j=0}^{n+1}(1/2)^j/8) - \sum_{j=1}^{n}(1/2)^j\right)|C_n|\right] \quad (4.120)$$
$$\le 2^{-n} \cdot 4r_n^2 \exp\left(-(1/2)^{n+1}\tilde{\beta}r_n/8\right)$$

Using that $r_n \ge L_n$ for any $C_n \in \mathfrak{D}_n^\eta$ and choosing $r_n = L_n$ proves the proposition for the weights ψ_n^η, $n = 1, 2, \ldots$.

Equation (4.58) reads that $\phi_n^\eta(C) = \psi_j^\eta(C)$ whenever $C \in \mathfrak{D}_j^\eta$ and $j \le n$. Using further that $\mathfrak{C}_n^\eta = \mathfrak{C}_0^\eta \cup \mathfrak{D}_1^\eta \cup \cdots \cup \mathfrak{D}_n^\eta$ and summing up the cluster weights of the clusters of all orders yields inequality (4.64), which finishes the proof.

4.11.3 Proof of Proposition 4.25

Let n_0 be the same as in Section 4.8.3. Due to the second part of Proposition 4.24,

$$\sup_x \sum_{\substack{C\ni x,\ |C|\ge r_0\\ C\in\mathfrak{C}^\eta_{n_0}}} |\phi^\eta_{n_0}(C)|\exp\left[(\beta/4l_0)\right]|C| \le 2\exp\left(-(3\beta/4l_0)r_0\right)$$

According to the definition of the corner-aggregates, we have

$$\sum_{\substack{C\not\sim\mathcal{K}_{\infty,i}\\ C\in\mathfrak{C}^\eta_{n_0}}} |\phi^\eta_{n_0}(C)|\exp\left[(\beta/4l_0)|C|\right] \le 2l_\infty^2\exp\left(-l_\infty(3\beta/2l_0)\right) \le 2^{-3}\exp\left(-l_\infty(\beta/l_0)\right)$$

Applying Proposition 4.50, we obtain

$$\sum_{\substack{C\not\sim\mathcal{K}_{\infty,i}\\ C\in\mathfrak{D}^\eta_\infty}} |\psi^\eta_\infty(C)|\exp\left[(\beta/4l_0)\right]|C| \le 2^{-2}\exp\left(-l_\infty(\beta/l_0)\right)$$

which implies

$$\sum_{D\in\mathfrak{D}^\eta_\infty} |\psi^\eta_\infty(D)| \le \exp\left(-(3\beta/2l_0)l_\infty\right)$$

4.11.4 Proof of Lemma 4.26

Let $\eta\in\Omega^{**}$ and $\mathcal{K}^\eta_{n,\alpha}$ be an n-aggregate, $n=1,2,\ldots$. Recall that

$$\hat{\mathcal{Z}}^\eta_{n,\alpha} = \sum_{\partial\in\mathscr{D}^\eta_{n,\alpha}} \hat\rho^\eta(\partial) \tag{4.121}$$

where

$$\hat\rho^\eta(\partial) = \prod_{\Gamma\in\partial}\rho^\eta(\Gamma)\,\exp\Big(-\sum_{\substack{C\in\mathfrak{C}^\eta_{n-1}\\ C\not\sim\partial;\ |C|<L_n}}\phi^\eta_{n-1}(C)\Big) \tag{4.122}$$

Using the η-uniform bounds

$$\rho^\eta(\Gamma)\le\exp[-2\beta(|\Gamma|-|\partial\Gamma|)] \tag{4.123}$$

and

$$\sup_{x^\star}\sum_{\substack{C\in\mathfrak{C}^\eta_{n-1}\\ x^\star\in C}}|\phi^\eta_{n-1}(C)| \le \exp\big[-\frac{3\beta}{l_0}\big] \tag{4.124}$$

for all $n = 1, 2, \ldots$, one can subsequently write (for simplicity, we use the shorthand $\varepsilon = \exp\big[-\frac{3\beta}{l_0}\big]$ below):

$$\begin{aligned}
\hat{\mathcal{Z}}^{\eta}_{n,\alpha} &\leq \sum_{\partial \in \mathscr{D}^{\eta}_{n,\alpha}} \prod_{\Gamma \in \partial} \exp[-(2\beta - \varepsilon)|\Gamma| + 2\beta|\partial\Gamma|] \\
&\leq e^{(\varepsilon + 4e^{-2\beta+\varepsilon})|\partial \mathcal{K}^{\eta}_{n,\alpha}|} \sum_{\partial \in \mathscr{D}^{\eta}_{n,\alpha}} \prod_{\Gamma \in \partial} \exp[-(2\beta - \varepsilon)|\Gamma| + (2\beta - \varepsilon - 4e^{-2\beta+\varepsilon}))|\partial\Gamma|] \\
&\leq e^{(\varepsilon + 4e^{-2\beta+\varepsilon})|\partial \mathcal{K}^{\eta}_{n,\alpha}|} \sum_{\partial \in \mathscr{D}^{\eta}_{n,\alpha}} \prod_{\Gamma \in \partial} \prod_{\gamma \in \Gamma} \exp[-(2\beta - \varepsilon)|\gamma| + (2\beta - \varepsilon - 4e^{-2\beta+\varepsilon})|\partial\gamma|] \\
&\leq e^{(\varepsilon + 4e^{-2\beta+\varepsilon})|\partial \mathcal{K}^{\eta}_{n,\alpha}|} \Big\{ 1 + \sum_{\gamma \ni p} \exp[-(2\beta - \varepsilon)|\gamma| + (2\beta - \varepsilon - 4e^{-2\beta+\varepsilon}))|\partial\gamma|] \Big\}^{|\partial \mathcal{K}^{\eta}_{n,\alpha}|}
\end{aligned} \tag{4.125}$$

where the last sum runs over all pre-contours (= connected components of contours) such that a fixed dual bond $p = \langle x, y \rangle^{\star}$, $d(x, \Lambda^c) = d(y, \Lambda^c) = 1$ is an element of γ and it is the leftmost bond with these properties, w.r.t. a fixed orientation on the boundary. To estimate this sum, we associate with each pre-contour γ a path (= sequence of bonds; not necessarily unique) starting at p. Every such a path consists of steps choosing from three of in total four possible directions. One easily realizes that, for every such a path, the total number of steps to the right is bounded from below by $|\partial\gamma|$. Hence, the last sum in (4.125) is upper-bounded via the summation over all paths started at p, so that to each step going to the right (respectively to the left/up/down) one assigns the weight $e^{-4e^{-2\beta+\varepsilon}}$ (respectively $e^{-2\beta+\varepsilon}$), which yields

$$\begin{aligned}
&\sum_{\gamma \ni p} \exp[-(2\beta - \varepsilon)|\gamma| + (2\beta - \varepsilon - 4e^{-2\beta+\varepsilon}))|\partial\gamma|] \\
&\qquad \leq e^{-2\beta+\varepsilon} \sum_{n=1}^{\infty} \big(2e^{-2\beta+\varepsilon} + e^{-4e^{-2\beta+\varepsilon}}\big)^n \leq 2e^{-2\beta+\varepsilon}
\end{aligned} \tag{4.126}$$

All in all, one obtains

$$\hat{\mathcal{Z}}^{\eta}_{n,\alpha} \leq e^{(\varepsilon + 6e^{-2\beta+\varepsilon})|\partial \mathcal{K}^{\eta}_{n,\alpha}|} \tag{4.127}$$

proving the first part of the statement.

The proof of the second part is trivial by counting the number of all configurations in the square volume with side $2l_{\infty}$. Note that the latter contains all contours $\Gamma \in \partial$ for any configuration $\partial \in \mathscr{D}^{\eta}_{\infty,i}$ and that the weights of all clusters renormalizing the contour weights are summable due to Proposition 4.24.

4.11.5 Proof of Lemma 4.28

Due to Proposition 4.50, it is enough to show that the inequality

$$\sum_{\mathcal{B}:\, x^* \in \mathrm{Supp}(\mathcal{B})} |w_t(\mathcal{B} \,|\, \eta_{\underline{\partial\Lambda_C}})| \, \exp\big(\frac{1}{2}\beta^2 t^2 |\,\mathrm{Supp}(\mathcal{B})|\big) \le \frac{1}{2}\beta^2 t^2 \tag{4.128}$$

holds true for all $|t| \le t_0$, with a constant $t_0 > 0$. Remark that the RHS of the last equation is not optimal and can be improved, as obvious from the computation below.

In order to prove (4.128), we use the symmetric representation (4.92) of the weight $w_t(\mathcal{B} \,|\, \eta_{\underline{\partial\Lambda_C}})$, the lower bound $\Psi_0(t) \ge e^{-\alpha}$ which is true for any $\alpha > 0$ provided that $|t| \le t_1(\alpha)$ with a constant $t_1(\alpha) > 0$, and the estimate

$$\begin{aligned} &\Big|T\Big\{t\big[2\beta \sum_{x \in \underline{\mathrm{Supp}(\mathcal{B})}} \eta_x + \frac{1}{2}\sum_{B\in\mathcal{B}} \bar{U}^\eta(B)\big]\Big\}\Big| \\ &\qquad \le \begin{cases} t\Big[2\beta \sum_{x\in B} |\eta_x| + \frac{1}{2}|\bar{U}^\eta(B)|\Big] & \text{for } \mathcal{B} = \{B\} \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{4.129}$$

which will be enough in order to get the t^2 factor in what follows. Using Proposition 4.24 and Lemma 4.26, we get a uniform upper bound $|\bar{U}^\eta(B)| \le c|B|$ with a constant $c > 0$ such that $c \downarrow 0$ for $\beta \uparrow \infty$. Hence, in the case $\mathcal{B} = \{B\}$ we have

$$\begin{aligned} &|w_t(\mathcal{B} = \{B\}) \,|\, \eta_{\underline{\partial\Lambda_C}}| \\ &\qquad \le e^{\alpha|B|} t^2 \boldsymbol{E}\Big[\big(2\beta \sum_{x\in B} |\eta_x| + \frac{1}{2}\sum_{B\in\mathcal{B}} |\bar{U}^\eta(B)|\big) \prod_{B\in\mathcal{B}} |\bar{U}^\eta(B)| \,\Big|\, \eta_{\underline{\partial\Lambda_C}}\Big] \\ &\qquad \le e^{\alpha|B|} t^2 (2\beta + \frac{c}{2})|B| \, \boldsymbol{E}\Big[|\bar{U}^\eta(B)| \,\Big|\, \eta_{\underline{\partial\Lambda_C}}\Big] \end{aligned} \tag{4.130}$$

Note that the above uniform upper bound on $|\bar{U}^\eta(B)|$ is not sufficient to get a sensible estimate on the conditional expectation. However, a more detailed upper bound can be obtained. Without loss of generality, we can assume that $B \cap \partial\Lambda_C = \emptyset$, so that the conditioning on $\eta_{\underline{\partial\Lambda_C}}$ can be omitted. First, assume there is an aggregate[2] $\mathcal{K}^\eta_\alpha$ such that $\mathrm{Dom}(\mathcal{K}^\eta_\alpha) = B$. Then, Lemma 4.26 gives the estimate $\log \hat{\mathcal{Z}}^\eta_\alpha \le c_7|B|$ and, since $|\partial\mathcal{K}^\eta_\alpha| \ge |B|/2$, Proposition 4.20 reads that the probability of such an event is bounded by $\exp(-\frac{c_5}{2}|B|)$. Second, assume there is a family of aggregates $(\mathcal{K}^\eta_{\alpha_i})_i$ (of possibly different orders) such that $D^\eta := \cup_i \mathrm{Dom}(\mathcal{K}^\eta_{\alpha_i}) \subset B$. Then, any cluster C such that

[2] For simplicity, we suppress the subscript n here.

$\mathrm{Dom}(C) = B$ has the length $|C| \geq |B \setminus D^\eta|$ and Proposition 4.24 gives the estimate

$$\sum_{\substack{C \in \cup_n \mathfrak{C}_n^\eta \\ \mathrm{Dom}(C)=B}} |\phi_n^\eta| \leq \exp\Big(-\frac{\beta}{2l_0}|B \setminus D^\eta|\Big)$$

Moreover, the probability that $D^\eta = D$ for a fixed set D is bounded by $e^{-\frac{c_5}{2}|D|}$. Note, however, that the above two scenarios are possible only provided that $|B| \geq l_1$, otherwise we only get a contribution from 0-clusters, the sum of which is bounded by $e^{-\frac{\beta}{2l_0}|B|}$. All in all, we obtain

$$\begin{aligned} \boldsymbol{E}\Big[|U^\eta(B)|\Big] &\leq c_7|B|\, e^{-\frac{c_5}{2}|B|}\mathbf{1}_{|B|\geq l_1} + e^{-\frac{\beta}{2l_0}|B|} + \mathbf{1}_{|B|\geq l_1}\sum_{D\subset B} e^{-\frac{c_5}{2}|D|-\frac{\beta}{2l_0}|B\setminus D|} \\ &\leq e^{-\frac{\beta}{2l_0}|B|} + \mathbf{1}_{|B|\geq l_1}(c_7+1)|B|\, e^{-\frac{c_5}{4}|B|} \end{aligned} \tag{4.131}$$

provided that β/l_0 is large enough. Recall that c_5 does not depend on l_0, which means that the latter can be adjusted as large as necessary. Using the same argument for $U^{-\eta}(B)$ and substituting (4.131) into (4.130), we get

$$\begin{aligned} &\sum_{\substack{B\ni x \\ B\subset\partial\Lambda}} |w_t(\mathcal{B}=\{B\})\,|\,\eta_{\underline{\partial\Lambda_C}})|\, e^{\tau|B|} \leq 2\cdot t^2(2\beta+\frac{c}{2})\sum_{\substack{B\ni x \\ B\subset\partial\Lambda}} |B|\, e^{(\tau+\alpha)|B|} \\ &\qquad \times\Big[e^{-\frac{\beta}{2l_0}|B|} + \mathbf{1}_{|B|\geq l_1}(c_7+1)|B|\, e^{-\frac{c_5}{4}|B|}\Big] \leq \tau'\beta t^2 \end{aligned} \tag{4.132}$$

which is true for any $\tau' > 0$ provided that τ and α are chosen sufficiently small and l_0 (and hence l_1) sufficiently large. This argument can easily be generalized by taking into account all collections $\mathcal{B}$, $\mathrm{card}(\mathcal{B}) > 1$. Hence, the proof of (4.128) is completed by choosing $\tau = \frac{1}{2}\beta^2 t^2$, under the condition $|t| \leq t_0$ with $t_0 = t_0(\beta)$ being small enough.

4.12 High field

Let us consider again the Ising model but now with a high boundary field:

$$H_{\Lambda,\lambda}^\eta(\sigma) = -\beta\sum_{\langle x,y\rangle\subset\Lambda}(\sigma_x\sigma_y - 1) - \beta\lambda\sum_{\substack{\langle x,y\rangle \\ x\in\Lambda,\ y\in\Lambda^c}}\sigma_x\eta_y \tag{4.133}$$

where we take $\lambda \geq 5$. The corresponding Gibbs measure we denote by $\mu_{\Lambda,\lambda}^\eta$. Assuming $\lambda \to \infty$ (independently of the volume size) the field determines the spin configuration of the non-corner sites next to the boundary. Together with the corner spins these sites form the inner boundary $\partial\Lambda^i$ of Λ:

Definition 4.35. *The inner boundary $\partial\Lambda^i$ is the set of all bonds $<x,y>\in\Lambda^\star$ such that $d(<x,y>,\partial\Lambda)=1$.*

The set $\underline{\partial\Lambda^i}$ corresponds to the set $\{x\in\Lambda:\ d(x,\Lambda^c)=1\}$.

The set $\Lambda^i=\Lambda\setminus\{x\in\Lambda:\ d(x,\Lambda^c)=1\}$.

For the corner spins it can happen that the fields from its nearest neighbor spins are opposite to each other and therefore cancel out. We need to separate these spins, because we can not let 'these spins follow the field'; it simply does not make any sense. The union of these corner spins we denote by x_{Ca}:

Definition 4.36. *The set x_{Ca} is the set of corner spins for which the adjacent fields cancel each other out.*

Remark 4.37. *For shortening notation we sometimes write for the field η_y adjacent to the spins $x\in\underline{\partial\Lambda^i}\setminus x_{Ca}$ simply η_x. Because for these spins the field next to it has only one value we are allowed to do this.*

Now we try to find the nature of the Gibbs measures for the infinite-field case. Suppose that for a configuration σ_Λ there is a spin σ_i with $i\in\underline{\partial\Lambda^i}\setminus x_{Ca}$ for which $\sigma_i=-\eta_i$. Automatically in the limit $\lambda\to\infty$ the energy becomes infinite by equation (4.133). Therefore this occurs only with zero probability. So the boundary field functions as a boundary condition for the spins σ_i for which $d(\Lambda^c,i)=2$. Apart from some minor finite contribution of the spins of the sites in x_{Ca}, the infinite boundary field Ising model on Λ, where $\lambda=\infty$, is equivalent to the Ising model with random boundary conditions for Λ^i, where $\lambda=1$. Because we consider 2 dimensions only: $|x_{Ca}|\leq 4$, which is a finite number.

With some rewriting we easily see the following:

$$\lim_{\lambda\to\infty} Z^\eta_{\Lambda,\lambda}=$$
$$\lim_{\lambda\to\infty}\exp\left(\beta(\lambda-1)|\underline{\partial\Lambda^i}|\right)\exp\Big(\beta\sum_{\substack{<x,y>:\\ x,y\in\underline{\partial\Lambda^i}\setminus x_{Ca}}}\eta_x\eta_y\Big)\sum_{\sigma_{x_{Ca}}}\exp\Big(\beta\sum_{\substack{<x,y>:\\ x\in x_{Ca},\ y\in\underline{\partial\Lambda^i}}}\sigma_x\eta_y\Big)Z^\eta_{\Lambda^i,1} \tag{4.134}$$

Note that the factor before the partition function $Z^\eta_{\Lambda^i,1}$ only depends on λ, η and $\sigma_{x_{Ca}}$. Of course this holds only in the limit $\lambda\to\infty$. This makes that when we calculate the Gibbs mean of a spin-function which does not depend on x_{Ca} we obtain

$$\lim_{\lambda\to\infty}<F(\sigma_{\Lambda\setminus x_{Ca}})>_{\mu^\eta_{\Lambda,\lambda}}=<F(\eta_{\underline{\partial\Lambda^i}\setminus x_{Ca}},\sigma_{\Lambda^i})>_{\mu^\eta_{\Lambda^i,1}} \tag{4.135}$$

Because the value of the spins on $\underline{\partial\Lambda^i} \setminus x_{Ca}$ is determined by η, we only need to use the Gibbs measure restricted to Λ^i with boundary condition η instead of the full volume Gibbs measure for Λ with the infinite boundary field.

Now we take $\lambda \geq 5$ but finite. By our assumption on λ it is more beneficial for spins on $\underline{\partial\Lambda^i} \setminus x_{Ca}$ to follow the field instead of following the adjacent spins. This suggests to expand differently than before. Then we expanded around the two ground states $\sigma \equiv +1$ and $\sigma \equiv -1$ which appear when we have zero-field: *the zero-field expansion*. Now we expand from the two ground states which appear in the infinite boundary field case.

Denote $\sigma_\Lambda = \sigma_{\underline{\partial\Lambda^i}} \vee \sigma_{\Lambda^i}$. For the ground-states restricted to $\Lambda \setminus x_{Ca}$ it holds: $\sigma_{\underline{\partial\Lambda^i}\setminus x_{Ca}} \equiv \eta$ combined with $\sigma_{\Lambda^i} \equiv \pm 1$. For the corner spins x_{Ca} we set the spin values equal to +1 or -1, equal to the chosen ground state value in the interior Λ^i. This expansion we refer to as *the high-field expansion*. The corresponding ensembles we again refer to as the $+$ and $-$ ensemble.

4.12.1 Contour representation

In this setup there will be two types of contours. The first type of contours are contours Γ for which $\Gamma \subset \Lambda^{i\star}$ and therefore do not contain sites of $\partial\Lambda^i$. These contours we define as before but with the role of $\partial\Lambda$ replaced by $\partial\Lambda^i$. The second type of contours represent the part of the spins on $\underline{\partial\Lambda^i}$ which do not follow the field or where the spin-values are ∓ 1 on the corner sites x_{Ca}. These contours γ are connected sets of sites on the inner boundary:

Definition 4.38. *A contour γ is a set of sites of $\underline{\partial\Lambda^i}$ such that*

1. *For every site $x \in \gamma$ and $x \notin x_{Ca}$: $\sigma_x = -\eta_y$, and $\sigma_x = \mp 1$ when $x \in x_{Ca}$, where $y \in \Lambda^c$ and $d(x,y) = 1$.*

2. *For every site $x \in \underline{\partial\Lambda^i}$ and $x \notin x_{Ca}$ with $d(x,\gamma) = 1$: $\sigma_x = \eta_y$, $y \in \Lambda^c$ and $d(x,y) = 1$, and above with $\sigma_x = \pm 1$ in case $x \in x_{Ca}$.*

3. *The length $|\gamma|$ is the number of sites in γ.*

In this way every collection of contours $\Gamma \cup \gamma$ correspond to two flip-related (except for the part in $\underline{\partial\Lambda^i} \setminus x_{Ca}$) spin configurations. This gives a natural separation into the $+$ and $-$ ensemble. The two corresponding partition functions are:

$$Z^\eta_{\Lambda,\lambda} = Z^{+,\eta}_{\Lambda,\lambda} + Z^{-,\eta}_{\Lambda,\lambda},\ Z^{\pm,\eta}_{\Lambda,\lambda} = \mathrm{pref}^{\pm,\eta}_\lambda(\emptyset) \sum_{\partial_1 \subset \underline{\partial\Lambda^i}} \prod_{\gamma\in\partial_1} \rho^{\pm,\eta}_\lambda(\gamma) \sum_{\partial_2 \subset \mathcal{K}_{\Lambda^i}} \prod_{\Gamma\in\partial_2} \rho^{\pm,\eta}(\Gamma) \tag{4.136}$$

with

$$\mathrm{pref}_{\lambda}^{\pm,\eta}(\emptyset) = \exp\left[\beta(\lambda-2)|\partial\Lambda^i| + \beta \sum_{<x,y>:\, x,y\in\underline{\partial\Lambda^i}\setminus x_{Ca}} \eta_x\eta_y\right] \exp\Big(\pm\beta\Big(\sum_{x\in\underline{\partial\Lambda^i}\setminus x_C}\eta_x + \sum_{\substack{y\in\underline{\partial\Lambda^i}\\ d(y,x_{Ca})=1}}\eta_y\Big)\Big) \quad (4.137)$$

We define for convenience

$$Z_{\partial\Lambda,\lambda}^{\pm,\eta} \equiv \sum_{\partial_1\subset\underline{\partial\Lambda^i}} \prod_{\gamma\in\partial_1} \rho_{\lambda}^{\pm,\eta}(\gamma) \quad (4.138)$$

Then

$$Z_{\Lambda,\lambda}^{\pm,\eta} = Z_{\partial\Lambda,\lambda}^{\pm,\eta}\cdot \mathrm{pref}_{\lambda}^{\pm,\eta}(\emptyset) \sum_{\partial_2\subset\mathcal{K}_{\Lambda^i}} \prod_{\Gamma\in\partial_2} \rho^{\pm,\eta}(\Gamma) \quad (4.139)$$

The weights of the contours are defined as follows:

$$\rho^{\pm,\eta}(\Gamma) = \exp\Big(-2\beta\Big(|\Gamma| \pm \sum_{i\in\underline{\partial\Lambda}:\ d(i,\underline{\partial\Gamma^{\mp}})=1}\eta_i\Big)\Big) \quad (4.140)$$

just as before and

$$\rho_{\lambda}^{\pm,\eta}(\gamma) = \exp\Bigg[-2\beta\Bigg(\lambda(|\gamma\setminus(\gamma\cap x_{Ca})| + |\gamma\cap x_C\setminus x_{Ca}|) \pm \sum_{i\in\gamma\setminus\gamma\cap x_C}\eta_i + \sum_{\substack{<x,y>\ x\in\gamma\\ y\in\underline{\partial\Lambda^i}:\ y\notin\gamma}} \begin{cases}\eta_x\eta_y & \text{if } x,y\notin x_{Ca}\\ \pm\eta_x & \text{if } y\in x_{Ca}\\ \pm\eta_y & \text{if } x\in x_{Ca}\end{cases}\Bigg)\Bigg] \quad (4.141)$$

Remark 4.39. *The λ-dependent part of $\rho_{\lambda}^{\pm,\eta}(\gamma)$ does not depend on the ensemble. This is also true for the pre-factors. It suggests that the restriction $\lambda \geq 5$ might not be necessary.*

From now on we restrict ourselves to the $+$-ensembles. We set $Z_{\Lambda,\lambda}^{\eta} \equiv Z_{\Lambda,\lambda}^{+,\eta}$ and $\rho^{\eta}(\Gamma) \equiv \rho^{+,\eta}(\Gamma)$. The $-$-ensemble we can treat in a way similar to the $-$-ensemble of the zero-field expansion.

4.12.2 Partitioning contour families

Because of the form of the weights of the contours Γ and γ we expand these two types separately. Note that the weights of the contours Γ do coincide with the corresponding weights of the zero-field expansion of the volume Λ^i with boundary condition η. Therefore we treat the contours Γ with the zero-field expansion for Λ^i.

Except for the contours $\gamma \in x_{Ca}$ we can expand every contour γ in a single step. This because the weights of the contours γ behave well-damped. By our assumption $\lambda \geq 5$

$$\rho_\lambda^\eta(\gamma) \leq \exp\left(-2\beta((\lambda-1)|\gamma \setminus (\gamma \cap x_{Ca})| - 2)\right) \leq \exp\left(-2\beta|\gamma|\right) \tag{4.142}$$

except for all $\gamma \in x_{Ca}$.

Remark 4.40. *Equation* (4.142) *shows us that for γ with $|\gamma| \geq 3$ or γ not containing any point of x_{Ca}*

$$\rho_\lambda^\eta(\gamma) \leq \exp\left(-2\beta|\gamma|/l_0\right) \textit{ whenever } (\lambda-1)|\gamma| \geq 1/l_0 + 2 \tag{4.143}$$

This means in particular that we can use the high field expansion for $\lambda \geq 3 + 1/l_0$. For intermediate λ (i.e. $1 < \lambda < 3 + 1/l_0$) the way of proceeding is not clear.

Because of the uniform decay of the contours in (4.142) we are allowed to cluster expand:

Proposition 4.41. *There is a constant c_5 such that for β large enough*

$$\sup_{x \in \underline{\partial\Lambda^i}} \sum_{\substack{C \in \mathfrak{C}_0^\eta \\ C \ni x}} |\phi_{0,\lambda}^\eta(C)| \exp\left(2\beta - c_5\right)|C| \leq 1 \tag{4.144}$$

To shorten notation we have defined

Definition 4.42. *The set $\mathfrak{C}_0^\eta$ is the union of all clusters C such that for all $\gamma \in C$: $\gamma \subset \underline{\partial\Lambda^i}$ and $\gamma \notin x_{Ca}$.*

Proof. The proof goes in a rather standard way. Because the contours are sets of sites which are adjacent to each other the entropy is limited.

$$\sup_{x \in \underline{\partial\Lambda^i}} \sum_{\gamma \ni x} \rho_\lambda^\eta(\gamma) \exp\left(2\beta - c_5 + 1\right)|\gamma| \leq \sum_{n=1}^{\infty} n \exp\left(1 - c_5\right)n \leq 1 \tag{4.145}$$

for c_5 large enough. Now apply Proposition 4.50 to obtain the Proposition. □

We put this cluster expansion into the partition function $Z^+_{\Lambda,\lambda}$ given by (4.138) and we obtain

$$Z^\eta_{\partial\Lambda,\lambda} = \exp\Big(\sum_{C\in\mathfrak{C}^\eta_0}\phi^\eta_{0,\lambda}(C)\Big)\sum_{\partial\subset x_{Ca}}\prod_{\gamma\in\partial}\rho^\eta_\lambda(\gamma)\exp\Big(\sum_{C\in\mathfrak{C}^\eta_0:\ C\not\sim\partial}\phi^\eta_{0,\lambda}(C)\Big) \tag{4.146}$$

Now we put the pre-factor at the left-hand site for notational convenience and we rewrite the above to

$$\exp\Big(-\sum_{C\in\mathfrak{C}^\eta_0}\phi^\eta_{0,\lambda}(C)\Big)Z^\eta_{\partial\Lambda,\lambda} = Z^\eta_{Ca}\Big[\frac{1}{Z^\eta_{Ca}}\sum_{\partial\subset x_{Ca}}\prod_{\gamma\in\partial}\rho^\eta(\gamma)\exp\Big(\sum_{C\in\mathfrak{C}^\eta_0:\ C\not\sim\partial}\phi^\eta_{0,\lambda}(C)\Big)\Big] \tag{4.147}$$

where

$$Z^\eta_{Ca} = \sum_{\partial\subset x_{Ca}}\prod_{\gamma\in\partial}\rho^\eta(\gamma)$$

We use the Mayer expansion as we did before in the $\lambda = 1$ boundary field case. To every family $\mathcal{C}\subset\mathfrak{C}^\eta_0$ we set the weight

$$w^\eta_\lambda(\mathcal{C}) = \frac{1}{Z^\eta_{Ca}}\sum_{\partial\subset x_{Ca}}\prod_{\gamma\in\partial}\rho^\eta(\gamma)\prod_{C\in\mathcal{C}}\Big(\exp\Big(\phi^\eta_{0,\lambda}(C)^{(1)}_{\{C\not\sim\partial\}}\Big)-1\Big) \tag{4.148}$$

Then

$$(4.147) = Z_{Ca}\sum_{\mathcal{C}\in\mathfrak{C}^\eta_0} w^\eta_\lambda(\mathcal{C}) \tag{4.149}$$

We define

Definition 4.43. *Any pair of clusters C_1 and C_2 of $\mathfrak{C}^\eta_0$ is called corner-incompatible, $C_1 \overset{Ca}{\not\leftrightarrow} C_2$, whenever there is a corner site x in x_{Ca} such that for the contour $x: x\not\sim C_1$ and $x\not\sim C_2$. And any pair of families of clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ is corner-incompatible, $\mathcal{C}_1 \overset{Ca}{\not\leftrightarrow} \mathcal{C}_2$, whenever there are clusters $C_1\in\mathcal{C}_1$, $C_2\in\mathcal{C}_2$ such that there is a corner site x in x_{Ca} so the contour $x: x\not\sim C_1$ and $x\not\sim C_2$.*

One can check the properties of the weight (compare lemma 4.23)

Lemma 4.44. *For any set of clusters $\mathcal{C}\in\mathfrak{C}^\eta_0$*

1. $\sup_\eta |w^\eta_\lambda(\mathcal{C})| \le \Big|\prod_{C\in\mathcal{C}}(e^{|\phi^\eta_{0,\lambda}(C)|}-1)\Big| \le 2\prod_{C\in\mathcal{C}}|\phi^\eta_{0,\lambda}(C)|$,
2. *If $\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2$ with $\mathcal{C}_1 \overset{Ca}{\leftrightarrow} \mathcal{C}_2$ then $w^\eta_\lambda(\mathcal{C}) = w^\eta_\lambda(\mathcal{C}_1)w^\eta_\lambda(\mathcal{C}_2)$.*

Equation (4.149) represents the partition function of a polymer model. The polymers are the sets of clusters $\mathcal{C} \in \mathfrak{C}_0^\eta$, which are incompatible if and only if they are corner-incompatible. We cluster expand this polymer model to obtain the second and final cluster expansion. We use $\mathfrak{D}_{Ca}^\eta$ for the set of all clusters in this polymer model and $\psi_{Ca}^\eta(D)$ for the weight of a cluster $D \in \mathfrak{D}_{Ca,\lambda}^\eta$. Then we get

$$\begin{aligned} (4.149) &= Z_{Ca}^\eta \exp\big(\sum_{D\in\mathfrak{D}_{Ca}^\eta} \psi_{Ca,\lambda}^\eta(D)\big), \\ Z_{\partial\Lambda}^\eta &= Z_{Ca}^\eta \exp\Big(\sum_{C\in\mathfrak{C}_0^\eta} \phi_{0,\lambda}^\eta(C)\Big) \exp\Big(\sum_{D\in\mathfrak{D}_{Ca}^\eta} \psi_{Ca,\lambda}^\eta(D)\Big) \end{aligned} \tag{4.150}$$

The cluster weights turn out to behave well damped. The corner partition function Z_{Ca}^η is well controlled:

Proposition 4.45. *There exist a constant c_5 (the same as in Proposition 4.41) such that for β large enough*

$$\sup_{x\in\underline{\partial\Lambda^i}} \sum_{\substack{D\in\mathfrak{D}_{Ca}^\eta \\ D\not\sim x}} |\psi_{Ca,\lambda}^\eta(D)| \exp\left(2\beta - c_5 - 6\right)|D| \leq 1 \tag{4.151}$$

Proof. Because of Lemma 4.44 we can use our generalized KP-criterion defined in Proposition 4.50 (2). This proposition tells us that whenever we can prove the following inequality

$$\sup_{C_0\in\mathfrak{C}_0^\eta} \sum_{\substack{C\not\sim C_0 \\ C\in\mathfrak{C}_0^\eta}} 2|\phi_{0,\lambda}(C)| \exp\left(2|C| + (2\beta - c_5 - 4)|C|\right) \leq |C_0| \tag{4.152}$$

automatically

$$\sum_{D\not\sim C_0} |\psi_{Ca,\lambda}^\eta(D)| \exp\left(2\beta - c_5 - 4\right)|D| \leq |C_0| \tag{4.153}$$

follows. For each point x there are only two contours $\gamma \ni x$ with $|\gamma| = 2$. When $D \not\sim x$ then D must be incompatible with at least one of these two contours. Then by using (4.152) the Proposition follows.

By Proposition 4.41 it holds:

$$\sup_{C_0\in\mathfrak{C}_0^\eta} \sum_{\substack{C\not\sim C_0 \\ C\in\mathfrak{C}_0^\eta}} |\phi_{0,\lambda}^\eta(C)| \exp\left(2\beta - c_5\right)|C| \leq |C_0| + 2 \tag{4.154}$$

making

$$\sup_{C_0\in\mathfrak{C}_0^\eta}\sum_{\substack{C\not\sim C_0\\ C\in\mathfrak{C}_0^\eta}} 2|\phi^\eta_{0,\lambda}(C)|\exp\left(2\beta-c_5-2\right)|C|\leq|C_0| \tag{4.155}$$

proving (4.152). □

Any cluster in $D\in\mathfrak{D}^\eta_{Ca}$ is corner incompatible. Because there are at most 4 corners in the set x_{Ca}, immediately it follows

Corollary 4.46.

$$\sum_{D\in\mathfrak{D}^\eta_{Ca}}|\psi^\eta_{Ca,\lambda}(D)|\exp\left(2\beta-c_5-6\right)|D|\leq 4$$

Proposition 4.47. *For the partition function Z^η_{Ca} it holds*

$$\sup_\eta Z^\eta_{Ca}\leq 2\exp\left(16\beta\right) \tag{4.156}$$

Proof. By (4.141) it holds for every contour $\gamma\in x_{Ca}:\rho^\eta(\gamma)\leq\exp 4\beta$. So

$$Z^\eta_{Ca}=\sum_{\partial\subset x_{Ca}}\prod_{\gamma\in\partial}\rho^\eta(\gamma)\leq\sum_{\partial\subset x_{Ca}}\exp\left(4\beta|\partial|\right)\leq 2\exp\left(16\beta\right) \tag{4.157}$$

for β large enough. □

Now we try to find a good estimate for the free energy difference. It turns out that the log-difference of the pre-factors appearing in (4.137) is of a convenient nature:

$$\begin{aligned}\log\mathrm{pref}^+_\lambda(\emptyset)-\log\mathrm{pref}^-_\lambda(\emptyset)=2\beta\Bigg[\sum_{\underline{\partial\Lambda^i}\setminus x_C}\eta_x+\sum_{\substack{y\in\underline{\partial\Lambda^i}\\ d(y,x_{Ca})=1}}\eta_y\Bigg]=\\ E^+_{\Lambda^i}(\emptyset)-E^-_{\Lambda^i}(\emptyset)+2\beta\sum_{\substack{y\in\underline{\partial\Lambda^i}\\ d(y,x_{Ca})=1}}\eta_y\end{aligned} \tag{4.158}$$

In fact they decompose nicely into the pre-factor difference of the log-differen-ce of the partition functions of the interior Λ^i and an additional part coming solely from the boundary $\underline{\partial\Lambda^i}$.

So it holds by (4.150), (4.158) and the above

$$\begin{aligned}\log Z^+_{\Lambda,\lambda}-\log Z^-_{\Lambda,\lambda}=\log Z^+_{\Lambda^i,1}-\log Z^-_{\Lambda^i,1}+2\beta\sum_{\substack{y\in\underline{\partial\Lambda^i}\\ d(y,x_{Ca})=1}}\eta_y+\log Z^+_{Ca}-\log Z^-_{Ca}\\ \sum_{C\in\mathfrak{C}_0^+}\phi^+_{0,\lambda}(C)-\sum_{C\in\mathfrak{C}_0^-}\phi^-_{0,\lambda}(C)+\sum_{D\in\mathfrak{D}^+_{Ca}}\psi^\eta_{Ca,\lambda}(D)-\sum_{D\in\mathfrak{D}^-_{Ca}}\psi^\eta_{Ca,\lambda}(D)\end{aligned} \tag{4.159}$$

Now we put in Corollary 4.46 and Proposition 4.47 to obtain

$$\log Z^+_{\Lambda,\lambda} - \log Z^-_{\Lambda,\lambda} = \log Z^+_{\Lambda^i,1} - \log Z^-_{\Lambda^i,1} + \sum_{C\in\mathfrak{C}^+_0} \phi^+_{0,\lambda}(C) - \sum_{C\in\mathfrak{C}^-_0} \phi^-_{0,\lambda}(C) + O(\beta) \tag{4.160}$$

For the clusters of $\mathfrak{C}^\pm_0$ we now need to define the domain.

Definition 4.48.

1. *The 'interior'* $\mathrm{Int}(\gamma)$ *of a contour* $\gamma \in \underline{\partial\Lambda^i}$ *equals the set of unit cubes which has as center points the sites of* γ*,*

2. *The domain* $\mathrm{Dom}(C)$ *of a cluster* $C \in \mathfrak{C}^\pm_0$ *equals* $\mathrm{Dom}(C) = \cup_{\gamma\in C} \mathrm{Int}(\gamma)\cap\partial\Lambda^i$.

Then we define $U^\eta(B)$ for a set $B \subset \partial\Lambda^i$ as in (4.83) but with $\phi^\eta_0(C)$ replaced by $\phi^\eta_0(C) + \phi^+_{0,\lambda}(C)$. In the same way we replace $\phi^{-\eta}_0(C)$ by $\phi^{-\eta}_0(C) + \phi^-_{0,\lambda}(C)$.

When we consider the upper bounds for both the cluster expansions we easily realize that still it holds:

$$|U^\eta_s(B)| = \Big| \sum_{\substack{C\in\mathfrak{C}^+_0:\\ \mathrm{Dom}(C)=B}} \phi^+_{0,\lambda}(C) + \sum_{\substack{C\in\mathfrak{C}^\eta_0:\\ \mathrm{Dom}(C)=B}} \phi^\eta_0(C) \Big| \le e^{-\beta|B|/l_0} \tag{4.161}$$

For the free energy we can make use of the characteristic function expansion. In this expansion we do not put the $O(\beta)$ term. When β is finite or not growing too fast compared to the volume size L, we can neglect it. All the properties of the zero-field do follow for the free energy. Because for $L \to \infty$ the boundary moves to infinity the nature of the high-field states is the same.

This shows that for $\lambda \ge 5$ the random boundary field is only a small perturbation of the case of infinite random boundary field where $\lambda = \infty$. When the volume goes to infinity the scenario of the high-field expansion turns into the scenario of the zero-field expansion. In this sense the Ising model with a random high boundary field is equivalent to the Ising model with ordinary random boundary conditions.

4.13 Concluding remarks and some open questions

Our result that a typical boundary condition (w.r.t. a symmetric distribution) suppresses both mixed and interface states explains why these states are typically not observed in experimental situations without a special preparation. To a certain extent it justifies the standard interpretation of extremal invariant Gibbs measures as pure phases.

Although this result, which finally solves the question raised in [63], is only about the 2-dimensional Ising ferromagnet, and thus seemingly of limited interest, it is our

opinion that the perturbation approach developed in this chapter is actually very robust (compare [39]). As we have observed at various points in this chapter, there seems to be no barrier except some technical ones to extend the analysis to the Ising model with random boundary conditions in higher dimensions. In fact, there might be extensions of our approach into various different directions.

Instead of Ising spins one can consider the same model with Potts spins. The set-up is a more general Pirogov-Sinai one in which the number of extremal Gibbs measures could be larger than two. The contour analysis still can be performed, but we need to add labels to the contours, representing the different spin-values in their exterior and their interior. We conjecture that the same results as for Ising model hold. Again there is chaotic size dependence, but now the measure oscillates between q different Gibbs-measures μ^q. These are generalizations of the Ising measures μ^+ and μ^-, having the same type of island structures.

One can consider a more general symmetric distribution of the random field, possibly with some weak dependence. This can serve as a model of a high-temperature environment. We conjecture that the additional interaction between the fields can also be analyzed by a similar expansion technique, leading to the same result.

One can also extend the model by taking non-symmetric distributions for the η's. In general we do not expect chaotic size dependence. However, can we choose a non-symmetric distribution such that there is still chaotic size dependence, but with a slight preference for one of the two extremal states μ^+ and μ^-?

Instead of nearest neighbor interactions one can consider (long-range) Kac-potentials. This needs a refinement of the definitions of the contours. Because of the long-range nature of the interaction, the spins tend to behave like the spins in the mean-field model. Now, the contours are formed by blocks of spins with size of order of the interaction, which behave in a significantly different way than in the mean-field case. After this blocking the model turns into a short-range model with a boundary field which is typically weak. Again large boundary fields do appear on all scales, so a multi-scale expansion is needed. A special interest in the Kac-model is due to the possibility of extending the contour analysis up to the critical temperature.

Another possible extension could be to finite-range Hopfield-type models, in which periodic or fixed boundary conditions lack a coherence property with respect to the possible Gibbs measures, and thus are expected to behave as random ones [69]. Actually, our result can be translated in terms of the Mattis (= single-pattern Hopfield) model with fixed boundary conditions, proving the chaotic size-dependence there.

More generally, in principle the phenomenon of the exclusion of interface states for typical boundary conditions might well be of relevance for spin glass models of Edwards-Anderson type, which has indeed been one of our main motivations. Our result illustrates in a simple way how the Newman-Stein metastate program, designed

for the models exhibiting the chaotic size-dependence, can be realized. The number of states, as well as the number of "physically relevant" states for short-range spin glasses has been an issue of contention for a long time. In this chapter, we have provided a very precise distinction between the set of all Gibbs states, the set of all extremal Gibbs measures, and the set of "typically visible" ones, without restricting *a priori* to the states with a particular symmetry. We hope the provided criterion might prove useful in a more general context.

We mention that the restriction to sparse enough sequences of volumes is essential to obtain almost sure results. Actually, for a regular sequence of volumes, we expect all mixtures (in dimension three all translation-invariant Gibbs measures) to be almost sure limit points, although in a null-recurrent way. This still would mean that the metastate would not be affected, and that it would be concentrated on the plus and minus measures. See also the discussion in [28]. However, proving this conjecture goes beyond the presented technique and remains an open question.

A different but very intriguing problem is to analyze the $d = 3$ random field Ising model with free or periodic boundary conditions. In order to have a phase transition the field has to be typically small enough. On the other hand, when the field takes typically high values, then the spins tend to follow the direction of the field and there is only one Gibbs state. The famous paper by Bricmont and Kupiainen [14] considers the cases of $+$ and $-$ boundary conditions. They show that the resulting Gibbs measures are different. We conjecture that for free boundary conditions the same type of chaotic size dependence as in the random Ising boundary field model does occur; on sparse enough sequences of volumes the Gibbs measure randomly oscillates between two infinite-volume Gibbs states μ^+ and μ^- with probability 1, for $\beta \geq \beta_0$ large enough. In [14] a delicate multi-scale block spin approach is used. We conjecture that our techniques, avoiding a systematic blocking procedure, can be used as well. Work in this direction is in progress.

4.14 Appendix on cluster models

In this section we present a variant of the familiar result on the convergence of the cluster expansion for polymer models, which proves useful in the cases when the summation over polymers becomes difficult because of their high geometrical complexity. Such a situation arises, for example, in the applications of the cluster expansion to the study of the convergence of high-temperature (Mayer) series in lattice models with an infinite-range potential. Since the Mayer expansion techniques are by no means restricted to the high-temperature regimes (note e.g. its application in the RG schemes for low-temperature contour models), the result below can be applied in a wide class of

problems under a perturbation framework. In our context, we use the result to provide upper bounds on the weights ψ_n^η of n-clusters, see Section 4.11.2.

We consider an abstract cluster model defined as follows. Let $G = (S, \nsim)$ be a finite or countable non-oriented graph and call its vertices *polymers*. Any two polymers $X \nsim Y$ are called *incompatible*, otherwise they are *compatible*, $X \sim Y$. By convention, we add the relations $X \nsim X$ for all $X \in S$. Any non-empty finite set $\Delta \subset S$ is called a *cluster* whenever there exists no decomposition $\Delta = \Delta_1 \cup \Delta_2$ such that Δ_1 and Δ_2 are non-empty disjoint sets of polymers and $\Delta_1 \sim \Delta_2$, where the latter means that $X \sim Y$ for all $X \in \Delta_1$ and $Y \in \Delta_2$. Let $\mathcal{P}(S)$ denote the set of all finite subsets of S and $\mathcal{C}(S)$ denote the set of all clusters. A function $g : \mathcal{P}(S) \mapsto \mathbb{C}$ is called a *weight* whenever

i) $g(\emptyset) = 1$,

ii) If $\Delta_1 \sim \Delta_2$, then $g(\Delta_1 \cup \Delta_2) = g(\Delta_1)\, g(\Delta_2)$.

If the extra condition

iii) $g(\Delta) = 0$ whenever there is an $X \in \Delta$ such that $X \nsim \Delta \setminus \{X\}$

holds true, then we obtain the familiar *polymer model*. In the sequel we do not assume Condition iii) to be necessarily true, unless stated otherwise.

Note a simple duality between the classes of polymer and cluster models: Any cluster model over the graph $G = (S, \nsim)$ is also a polymer model over the graph $G' = (\mathcal{C}(S), \nsim)$. The other inclusion is also trivially true. A natural application of this duality is to the polymer models with a complicated nature of polymers. Such polymers can often be represented as clusters in a new cluster model with the polymers being simpler geometric objects.

To any set $A \in \mathcal{P}(S)$ we assign the *partition function* $Z(A)$ by

$$Z(A) = \sum_{\Delta \subset A} g(\Delta) \tag{4.162}$$

The map between the functions g and Z is actually a bijection and the last equation can be inverted by means of the Möbius inversion formula. In particular, we consider the function $g^T : \mathcal{P}(S) \mapsto \mathbb{C}$ such that the Möbius conjugated equations

$$\log Z(A) = \sum_{\Delta \subset A} g^T(\Delta) \qquad g^T(\Delta) = \sum_{A \subset \Delta} (-1)^{|\Delta \setminus A|} \log Z(A) \tag{4.163}$$

hold true for all $A \in \mathcal{P}(S)$ and $\Delta \in \mathcal{P}(S)$, respectively. The function g^T is called a *cluster* weight, the name being justified by the following simple observation:

Lemma 4.49. *For any cluster model, $g^T(\Delta) = 0$ whenever Δ is not a cluster.*

A familiar result about the polymer model is the exponential decay of the cluster weight g^T under the assumption on a sufficient exponential decay of the weight g, see [50, 57]. We use the above duality to extend this result to the cluster models, formulating a new condition that can often be easily checked in applications.

Proposition 4.50. *Let positive functions $a, b : S \mapsto \mathbb{R}^+$ be given such that either of the following conditions is satisfied:*

1. ***(Polymer model)***
   Condition iii) is fulfilled and[3]
   $$\sup_{X\in S} \frac{1}{a(X)} \sum_{Y \nsim X} e^{(a+b)(Y)} |g(Y)| \leq 1 \tag{4.164}$$

2. ***(Cluster model)***
   There is $z : S \mapsto \mathbb{R}^+$ satisfying the condition
   $$\sup_{X\in S} \frac{1}{a(X)} \sum_{Y \nsim X} e^{(2a+b)(Y)} z(Y) \leq 1 \tag{4.165}$$
   such that $|g(\Delta)| \leq \prod_{X\in\Delta} z(X)$ for all $\Delta \in \mathcal{P}(S)$.

Then,
$$\sup_{X\in S} \frac{1}{a(X)} \sum_{\Delta \nsim X} e^{\sum_{Y\in\Delta} b(Y)} |g^T(\Delta)| \leq 1 \tag{4.166}$$

Proof. (1) For the case of the polymer models, see [50] or better [57] for the proof.
(2) To prove the statement for a cluster model, we represent it as a polymer model over the graph $(\mathcal{C}(S), \nsim)$ and make use of the above result. Hence, it is enough to show the inequality
$$\sum_{\substack{\Delta\in\mathcal{C}(S)\\ \Delta\nsim X}} e^{\sum_{Y\in\Delta}(a+b)(Y)} |g(\Delta)| \leq a(X) \tag{4.167}$$
for all $X \in S$. Indeed, then one gets
$$\sum_{\substack{\Delta\in\mathcal{C}(S)\\ \Delta\nsim \Delta_0}} e^{\sum_{Y\in\Delta}(a+b)(Y)} |g(\Delta)| \leq \sum_{Y\in\Delta_0} a(Y) \tag{4.168}$$
for all $\Delta_0 \in \mathcal{C}(S)$ and the statement about the polymer models yields
$$\sum_{\substack{\Delta^*\in\mathcal{C}(\mathcal{C}(S))\\ \Delta^*\nsim \Delta_0}} e^{\sum_{\Delta\in\Delta^*}\sum_{Y\in\Delta} b(Y)} |g^T(\Delta^*)| \leq \sum_{Y\in\Delta_0} a(Y) \tag{4.169}$$

[3] We use the convention $\frac{0}{0} = 0$ here.

where the sum on the LHS is over all clusters incompatible with Δ_0 in the polymer model with the set of polymers $\mathcal{C}(S)$. Since the weights $g^T(\Delta)$ of the clusters in the original cluster model are related to the cluster weights $g^T(\Delta^*)$ in the polymer model under consideration as

$$g^T(\Delta) = \sum_{\Delta^*:\, \cup_{\Delta' \in \Delta^*} \Delta' = \Delta} g^T(\Delta^*) \tag{4.170}$$

we immediately get

$$\sum_{\Delta \nsim X} e^{\sum_{Y \in \Delta} b(Y)} |g^T(\Delta)| \leq \sum_{\substack{\Delta^* \in \mathcal{C}(\mathcal{C}(S)) \\ \Delta^* \nsim X}} e^{\sum_{\Delta \in \Delta^*} \sum_{Y \in \Delta} b(Y)} |g^T(\Delta^*)| \leq a(X) \tag{4.171}$$

which is inequality (4.166).

Using the notation $\hat{z}(X) := z(X)\, e^{a(X)+b(X)}$ and

$$\mathfrak{Z}_X(A) = \sum_{\substack{\Delta \in \mathcal{C}(A) \\ \Delta \ni X}} \prod_{Y \in \Delta} \hat{z}(Y) \tag{4.172}$$

for any $A \in \mathcal{P}(S)$ and $X \in A$, inequality (4.167) follows from the next two lemmas. □

Lemma 4.51. *The function $\mathfrak{Z}_X(A)$ satisfies the recurrence inequality*

$$\mathfrak{Z}_X(A) \leq \hat{z}(X) \exp\Big[\sum_{\substack{Y \nsim X \\ Y \in A \setminus \{X\}}} \mathfrak{Z}_Y(A \setminus \{X\}) \Big] \tag{4.173}$$

Proof. For any cluster Δ we split $\Delta \setminus \{X\}$ into connected components, i.e. a family of clusters (Δ_j), and subsequently write:

$$\begin{aligned}
\mathfrak{Z}_X(A) &= \hat{z}(X) \sum_{\substack{\Delta \in \mathcal{C}(A) \\ \Delta \subset A \setminus \{X\}}} \prod_j \prod_{Y \in \Delta_j} \hat{z}(Y) \\
&\leq \hat{z}(X) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{Y_1, \dots, Y_n \in A \setminus \{X\} \\ \forall j:\, Y_i \nsim X}} \prod_{j=1}^{n} \sum_{\substack{\Delta_j \subset A \setminus \{X\} \\ \Delta_j \ni Y_j}} \prod_{Y \in \Delta_j} \hat{z}(Y) \\
&= \hat{z}(X) \sum_{n=0}^{\infty} \frac{1}{n!} \Big[\sum_{\substack{Y \nsim X \\ Y \in \mathcal{A} \setminus \{X\}}} \mathfrak{Z}_Y(A \setminus \{X\}) \Big]^n \\
&= \hat{z}(X) \exp\Big[\sum_{\substack{Y \nsim X \\ Y \in A \setminus \{X\}}} \mathfrak{Z}_Y(A \setminus \{X\}) \Big]
\end{aligned} \tag{4.174}$$

□

Lemma 4.52. *Assume that*

$$\sum_{Y \nsim X} \hat{z}(Y) e^{a(Y)} \leq a(X) \tag{4.175}$$

Then

$$\sum_{Y \nsim X} \mathfrak{Z}_Y(S) \leq a(X) \tag{4.176}$$

Proof. We prove the inequality

$$\mathfrak{Z}_X(A) \leq \hat{z}(X) e^{a(X)} \tag{4.177}$$

for all $A \in \mathcal{P}(S)$ and $X \in A$, by induction in the number of polymers in the set A. Assuming that this bound is satisfied whenever $|A| < n$, we can estimate $\mathfrak{Z}_X(A)$ for $|A| = n$ by using Lemma 4.51, condition (4.175), and the induction hypothesis as follows:

$$\mathfrak{Z}_X(A) \leq \hat{z}(X) \exp\Big[\sum_{Y \nsim X} \hat{z}(Y) e^{a(Y)}\Big] \leq \hat{z}(X) e^{a(X)} \tag{4.178}$$

As the statement is obvious for $|A| = 1$, the lemma is proven. □

4.15 Appendix on interpolating local limit theorem

We present here a simple general result that can be useful in the situations where a full local limit theorem statement is not available due to the lack of detailed control on the dependence among random variables the sum of which is under consideration. For a detailed explanation of the central and the local limit theorems as well as the analysis of characteristic functions in the independent case, see e.g. [23]. Here, under only mild assumptions, we prove an asymptotic upper bound on the probabilities in a regime that interpolates between the ones of the central and the local limit theorem. Namely, we have the following result that is a simple generalization of Lemma 5.3 in [28]:

Proposition 4.53. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and denote by $\psi_n(t)$ the corresponding characteristic functions, $\psi_n(t) = \mathbb{E}\, e^{itX_n}$. If $(A_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ are strictly positive sequences of reals satisfying the assumptions*

i) $\overline{\lim}_{n\to\infty} A_n \int_{-\tau_n}^{\tau_n} \mathrm{d}t |\psi_n(t)| \leq 2\pi$

ii) There is $k > 1$ such that $\lim_{n\to\infty} \frac{A_n}{\delta_n^k \tau_n^{k-1}} = 0$

then

$$\overline{\lim_{n\to\infty}} \frac{A_n}{\delta_n} \boldsymbol{P}\{a\delta_n \le X_n \le b\delta_n\} \le b-a \tag{4.179}$$

for any $a < b$.

Remark 4.54. *Note that:*

1. *Up to a normalization factor, Condition i) of the proposition only requires A_n to be chosen as*
$$A_n = \mathcal{O}\Big(\Big[\int_{-\tau_n}^{\tau_n} \mathrm{d}t\, |\psi_n(t)|\Big]^{-1}\Big) \tag{4.180}$$

2. *If there is ε_1 such that $A_n\tau_n \le n^{\varepsilon_1}$ eventually in n, then Assumption ii) of the proposition is satisfied whenever $\delta_n\tau_n \ge n^{\varepsilon_2}$ with a constant $\varepsilon_2 > 0$.*

3. *The choice $\delta_n = A_n$ (if available) gives an upper-bound on the probabilities in the regime of the central limit theorem. On the other hand, $\delta_n = $ const corresponds to the regime of the local limit theorem. However, for the latter choice it can be difficult to check the assumptions, and that is why one has to allow for a sufficient scaling of δ_n, see Part (2) of this remark.*

4. *Much more information about the distribution of the random variables X_n would be needed in order to get any* lower bounds *on the probabilities (except for the case $\tau_n = \infty$ in which a full local limit theorem can be proven). This is a hard problem that we do not address here.*

Proof. Let sequences $(A_n), (\tau_n), (\delta_n)$ be given such that the assumption of the proposition is true and take an arbitrary positive function $h \in C^\infty(\mathbb{R})$ for which i) $h(x) = 0$ for any $x \notin (-\varepsilon, \varepsilon)$ and ii) $\int_{-1}^{1} \mathrm{d}x\, h(x) = 1$. Using the notation G_n for the distribution function of X_n, we consider its 'regularized version' $\bar{G}_n$ defined by the Lebesgue density

$$\frac{\mathrm{d}\bar{G}_n(x)}{\mathrm{d}x} = \int_{-\infty}^{\infty} \mathrm{d}G_n(y)\, h_n(x-y) \tag{4.181}$$

where $h_n(x) := \frac{1}{\delta_n} h(\frac{x}{\delta_n})$. Obviously, $\frac{\mathrm{d}\bar{G}_n}{\mathrm{d}x} \in C^\infty(\mathbb{R})$ and it can be expressed by the Fourier integral as follows:

$$\begin{aligned} \frac{\mathrm{d}\bar{G}_n(x)}{\mathrm{d}x} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-itx} \psi_n(t) \int_{-\infty}^{\infty} \mathrm{d}y\, e^{ity} h_n(y) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-itx} \psi_n(t)\, \hat{h}(t\delta_n) \end{aligned} \tag{4.182}$$

where $\hat{h}(t) := \int_{-\infty}^{\infty} \mathrm{d}x\, e^{itx} h(x)$ and we have used that $\psi_n(t)\, \hat{h}(t\delta_n) \in L^1(\mathbb{R})$ following from Assumption i) of the proposition and from the bounds $|\psi_n(t)|$,$|\hat{h}(t)| \leq 1$. Moreover, if $k > 1$ is such that Assumption ii) holds, then, using the bound $|\hat{h}(t)| \leq c|t|^{-k}$ which is true with some constant c for all $t \in \mathbb{R} \setminus \{0\}$, we obtain the estimate

$$\begin{aligned}\overline{\lim_{n\to\infty}} \sup_x A_n \frac{\mathrm{d}G_n(x)}{\mathrm{d}x} &\leq \frac{1}{2\pi} \overline{\lim_{n\to\infty}} \Big(A_n \int_{-\tau_n}^{\tau_n} \mathrm{d}t\, |\psi_n(t)| + \int_{\mathbb{R}\setminus[-\tau_n,\tau_n]} \mathrm{d}t\, |\hat{h}(t\delta_n)| \Big) \\ &\leq 1 + \frac{1}{\pi} \frac{c}{k-1} \overline{\lim_{n\to\infty}} \frac{A_n}{\delta_n^k \tau_n^{k-1}} = 1\end{aligned} \tag{4.183}$$

Finally, by using the inequality

$$\begin{aligned}\boldsymbol{P}\{a\delta_n \leq X_n \leq b\delta_n\} &= \int_{a\delta_n}^{b\delta_n} \mathrm{d}G_n(y) \int_{-\infty}^{\infty} \mathrm{d}x\, h_n(x-y) \\ &\leq \int_{(a-\varepsilon)\delta_n}^{(b+\varepsilon)\delta_n} \mathrm{d}x \int_{a\delta_n}^{b\delta_n} \mathrm{d}G_n(y)\, h_n(x-y) \\ &\leq \int_{(a-\varepsilon)\delta_n}^{(b+\varepsilon)\delta_n} \mathrm{d}\bar{G}_n(x)\end{aligned} \tag{4.184}$$

we get

$$\overline{\lim_{n\to\infty}} \frac{A_n}{\delta_n} \boldsymbol{P}\{a\delta_n \leq X_n \leq b\delta_n\} \leq (b-a+2\varepsilon)\, \overline{\lim_{n\to\infty}} \sup_x A_n \frac{\mathrm{d}G_n(x)}{\mathrm{d}x} \leq b-a+2\varepsilon \tag{4.185}$$

and the proposition follows by taking the limit $\varepsilon \to 0$. □

Appendix A

Cluster expansions for Ising-type models

In this appendix we present more background information about cluster expansions. The cluster expansion is a frequently used perturbation expansion in statistical mechanics. There is a vast amount of literature, for reviews see [10, 17, 30, 49, 72, 75] and many others. First we look at high temperature models and introduce the Mayer expansion which is an example of a high temperature cluster expansion. Then we look at the low temperature Ising model for which the objects in the cluster expansion are contours. Together with this expansion we review the more general setting of the expansion into polymers. Then we introduce multi-scale methods, which enlarge the area to which the single cluster expansion is restricted.

A.1 High temperature results

A.1.1 1D Ising model by Mayer expansion

Consider the 1D Ising model without boundary field and with free boundary conditions. It has as Hamiltonian:

$$-\beta H_N = -\beta \sum_{i=1}^{N-1} (\sigma_i \sigma_{i+1} - 1) \tag{A.1}$$

As we see from (A.1) the interactions are only between nearest neighboring sites. We assume β is small so that $T = 1/\beta$ is large. Every set of broken bonds corresponds uniquely to two flip-related configurations σ and $-\sigma$. We set Λ_b equal to the set of bonds $b = (i, i+1)$:

$$\Lambda_b = \{(i, i+1),\ 1 \leq i \leq N-1\} \tag{A.2}$$

The partition function becomes

$$Z_N = 2 \sum_{B \subset \Lambda_b} \prod_{b \in B} \exp(-2\beta) \tag{A.3}$$

We expand the partition function by the Mayer expansion technique. This technique is represented by the following identity:

$$\sum_{\mathscr{A} \subset \Lambda} \prod_{A \in \mathscr{A}} (\exp f_\beta(A) - 1) = \exp\Big(\sum_{A \subset \Lambda} f_\beta(A) \Big) \tag{A.4}$$

The symbol $\mathscr{A}$ represents a family of mutually different subsets $A \subset \Lambda$. One uses the Mayer expansion often for high temperature expansions [72]. When these expansions are to converge, it is needed that the absolute values of the functions $f_\beta(A)$ become small for the desired values of β. When $d = 1$, due to analytic continuation, it gives a solution everywhere. The expansion is an example of a cluster expansion. We have used a variant of it in Chapter 4 for high β.

To apply this technique we first rewrite (A.3) into the form of (A.4):

$$Z_N = 2 \sum_{B \subset \Lambda_b} \prod_{b \in B} \Big(\exp(\log(e^{-2\beta} + 1)) - 1 \Big) \tag{A.5}$$

When we compare (A.5) with (A.4) we see the following relations. The set $\Lambda \equiv \Lambda_b$ and the subsets $A \equiv \{b\}$, where b is a bond of Λ_b. The sets $\mathscr{A}$ are subsets $B \subset \Lambda_b$. The function $f_\beta(A) \equiv f(\{b\}) = \log(e^{-2\beta} + 1)$ for all bonds $b \in \Lambda_b$.

Now we apply the Mayer expansion (A.4) to (A.5) and obtain

$$Z_N = 2 \exp\Big(\sum_{b \in \Lambda_b} \log(e^{-2\beta} + 1) \Big) = 2 \Big(e^{-2\beta} + 1 \Big)^{N-1} \tag{A.6}$$

So the pressure equals

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N = \log(e^{-2\beta} + 1) = -\beta + \log(2\cosh(\beta)) \tag{A.7}$$

A.1.2 Uniqueness of Gibbs measure for high temperature or $d = 1$

For more general models one always need some regularity of the interactions.

Definition A.1. *An interaction Φ is called a regular interaction whenever*

$$\forall x \in \mathbb{Z}^d : \ \sum_{A \ni x} \|\Phi_A\|_\infty \leq C < \infty \tag{A.8}$$

For a subclass of these interactions the following theorem does hold:

Theorem A.2 (Dobrushin uniqueness criterion). *Assume Φ is a regular interaction. Whenever it moreover holds:*

$$\sup_{x\in\mathbb{Z}^d}\sum_{A\ni x}(|A|-1)\|\Phi_A(\sigma)\|_\infty<\beta^{-1} \tag{A.9}$$

then there is only one limiting infinite-volume Gibbs measure.

So for regular interactions, when the temperature β^{-1} is high enough, there is always an unique Gibbs measure.

Remark A.3. *The original uniqueness criterion is stated for slightly more general interactions. It can also be used for getting more information about the unique Gibbs measure, e.g. its decay of correlations, which goes like the potential. For all of this and more see [10, 21, 72].*

Corollary A.4. *Assume $\beta<1/2d$. Then for the d-dimensional Ising model the Gibbs measure is unique irrespectively of the 1-site interactions.*

Proof. In (A.9) only sets A are summed with $|A|\geq 2$. With this condition the interactions of the Ising model only cover bonds $\{i,j\}$ for which $\Phi_{\{i,j\}}\leq 1$. Then inequality (A.9) turns into our assumption, where $2d$ is the number of nearest neighbors. We apply Theorem A.2 to obtain the result. □

For one-dimensional translational-invariant interactions the criterion for uniqueness turns out to be easier:

Proposition A.5. *Assume Φ is a regular translational-invariant interaction on $\mathbb{Z}$. Then for having an unique Gibbs measure at every $\beta<\infty$ it is sufficient that*

$$\sum_{\substack{A\ni 0\\ A\subset\mathbb{Z}}}\frac{|\phi_A|diam\,(A)}{|A|}<\infty \tag{A.10}$$

Corollary A.6. *The 1 dimensional Ising model* (A.1) *has an unique limiting Gibbs measure for $\beta<\infty$.*

Proof. From (A.1)

$$-\beta H_N(\sigma)=-\beta\sum_{A\subset\{1,\cdots,N\}}\Phi_A(\sigma) \tag{A.11}$$

with

$$\Phi_A(\sigma) = \begin{cases} \sigma_i\sigma_{i+1} - 1 & \text{if } A = \{i, i+1\}, \\ 0 & 0 \text{ otherwise} \end{cases} \tag{A.12}$$

In particular Φ_A is translational invariant. Without losing generality we assume N is odd. Then we can translate uniquely every set A point-wise by transformation $\tau : \tau(i) = i - (N+1)/2$. The set $\mathcal{N} = \{1, \cdots, N\}$ translates to $\tau\mathcal{N} = \{-(N-1)/2, \cdots, 0, \cdots, (N-1)/2\}$. Now we are allowed to apply the Dobrushin uniqueness criterion in the above Proposition for the interactions Φ_A.

$$\sum_{A \ni i} \frac{|\Phi_A(\sigma)|\text{diam}\,(A)}{|A|} \leq \sum_{\tau A \ni 0} |\Phi_{\{-1,0\}}(\sigma)| + |\Phi_{\{0,1\}}(\sigma)| \leq 4 \tag{A.13}$$

so the Gibbs measure is unique. □

A.1.3 High temperature polymer expansions

The above theorems do not state whether or not the free energy is analytic or C^k. When we restrict to interactions with stronger decay properties it holds (e.g. [72])

Theorem A.7. *Assume $\Phi(.)$ is translation invariant and*

$$\sup_x \sum_{A \ni x} |A|^{k-1} \|\Phi(A)\|_\infty < \infty \tag{A.14}$$

Then the corresponding free energy is C^k in the region of (A.9)

When the interactions $\Phi(A)$ have an exponential decay in size $|A|$ the pressure is analytic and allows for a so-called convergent polymer expansion.

Theorem A.8. *Assume $\Phi(.)$ is translation invariant and that there is a $r > 0$ such that*

$$\sup_x \sum_{A \ni x} \beta e^{r|A|} \|\Phi(A)\|_\infty < e^r \ln\left(\frac{2}{1 + e^{-r}}\right) \tag{A.15}$$

Then the corresponding free energy is analytic in the region of (A.9) *and so are the correlation functions.*

Having (A.9) is no guarantee for having analyticity. There are examples of non-translation invariant interactions Φ for which $|\Phi(.)|_\infty$ is such that Theorem A.2 does hold, but for which the free energy is not analytic (not even defined). One needs to be

careful which interactions are taken into account [25]. The class defined by the regular interactions is too wide to state general results about analyticity.

In high temperature spin systems the spins behave almost independently. Now we consider the Hamiltonian defined by a more general potential $\Phi^\eta_\Lambda(A,\sigma_\Lambda)$ (free after [60]). As boundary condition we have η defined on Λ^c. We define

Definition A.9. *The support* $\mathrm{Supp}(\Phi^\eta_\Lambda(A))$ *is the union of the sets* A *for which* $\Phi^\eta_\Lambda(A,\sigma_\Lambda)\neq 0$ *for at least one configuration* σ_Λ.

The corresponding expansion is an expansion around the uniform measure over the spin configurations. For β small the proper normalization constant is different from the low temperature expansion. When we take $\beta\to 0$, the Gibbs measure becomes equal to the mentioned uniform measure. So the proper normalization is to put the partition function $Z^\eta_\Lambda=1$ for $\beta=0$. The partition function is equal to

$$Z^\eta_\Lambda=\frac{1}{2^{|\Lambda|}}\sum_{\sigma_\Lambda}\exp\Big(-\beta\sum_{A\cap\Lambda\neq\emptyset}\Phi^\eta_\Lambda(A,\sigma_\Lambda)\Big) \tag{A.16}$$

Indeed, when we take $\beta\to 0$ in (A.16) it turns into the partition function of the product measure over the spins, which is equal to one. Now we see that the Mayer expansion is in particular very useful for high temperature expansions. When we put (A.4) into (A.16) we obtain

$$Z^\eta_\Lambda=\frac{1}{2^{|\Lambda|}}\sum_{\sigma_\Lambda}\Big(e^{-\beta\sum_{A\cap\Lambda\neq\emptyset}\Phi^\eta_\Lambda(A,\sigma_\Lambda)}-1\Big)+1=\frac{1}{2^{|\Lambda|}}\sum_{\sigma_\Lambda}\sum_{\partial\subset\mathrm{Supp}\,\Phi^\eta_\Lambda}\prod_{A\in\partial}\Big(e^{-\beta\Phi^\eta_\Lambda(A,\sigma_\Lambda)}-1\Big) \tag{A.17}$$

This is the proper setting for a polymer expansion. If we rewrite (A.17) we see

$$Z^\eta_\Lambda=\sum_{\partial\subset\mathrm{Supp}\,\Phi^\eta_\Lambda}w^\eta_\Lambda(\partial) \tag{A.18}$$

with

$$w^\eta_\Lambda(\partial)=\begin{cases}1 & \text{if } \partial=\emptyset\\ \frac{1}{2^{|\Lambda|}}\sum_{\sigma_\Lambda}\prod_{A\in\partial}\Big(e^{-\beta\Phi^\eta_\Lambda(A,\sigma_\Lambda)}-1\Big) & \text{otherwise}\end{cases} \tag{A.19}$$

Note that $w^\eta_\Lambda(\partial)$ can be negative. Furthermore, because $\partial\subset\mathrm{Supp}\,\Phi^\eta_\Lambda$, it is decomposable into subsets A_i such that for all $i\neq j$: $d(A_i,A_j)\geq 1$ and $w^\eta_\Lambda(\partial)=\prod_i w^\eta_\Lambda(A_i)$. This makes the above a polymer model, where the polymers are connected sets of sites in $\mathrm{Supp}(\Phi^\eta_\Lambda)$. When the interactions $\beta\Phi^\eta_\Lambda(A,\sigma_\Lambda)$ are exponentially decaying in $|A|$

and β is small enough, the polymer expansion and therefore the free energy becomes analytic uniformly for such a boundary condition η. Because $e^{-x}-1 \leq 2|x|$ for $|x| \leq 1$

$$|w_\Lambda^\eta(\partial)| \leq 2 \prod_{A \in \partial} \beta |\Phi_\Lambda^\eta(A)|, \text{ with } |\Phi_\Lambda^\eta(A)| = \sup_{\sigma_\Lambda} |\Phi_\Lambda^\eta(A, \sigma_\Lambda)| \tag{A.20}$$

Now we can apply the generalized Kotecký-Preiss Proposition 4.50 (2), which we have proven in Section 4.14. By this we prove the next proposition.

Proposition A.10. *There exists a $\tau_0 > 0$ such that if we assume*

$$\beta |\Phi_\Lambda^\eta(A)| \leq \exp\left(-\tau |A|\right), \ \tau > \tau_0 \tag{A.21}$$

then

$$\sup_{A_0 \subset \mathrm{Supp}(\Phi_\Lambda^\eta)} \sum_{\mathcal{C} \not\sim A_0} e^{(\tau - \tau_0)|\mathcal{C}|} |\phi_\Lambda^\eta(\mathcal{C})| \leq |A_0| \tag{A.22}$$

Proof. The condition needed for the generalized Kotecký-Preiss Proposition reads

$$\sup_{A_0 \subset \mathrm{Supp}(\Phi_\Lambda^\eta)} \sum_{A \not\sim A_0} e^{(2+b)|A|} \beta |\Phi_\Lambda^\eta(A)| \leq \frac{1}{2} |A_0| \tag{A.23}$$

Then if we take $b = \tau - c_1 d - 4 = \tau - \tau_0 - 2$

$$\sum_{A \not\sim A_0} e^{(\tau - \tau_0)|A|} \beta |\Phi_\Lambda^\eta(A)| \leq \sum_{x \in A_0} \sum_{A \text{ connected: } A \ni x} e^{-\tau_0 |A|} \tag{A.24}$$

Because

$$\#\{A \text{ connected: } A \ni x, \ |A| = n\} \leq e^{c_1 d n} \tag{A.25}$$

and

$$\#\{x \in A^c, \ x \not\sim A\} \leq 2d|A| \tag{A.26}$$

if we choose $\tau_0 = (c_1 + \ln 2)d + 2$, then it holds:

$$\sum_{A \not\sim A_0} e^{(\tau - \tau_0)|A|} \beta |\Phi_\Lambda^\eta(A)| \leq 2d|A_0| \sum_{n=1}^{\infty} e^{-(2 + d \ln 2)n} \leq \frac{1}{2} |A_0| \tag{A.27}$$

i.e. the needed condition (A.23). Now we apply Proposition 4.50, the generalized KP (2), to condition (A.23) to obtain the result. □

Now we combine Theorem A.8 and above Proposition A.10 and obtain

Corollary A.11. *Assume Φ such that for all η and Λ*

$$\beta|\Phi^{\eta}_{\Lambda}(A)| \leq \exp\left(-\tau|A|\right),\ \tau \geq \tau_0 + 1 \tag{A.28}$$

then the Gibbs measure is unique. The free energy is analytic in the region of (A.9)*, and so are the correlation functions. The corresponding partition function is given by the polymer expansion of which the cluster weights satisfy the bounds given by Proposition A.10.*

Proof. The condition (A.28) of the corollary is far in the region of (A.9), so the Gibbs measure is unique by Theorem A.2. Furthermore Proposition A.10 does hold, giving the convergent polymer expansion. Now use condition (A.23) which is also true for any $x \in \mathbb{Z}^d$ if we take $A_0 = \{x\}$. This makes that condition (A.15) does hold. We apply Theorem A.15 to obtain the analyticity of the free energy and the correlation functions. □

A.2 Low temperature expansions of 2D Ising model

A.2.1 Upper bound on the pressure by cluster expansion

As an another example we apply the cluster expansion techniques of the previous chapter to the zero-field $2d$ Ising model with $+$ boundary conditions, at low T. The Hamiltonian equals

$$-\beta H_{\Lambda}(\sigma) = -\beta \left(\sum_{<x,y>\subset\Lambda} (\sigma_x\sigma_y - 1) + \sum_{<x,y>:\ x\in\Lambda,\ y\in\Lambda^c} (\sigma_x - 1) \right) \tag{A.29}$$

Note that we sum now over $\sigma_x\sigma_y - 1$ instead of $\sigma_x\sigma_y$. It turns out to be convenient for low temperature expansions. This shifts the energy by a constant, physically unobservable. In the Gibbs measure the constant also appears in the partition function, so it cancels. The partition function Z^{+}_{Λ} is obtained by the sum over all possible configurations σ. The boundary of Λ we define as

Definition A.12. *For a set $\Lambda \subset \mathbb{Z}^2$ the boundary $\partial\Lambda \subset \mathbb{Z}^{2\star}$ equals*

$$\partial\Lambda = \{< x, y >^{\star}:\ x \in \Lambda,\ y \in \Lambda^c\} \tag{A.30}$$

An equivalent way of writing the partition function Z^{+}_{Λ} is to rewrite it into a sum over sets of non intersecting contours (see Chapter 2).

We denote $\mathcal{K}$ as the set of all contours $\gamma \subset \mathbb{Z}^{2\star}$. With this we build the graph $G = (\mathcal{K}, \not\sim)$. The vertices are formed by the contours γ. Between every pair of vertices

γ and γ' there is an edge whenever $\gamma \not\sim \gamma'$. We say that $\gamma \not\sim \gamma'$ whenever γ touches γ'. Because of the rounding corner procedure the intersection set $(\gamma \cup \mathrm{Int}(\gamma)) \cap (\gamma' \cup \mathrm{Int}(\gamma'))$ is either empty or contains at least one dual bond. In other words $\gamma \not\sim \gamma'$ whenever the set of the two contours $\{\gamma, \gamma'\}$ does not correspond to any spin-configuration σ.

Then we put $\mathcal{P}(\mathcal{K})$ as the set of all finite subsets of $\mathcal{K}$. For a finite set of sites $\Lambda \subset \mathbb{Z}^2$ the set $\mathcal{K}_\Lambda \in \mathcal{P}(\mathcal{K})$ is the set of all closed contours $\gamma \subset \Lambda^\star \cup \partial\Lambda$. This is because of our chosen uniform $+$ boundary conditions. The set $\mathscr{D}_\Lambda$ is the set of all compatible families of contours out of $\mathcal{K}_\Lambda$. A family ∂ of contours γ is compatible whenever

$$\text{for all } \gamma \in \partial: \quad \text{every } \gamma' \in \partial \setminus \gamma \text{ has } \gamma' \sim \gamma \tag{A.31}$$

Or equivalently: the subgraph $(\partial, \not\sim)$ of $G = (\mathcal{K}, \not\sim)$ only consists of isolated vertices.

With these notions we rewrite the partition function Z^+_Λ:

$$Z^+_\Lambda = \sum_{\partial \subset \mathcal{K}_\Lambda} w_I(\partial) = \sum_{\partial \in \mathscr{D}_\Lambda} \prod_{\gamma \in \partial} \exp\left(-2\beta|\gamma|\right) \tag{A.32}$$

It is the sum over the weights $w_I(\partial)$ of every set of contours ∂, where we have defined the weight $w_I(.)$ as follows:

1. $w_I(\emptyset) = 1$,
2. $w_I(\gamma) = \exp\left(-2\beta|\gamma|\right)$,
3. $w_I(\partial) = \prod_{\gamma \in \partial} w_I(\gamma)$ whenever ∂ compatible,
4. $w_I(\partial) = 0$ whenever ∂ not compatible.

This representation (A.32) is an example of a so called polymer model. The contours γ form the polymers with the weight $w_I(\gamma)$. For the more general case we refer to Section 4.14.

To obtain the thermodynamic properties of the model one needs to calculate or at least get some bounds on the free energy F. The limiting free energy is defined as

$$F^+_{I,\Lambda} = \lim_{\Lambda \to \infty} \frac{\log Z^+_{I,\Lambda}}{|\Lambda|} \tag{A.33}$$

We see that it is convenient to rewrite the partition function $Z^+_{I,\Lambda}$ into a product of objects for which we have good bounds. This we can do by the so called cluster expansion and we obtain a well controlled expansion for the $\log Z^+_{I,\Lambda}$. The Mayer expansion is another example of a cluster expansion.

Note that in the partition function Z^+_Λ (A.32) every set of polymers appears only once in the sum. There is an equivalent way of cluster expanding where the order of polymers does matter. Every set of n polymers is counted $n!$ times. Then in the resulting clusters every polymer can appear more than once [17, 30, 72]. For an equivalent analytic approach but with a different language see [21]. For some extensions see [13]. Our way of cluster expanding is due to Kotecký and Preiss [50] and reduces a lot of combinatorics.

When we cluster expand equation (A.32) for Z^+_Λ it results into

$$\log Z^+_{I,\Lambda} = \sum_{C \in \mathfrak{C}_\Lambda} \phi^+(C) \tag{A.34}$$

The weights $\phi^+(C)$ we call cluster weights. They are given by the Möbius inversion formula

$$\phi^+(C) = \sum_{\partial \subset C} (-1)^{|C \setminus \partial|} \log \Big(\sum_{\substack{\partial' \subset \partial \\ \partial' \text{ compatible}}} \prod_{\gamma \in \partial'} \exp\left(-2\beta|\gamma|\right) \Big) \tag{A.35}$$

The set $\mathfrak{C}_\Lambda$ is the set of all clusters from $\mathcal{K}_\Lambda$. Clusters C here are sets of contours such that every contour in it is incompatible with at least one other contour from the cluster. Furthermore any cluster C does not contain two or more mutual compatible clusters.

The clusters can be of any shape. Some are like a chain: they contain many contours and are very long but not very wide. Others are more like a ball: they consist of a bunch of contours which are not going far from each other.

It is easy to see that

Lemma A.13.

$$\phi^+(C) = 0, \textit{ whenever } C \textit{ is not a cluster}$$

Proof. Note that the number of $\partial \subset C$ is even (we also count $\partial = \emptyset$ and $\partial = C$). So

$$\sum_{\partial \subset C} (-1)^{|C \setminus \partial|} = 0 \tag{A.36}$$

This makes $\phi^+(C)$ zero whenever C is not a cluster; if C is not a cluster then $C = C_1 \cup C_2$ with $C_1 \sim C_2$. By the Möbius inversion formula

$$\phi^+(C) = \sum_{\partial_1 \subset C_1} (-1)^{|C_1 \setminus \partial_1|} \sum_{\partial_2 \subset C_2} (-1)^{|C_2 \setminus \partial_2|} \left(\log Z(\partial_1) + \log Z(\partial_2)\right) \tag{A.37}$$

where

$$\log Z(\partial_i) = \log \Big(\sum_{\substack{\partial' \subset \partial_i \\ \partial' \text{ compatible}}} \prod_{\gamma \in \partial'} \exp\left(-2\beta|\gamma|\right) \Big) \tag{A.38}$$

Using (A.36)

$$\sum_{\partial_2 \subset C_2} (-1)^{|C_2 \setminus \partial_2|} \log Z(\partial_1) = \log Z(\partial_1) \sum_{\partial_2 \subset C_2} (-1)^{|C_2 \setminus \partial_2|} = 0 \tag{A.39}$$

because of independence on ∂_1. Then

$$\phi^+(C) = \text{(A.37)} = \sum_{\partial_2 \subset C_2} (-1)^{|C_2 \setminus \partial_2|} \sum_{\partial_1 \subset C_1} (-1)^{|C_1 \setminus \partial_1|} \log Z(\partial_2) = 0 \tag{A.40}$$

□

For estimates on the cluster weights $\phi^+(C)$ we need to fulfill the Kotecký-Preiss criterion. Then there is an uniform exponential bound. In our case it reads basically

Proposition A.14. *Suppose that for a positive function $f(\beta)$ it holds*

$$\sup_{\gamma_0 \in \mathcal{K}_\Lambda} \sum_{\gamma \not\sim \gamma_0} \exp\left[(f(\beta)+1)|\gamma|\right] \exp\left(-2\beta|\gamma|\right) \leq |\gamma_0| \tag{A.41}$$

Then

$$\sup_{\gamma_0 \in \mathcal{K}_\Lambda} \sum_{\substack{C \in \mathfrak{C} \\ C \not\sim \gamma_0}} \exp\left(f(\beta)|C|\right) |\phi^+(C)| \leq |\gamma_0| \tag{A.42}$$

We have put $a(\gamma) \equiv |\gamma|$ and $f(\beta)|\gamma| \equiv b(\gamma)$. Finally we have set $\sum_{\gamma \in C} b(\gamma) = \sum_{\gamma \in C} f(\beta)|\gamma| = f(\beta)|C|$. For the more general case we refer to Proposition 4.50 part (1).

Using this Proposition it holds

Corollary A.15. *For β large enough there exists a constant c_0 such that for any $x \in \Lambda$*

$$\sup_{\gamma_0 \in \mathcal{K}_\Lambda} \sum_{\substack{C \in \mathfrak{C} \\ \text{Int}(C) \ni x}} \exp\left[(2\beta - c_0)|C|\right] |\phi^+(C)| \leq 1 \tag{A.43}$$

Proof. By using Lemma 4.11 we prove this Proposition in the same way as we have proven Proposition 4.12. For any constant c_0 it holds:

$$\sup_{x^\star \in \Lambda^\star} \sum_{\gamma \ni x^\star} \exp\left(2\beta - c_0 + 1\right) \exp\left(-2\beta|\gamma|\right) \leq \sum_{n=4}^{\infty} \exp\left[-(c_0 - c_1 - 1)n\right] \leq \frac{1}{4} \tag{A.44}$$

when $c_0 > c_1 + 2$. Because the number of dual bonds of a contour equals it length the condition does hold for applying Proposition A.14. Now choose γ_0 as the contour with $|\gamma_0| = 4$ and $\mathrm{Int}(\gamma_0) \ni x$. Because γ_0 contains 4 dual bonds the corollary follows by above Proposition A.14. □

A.2.2 Site percolation

Now consider the following model on the lattice $\mathbb{Z}^2$. Take the finite set Λ. Associate with every site $i \in \Lambda$ the unit cube with site i in its center. We denote this unit cube also by i. To every cube i there is a variable σ_i which can be 0 or 1. Now we denote by A the set $A = \{i \in \Lambda : \ \sigma_i = 1\}$. Every configuration σ has an unique corresponding set A. The weight of σ we put equal to $w_s(A) = \exp(-8\beta|A|)$ and $w_s(\emptyset) = 1$. Note that this model is equivalent to independent site-percolation on Λ with $p = \exp(-8\beta)$.

The resulting partition function we obtain by summing over the weights of all subsets of Λ. This is equivalent with the sum over all possible configurations σ. It is equal to

$$Z_{s,\Lambda} = \sum_{A \subset \Lambda} w_s(A) = \sum_{A \subset \Lambda} \prod_{x \in A} \exp(-8\beta) = \sum_{A \subset \Lambda} \exp(-8\beta|A|) \tag{A.45}$$

Now we expand the partition function $Z_{s,\Lambda}$ by the Mayer expansion technique. Using (A.4) in (A.45) we obtain

$$Z_{s,\Lambda} = \sum_{A \subset \Lambda} \prod_{x \in A} \Big(\exp(\log(e^{-8\beta}+1) - 1 \Big) = \exp\Big(\sum_{x \in \Lambda} \log(e^{-8\beta}+1) \Big) = \Big(1 + e^{-8\beta}\Big)^{|\Lambda|} \tag{A.46}$$

Polymer representation

The above model has a polymer representation on $\mathbb{Z}^2$. The polymers are the unit cubes x. Cube $x \not\sim y$ if and only the associated sites x and y are nearest neighbors. For the *cluster model* we define the weight $W(A)$ of a set of mutually different polymers A as $W(A) = \exp(-8\beta|A|)$. Because for instance $w(\{x, y\}) \neq 0$ for $x \not\sim y$ this model is not a polymer model. The partition function $Z_{C,\Lambda}$ of this cluster model equals

$$Z_{C,\Lambda} = \sum_{A \subset \Lambda} w(A) = \sum_{A \subset \Lambda} \exp(-8\beta|A|) = Z_{s,\Lambda} \tag{A.47}$$

As mentioned in Section 4.14 we have an equivalent representation which turns the cluster model into a *polymer model*. Now the polymers are formed by the clusters of unit cubes x. Each polymer one can associate with a connected set of sites in $\mathbb{Z}^d$. Any family of mutual compatible polymers corresponds uniquely to a set of sites in $\mathbb{Z}^d$ which is in

general not connected anymore. We define the weight $W(A) = \prod_{a \in A} \exp(-8\beta|a|)$, where a is a connected set, whenever we can decompose A into mutual compatible connected sets a and $W(.) = 0$ otherwise. Then the partition function $Z_{p,\Lambda}$ of this polymer model is equal to

$$Z_{p,\Lambda} = \sum_{A \subset \Lambda} \prod_{a \in A} W(a) = \sum_{A \subset \Lambda} \prod_{a \in A} \exp(-8\beta|a|) = \sum_{A \subset \Lambda} \exp(-8\beta|A|) = Z_{s,\Lambda} \tag{A.48}$$

A.3 Nature of the clusters

In Subsection A.2.1 we have derived a cluster expansion for the 2D Ising model. Now we consider this expansion in more detail to obtain an lower bound. For large enough β

$$Z^+_{I,\Lambda} = \exp\Big(\sum_{C \in \mathfrak{C}_\Lambda} \phi^+(C) \Big) \tag{A.49}$$

Because of the product structure of the cluster expansion for $Z^+_{I,\Lambda}$:

$$Z^+_{I,\Lambda} = \exp\Big(\sum_{\substack{C \in \mathfrak{C}_\Lambda \\ |C|=4}} \phi^+(C) \Big) \exp\Big(\sum_{\substack{C \in \mathfrak{C}_\Lambda \\ |C| \geq 6}} \phi^+(C) \Big) \tag{A.50}$$

For clusters C with $|C| = 4$ the cluster weights are $\phi^+(C) = \log(1 + e^{-8\beta})$, compare (A.35). Clusters with $|C| = 4$ do contain only one contour which encloses one site. We can associate these clusters with the sites of Λ. This provides

$$\exp\Big(\sum_{\substack{C \in \mathfrak{C}_\Lambda \\ |C|=4}} \phi^+(C) \Big) = \Big(1 + e^{-8\beta}\Big)^{|\Lambda|} = Z_{s,\Lambda} \tag{A.51}$$

We see that, when we sum the cluster weights and take the exponential, we obtain the same expression as for the partition function (A.45) of the site percolation model with $p = e^{-8\beta}$.

Intuitively (A.51) seems to be a lower bound for $Z^+_{I,\Lambda}$ because less clusters are summed over. However in general the weight of a cluster $\phi(C)$ can be negative. So the factor of $Z^+_{I,\Lambda}$ in (A.50) coming from the exponential of the sum over the remaining clusters C with $|C| \geq 6$ does not need to be at least 1. By a more careful analysis we can show that the cluster-expansion part formed by the clusters C with $|C| = 4$, making up $Z_{s,\Lambda}$, is indeed a lower bound for the pressure. This is even true for small β, where the total cluster expansion diverges:

Lemma A.16. *For any* $\beta \geq 0$

$$Z_{s,\Lambda} \leq Z^{+}_{I,\Lambda} \tag{A.52}$$

Proof. Each set A we can uniquely decompose in mutually compatible subsets A_i which are clusters of cubes. Denote by ∂A_i the boundary of the sets A_i. This boundary consists only of closed curves γ in $\Lambda^\star \cup \partial\Lambda$.

Now we compare the site model to the 2d-Ising model. Take a finite set of sites $\Lambda \subset \mathbb{Z}^2$. Consider a set of cubes A with $A \cap \Lambda^c = 0$. Every site $i \in \Lambda$ which is covered by a unit cube i in set A we give the spin $\sigma_i = -1$. The boundary of the unit cube i we denote by ∂i. The spins outside A we set to $\sigma_i = +1$.

The closed curves which make the boundary ∂A are the contours γ which are formed by the broken bonds in $\Lambda^\star \cup \partial\Lambda$. So for every spin configuration we have two equivalent representations. The site representation is by a set A of cubes, the Ising representation is by a set $\{\gamma\} = \partial A$ of contours.

The Ising weight of this configuration A equals $w_I(A) \equiv w_I(\partial A)$ and $w_I(\partial A) = \exp\left(-2\beta \sum_{\gamma \in \partial A} |\gamma|\right)$. Comparing it with the site-model weight, which is $w_s(A) = \exp\left(-2\beta \sum_{i \in A} |\partial i|\right)$, we see that $w_s(A) \leq w_I(A)$. In the 2d Ising model only the dual bonds forming ∂A are accounted for in the weight of γ. But in the site model all the dual bonds in ∂i of every cube $i \in A$ are accounted for. □

Bond percolation

We have analyzed the part of $Z^{+}_{I,\Lambda}$ coming from the single-site enclosing clusters. Now we look at the clusters with $|C| = 6$. These clusters encloses two sites and contain only one contour. So these clusters we can associate with the bonds b in Λ_b: the set containing all nearest neighbor bonds in Λ. When we take the exponential of the sum of the corresponding cluster weights we obtain

$$\exp\Big(\sum_{\substack{C \in \mathfrak{C}_\Lambda \\ |C|=6}} \phi^+(C)\Big) = \Big(1 + e^{-12\beta}\Big)^{|\Lambda_b|} =$$

$$\sum_{A \subset \Lambda_b} \prod_{b \in A} \Big(\exp\left(\log\left(e^{-12\beta} + 1\right) - 1\right) = \sum_{A \subset \Lambda_b} \prod_{b \in A} e^{-12\beta} = Z_{b,\Lambda} \tag{A.53}$$

This is the partition function for independent 2d bond percolation with $p = e^{-12\beta}$. Together with the site-percolation part of $Z^{+}_{I,\Lambda}$ we see

$$Z^{+}_{I,\Lambda} = Z_{s,\Lambda} Z_{b,\Lambda} \exp\Big(\sum_{\substack{C \in \mathfrak{C}_\Lambda \\ |C| \geq 8}} \phi^+(C)\Big) \tag{A.54}$$

More general clusters

For the remaining clusters, which has $|C| \geq 8$, it can happen that the cluster weight becomes negative. Now a particular length corresponds to more classes of clusters. For instance the clusters with $|C| = 8$ we can divide in two classes. First there is the cluster which encloses three or four sites and therefore contains only one contour. The corresponding weight is $\log(1 + \exp(-16\beta))$. But now a cluster C can also consists of two contours γ, γ' which enclose two adjacent sites. Then the weight becomes negative: $\phi^+(C) = \log(1 + 2\exp(-8\beta)) - 2\log(1 + \exp(-8\beta)) < 0$. Furthermore we see that $\phi(C) = O(\exp(-16\beta)) = O(\exp(-2\beta|C|))$.

Bounds for pressure

Now we apply (A.15) to the cluster part of (A.54) to obtain an upper bound. When we combine it with the lower bound (A.46) of the two previous sections we obtain for the free energy for large enough β

$$\log(1 + e^{-8\beta}) \leq \frac{\log Z^+_{I,\Lambda}}{|\Lambda|} \leq \log(1 + e^{-8\beta}) + 2\log(1 + e^{-12\beta}) + O(\exp(-16\beta)) \tag{A.55}$$

Note that the lower bound does hold for any β. By symmetry the same holds for $Z^-_{I,\Lambda}$, which means we have proven:

Proposition A.17. *For the 2d-Ising model with uniform + or - boundary conditions it holds for β large enough*

$$\frac{\log Z^\pm_{I,\Lambda}}{|\Lambda|} = O(\exp(-8\beta)) \tag{A.56}$$

A.4 Multi-scale expansion for random systems

Until now in this chapter no randomness has been involved. Now we consider models which contain some randomness η and see if or in what way we still can apply the expansion techniques just introduced to the partition functions which now depend on η. Often they fail to behave in a controllable way. A more powerful expansion technique is needed. The set of bad polymers are the polymers which decay too slow compared to their volume size. These are responsible for the failure of the single cluster expansion. When the set of bad behaving polymers is small enough then the disorder functions like a small perturbation of the ordered model. For this setting the multi-scale expansion is invented. Even when the set of bad behaving polymers still contains polymers of logarithmic order in the system size, when this size is going to infinity we can use the

multi-scale expansion. However the probability of having these bad polymers must be very small and decrease fast enough when the volume size diverges.

For this multi-scale expansion we need to sort all polymers into classes. Each class consists of only sufficiently bad polymers for which the damping of the weights relatively to the size of the bad polymers lays in a carefully chosen interval. In this way the small probability in η of a polymer being in this class should balance the relatively large weight of the class in the sense of the Gibbs measure.

The remaining polymers which decay fast enough we treat in step zero of the expansion. This gives a partial cluster expansion consisting out of so called zero-clusters. Afterward the multi-scale expansion is performed by expanding step by step the bad polymers. After every step the bad contours appearing in a single class are expanded, together with the interaction with the clusters coming from the previous steps including the zero clusters.

The multi-scale technique originated from the KAM-theory, which originated in classical mechanics. The unperturbed system there is rewritable as a so-called integrable system $(\vec{\phi}, \vec{I}) \in \mathbb{T}^n \times \mathbb{R}^n$ with action variables I_i and angle variables ϕ_i. The Hamiltonian depends on the action variables only. The action variables I_i are constant in time. The angle variables $\phi_i(t)$ have as derivatives time-independent frequencies ω_i. When these frequencies are rationally independent then the motion is called quasi-periodic. During time the trajectory fills up dense the invariant tori $\mathbb{T}^n$.

KAM-theory deals with small perturbations of the Hamiltonian of the form $H_1(\vec{I}) + \epsilon H_2(\vec{I}, \vec{\phi})$. The frequencies need to be sufficiently non-rational, i.e. so-called diophantine conditions need to hold. Then, when the perturbation H_2 is small enough, for a large-measure set of action variables $\vec{I}$ the quasi-periodic tori present in the unperturbed system will be present in the perturbed system, although slightly deformed [4].

When we apply the KAM-theory directly, usually we have unsatisfactory restrictions. It can be improved significantly by performing first a couple of renormalizations on the Hamiltonian. It is written as a simplified Hamiltonian plus a perturbation. This perturbation gets smaller after each renormalization. These renormalizations are the multi-scale type of expansions appearing in the KAM-theory, e.g. [16].

In statistical physics a single cluster expansion (or more generally: the Pirogov-Sinai theory when more ground states are involved) can only be used whenever we have uniform decay in the polymer weights. Equivalently we need the *Peierls condition* to be fulfilled:

Condition A.18 (Peierls). *There exists a $\tau > 0$ independent of any polymer such that for all polymers γ*

$$W(\gamma) < \exp\left(-\tau\beta|\gamma|\right) \tag{A.57}$$

By $|\gamma|$ we denote the area which makes up polymer γ. For usual Ising contours this

is equal to the length of γ. This Peierls condition is an uniform condition on the decay of polymer weights. Because of the randomness, in general this uniformity does not hold. However often it holds for *almost* all polymers with τ depending on the randomness.

Then, because of the almost uniform decay, the model with disorder η seems to be a perturbation of the model without the disorder η. However, the perturbation can be big on all scales; for every scale there are unbalanced contours γ which do violate the Peierls condition (A.57). They appear only with small and fast decaying probability in $|\gamma|$. This is also the setting in the random boundary field Ising model we have treated in Chapter 4; there the disorder is present on all scales and the multi-scale technique is very useful.

Now we restrict ourselves to systems of contours on a lattice $\mathbb{Z}^d$. First we have to distinguish the contours which are well enough damped to be expanded in the 0-th step. For this extend we define a cut-off scale l_0. Every contour of which the weight $W(\gamma) \leq \exp(-|\gamma|/l_0)$ we call balanced.

Equivalently:

Definition A.19. *A contour γ is called balanced, whenever* (A.57) *does hold for $\tau = 1/l_0$. It is called unbalanced otherwise. The set $\mathcal{K}^{\eta}_{0,\Lambda}$ is the set of all balanced contours restricted to Λ*

It is needed that the set of balanced contours is large compared to the rest. Otherwise the model is not treatable by a perturbational multi-scale expansion. So we need strong large deviation estimates of the type of Proposition 4.20, which ensure that the fraction of the total domain of the aggregates of the unbalanced contours is going to zero when the volume gets large for almost all η. This total domain also covers the clusters inside the renormalized weights. Note that this region is larger than the union of the interiors of the unbalanced contours. Because these estimates are intimately related to the construction of the regions $\mathcal{K}_{i,\alpha}$ it is in general highly non-trivial to obtain useful upper bounds.

Assuming η is in the mentioned set of measure 1, we perform the multi-scale expansion. First we expand the set of balanced contours by a single cluster expansion, the zeroth step:

$$Z^{\eta}_{\Lambda} = Z^{b,\eta}_{\Lambda} Z^{u}_{>0} \overbrace{\Big\langle \exp\Big(\sum_{\substack{C \in \mathfrak{C}^{\eta}_{0} \\ C \not\sim \partial}} \phi^{\eta}_{0}(C) \Big) \Big\rangle_{\nu^{>0}_{\Lambda}}}^{\text{interaction 0-clusters}} \equiv Z^{b,\eta}_{\Lambda} Z^{\eta,1}_{\Lambda} \tag{A.58}$$

where

$$Z_\Lambda^{b,\eta} = \underbrace{\exp\left(-E^\eta(\emptyset)\right)}_{\emptyset\text{-configuration}} \underbrace{\exp\Big(\sum_{C\in\mathfrak{C}_0^\eta}\phi_0^\eta(C)\Big)}_{\text{expansion cluster weights}} \tag{A.59}$$

and

$$\nu_\Lambda^{>i}[.] = \frac{1}{Z_{>i}^u}\sum_{\partial\in\mathscr{D}_{>i}^\eta}\prod_{\Gamma\in\partial}\rho^\eta(\Gamma)[.],\ Z_{>i}^u = \sum_{\partial\in\mathscr{D}_{>i}^\eta}\prod_{\Gamma\in\partial}\rho^\eta(\Gamma) \tag{A.60}$$

As before we have denoted $\mathfrak{C}_\Lambda^0$ as the set of all clusters of balanced contours. We call its elements 0-clusters and $\mathscr{D}_{>0}^\eta$ is the set of all families of compatible unbalanced contours. The mean $<.>_\nu$ represents the dependence between the balanced and the unbalanced contours. The measure $\nu_\Lambda^{>0}$ is localized on the unexpanded part $\mathcal{K}\setminus\mathcal{K}_0$, which is formed by the unbalanced contours.

However, because the set $\mathcal{K}\setminus\mathcal{K}_0$ is still relatively large, we have not enough control over $Z_\Lambda^{u,\eta}$. It is not possible to cluster expand $Z_\Lambda^{u,\eta}$ by a convergent single-step cluster expansion. We need to define a cut-off scale l_1. With this cut off scale we separate the class of unbalanced contours, which behave well enough to expand in the first step. Because of the cluster interactions we also need that this class is not too close to the remaining unbalanced contours. The union of the first step contours is formed by the set $\mathcal{K}_1^\eta$.

Afterwards, the unexpanded part $\mathcal{K}\setminus(\mathcal{K}_0\cup\mathcal{K}_1)$ is further localized, allowing to separate a second class of contours $\mathcal{K}_2$ with less decay. After the separation of the contours in $\mathcal{K}_1$ the partition function Z_1^η turns into

$$Z_1^\eta = \underbrace{\sum_{\partial\in\mathscr{D}_{>1}^\eta}\prod_{\Gamma\in\partial}\rho^\eta(\Gamma)\exp\Big(\sum_{\substack{C\in\mathfrak{C}_0^\eta\\ C\not\sim\partial}}\phi_0^\eta(C)\Big)}_{\text{only expandable in higher steps}}\underbrace{\sum_{\partial^1\in\mathscr{D}_1^\eta}\prod_{\Gamma\in\partial^1}\rho^\eta(\Gamma)\exp\Big(\sum_{\substack{C\in\mathfrak{C}_0^\eta\\ C\not\sim\partial^1,\ C\sim\partial}}\phi_0^\eta(C)\Big)}_{\text{this we will expand now}} \tag{A.61}$$

After the first step expansion as in Section 4.8 it becomes

$$Z_1^\eta = \underbrace{\exp\Big(\sum_{D\in\mathfrak{D}_1^\eta}\psi_1^\eta(D)\Big)}_{\text{prefactor clusters 1-st step}}\underbrace{\prod_\alpha\hat{Z}_{1,\alpha}^\eta}_{\substack{\text{byproduct}\\ \text{renormalization}}}Z_{1>}^u\Big\langle\exp\Big(-\sum_{\substack{C\in\mathfrak{C}_1^\eta\\ C\not\sim\partial}}\phi_1^\eta(C)\Big)\Big\rangle_{\nu_\Lambda^{>1}} \tag{A.62}$$

When we compare (A.62) with (A.60) we see that the not expanded part changes from

$$Z^u_{>0}\Big\langle \exp\Big(-\sum_{\substack{C\in\mathfrak{C}^\eta_0\\ C\not\sim\partial,\ \partial\in\mathfrak{D}^\eta_{>0}}}\phi^\eta_0(C)\Big)\Big\rangle_{\nu^{>0}_\Lambda} \text{ to } Z^u_{1>}\Big\langle \exp\Big(-\sum_{\substack{C\in\mathfrak{C}^\eta_1\\ C\not\sim\partial,\ \partial\in\mathfrak{D}^\eta_{>1}}}\phi^\eta_1(C)\Big)\Big\rangle_{\nu^{>1}_\Lambda} \tag{A.63}$$

It has become more localized. After the first step expansion the unexpanded part has been restricted from $\mathcal{K}\setminus\mathcal{K}_0$ to $\mathcal{K}\setminus(\mathcal{K}_0\cup\mathcal{K}_1)$. Then, if we expand further until we are finished with highly localized terms, after several steps finally we end up with an unexpanded term like

$$Z^u_\infty\Big\langle \exp\Big(-\sum_{\substack{C\in\mathfrak{C}^\eta_\infty\\ C\not\sim\partial,\ \partial\in\mathfrak{D}^\eta_\infty}}\phi^\eta_\infty(C)\Big)\Big\rangle_{\nu^{\infty}_\Lambda} \tag{A.64}$$

If we have chosen the cut-off scales l_i and the renormalization scales L_i carefully enough, the sum of these terms should be small compared to the sum of the remainder. Furthermore for the cluster expansions in $\psi^\eta_i(D)$ to have a well controlled behavior like in Proposition 4.24, we need the following. The numbers $\log L_i$ should be large compared to the volume size of any component $\mathcal{K}_{i,\alpha}$. Otherwise the cluster expansions tend to diverge in l_i. In general the size $|\mathcal{K}_{i,\alpha}|$ has a polynomial upper bound in l_i. However the renormalization scales L_i should not be too large, because otherwise the large deviation estimates on $\mathcal{K}\setminus\mathcal{K}_0$ go down and the normalized partition functions $\hat{Z}^\eta_{i,\alpha}$ diverge too fast.

In the random boundary field Ising model we have treated in Chapter 4 we end up with the interactions with and between the corner components $\mathcal{K}_{\infty,i}$. As Proposition 4.25 and Lemma 4.26 show, the sum of these unexpanded terms is indeed small compared to the sum of the remainder. For Z^u_∞ we can only use rough estimates, because of the ill behavior of the involved contours. However, the area of $\mathcal{K}_\infty$ is small compared to the complete volume Λ, which is enough to compensate the lack of control over the corner-aggregate terms.

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[1] Chapter 3 is a revised version of this article

[2] Chapter 4 except for the high field part

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[39] For those readers who feel that we have invoked a heavy machinery to prove something which is physically quite plausible, we have some sympathy, but as a partial justification we'd like to quote from the recent book
Aspects of Ergodic, Qualitative and Statistical Theory of Motion, by G. Gallavotti, F. Bonetti and G. Gentile, Springer, 2004,
in particular what they have to say about cluster expansion methods (p257):
The proliferation of alternative or independent and different proofs, or of nontrivial extensions, shows that in reality the problem is a natural one and that the methods of studying it with the techniques of this section are also natural although they are still considered by many as not elegant (and not really natural) and they are avoided when possible or it is said that "it must be possible to obtain the same result in a simpler way" (often not followed by any actual work in this direction).

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Samenvatting

Denk eens aan een rivier, voortdurend stromende volgt hij zijn weg. De aard van de stroom verandert niet: het water van de rivier volgt altijd de weg van de minste weerstand. Stel je[3] eens voor dat het hard regent, het waterpeil van de rivier wordt hoger en hoger. Dan, opeens, is het water zo hoog gestegen dat de oevers overstromen. Daardoor kan het water in de nabijgelegen weilanden vloeien. Het zal nu wel even duren voordat de rivier zich heeft aangepast aan deze nieuwe situatie.

In de statistische fysica proberen we natuurlijke processen te modelleren. In het bijzonder proberen we de verbinding te leggen tussen de micro- en macroscopische eigenschappen. Voor de rivier koppelen we de globale stroming aan de beweging van de watermoleculen. Sommige universele wetten, betreffende enige globale (= macroscopische) kenmerken, beschrijven de algemene eigenschappen van de rivier, dit ondanks dat er een enorme hoeveelheid aan watermoleculen bij betrokken is. Daartoe gebruiken we kanstheoretische methoden (stochastiek) nodig om toe te passen op de onderliggende microscopische differentiaalvergelijkingen waaruit de beweging van de watermoleculen volgt. Als we dan kijken naar de macroscopische eigenschappen, weten we het globale gedrag van de rivier. Dit globale gedrag kan worden beschreven door vergelijkingen die alleen afhankelijk zijn van macroscopische eigenschappen. Het onderliggende microscopische deel is verdwenen door onze toegepaste stochastiek.

We pakken eens een dobbelsteen waarmee we een paar van de betrokken stochastische principes gaan demonstreren. Zoals we weten, hebben we dezelfde kans om een 1 of een 6 te gooien. Maar door ervaring weten we dat na een klein aantal worpen, de ontstane *relatieve frequenties* van de gegooide cijfers, significant van elkaar kunnen verschillen. Alleen als we een dobbelsteen heel vaak gooien, zullen de ontstane relatieve frequenties meer en meer gelijk worden. Hetzelfde resultaat verkrijgen we wanneer we in plaats van één dobbelsteen heel vaak, een heleboel dobbelstenen tegelijk gooien en dan pas te kijken naar de relatieve cijferfrequentie. Natuurlijk, het is evident dat met een dobbelsteen 1000 keer gooien, hetzelfde is als met 1000 dobbelstenen één keer gooien. Uiteindelijk zal de relatieve frequentie van elk van de zes cijfers $\frac{1}{6}$ naderen: gemiddeld komt elk cijfer een keer voorbij als we een dobbelsteen zes keer gooien. Deze relatieve frequentie wordt soms ook wel de *kans* op het cijfer genoemd. Met een kortere notatie noteren we dit als $P(i)$. Voor elk cijfer i op onze dobbelsteen geldt dat $P(i) = \frac{1}{6}$. De functie die aan elk cijfer i de bijbehorende kans toekent, noemen we de *waarschijnlijkheidsverdeling* van de eigenschap.

Al de materie om ons heen bestaat uit atomen. Elke gram van materie bevat zo

[3] Ik heb bewust gekozen voor een informeel getinte samenvatting. Wanneer U niet wenst getutoyeerd te worden dan mag U in plaats van je U lezen.

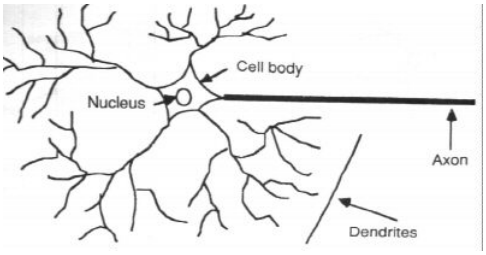
rond de 10^{23} atomen. Vaak kijkt men naar een collectie van een aantal globale (bulk) eigenschappen als toevoeging bij de atomaire eigenschappen. In de beschrijving van de materie beschrijven alle atomen bij elkaar met de bovengenoemde globale eigenschappen het *systeem*. Elke bepaalde realisatie van de bijbehorende atomaire waarden noemen we de *configuratie* van het systeem. Het bepalen van de configuratie heeft wel wat weg van het in één keer gooien van 10^{23} dobbelstenen.

Stel je nu voor dat je een macroscopische eigenschap wil meten, bijvoorbeeld de gemiddelde dichtheid. Omdat het aantal atomen heel groot is, is er geen noodzaak om in de waarschijnlijkheidsverdeling van de atomaire waarden de posities van alle afzonderlijke atomen bij te houden. Bij de rivier is de tijd die de globale aanpassingen in beslag nemen extreem lang, als je het vergelijkt met de tijdschaal van de lokale beweging van de watermoleculen. De manier waarop de eigenschappen van het systeem evolueren, noemen we de *dynamica* van het systeem. In het globale gedrag van de stroming van de rivier zijn de microscopische bewegingen van de watermoleculen uitgemiddeld.

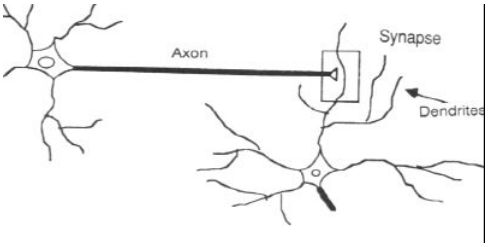
Neurale netwerken

Het eerste onderwerp van het proefschrift is over een model dat zijn oorsprong heeft in de theorie van neurale netwerken. In het bijzonder willen we graag het begrip geheugen beter begrijpen. Onze hersenen zijn opgebouwd uit biljoenen neuronen die op een zeer complexe manier met elkaar verbonden zijn. Deze structuur noemen we een *neuraal netwerk*. Het is moeilijk om rechtstreeks dit netwerk te bestuderen, omdat er in een relatief klein gebied erg veel neuronen betrokken zijn. Om te begrijpen hoe het geheugen werkt, is een gebruikelijke methode om een simpeler model te construeren dat nog steeds de hoofdkenmerken van het geheugen bevat. Net zoals het neurale netwerk van de hersenen moet het model voldoende robuust zijn: als er tussen neuronen signalen worden doorgestuurd dan kunnen er altijd kleine afwijkingen ontstaan. De hersenen zijn in staat om van het licht vervormde signaal de ruis te verwijderen en zo het pure signaal te reconstrueren. Voor een goed globaal overzicht van neurale netwerken zie [19]. Een neuron is opgebouwd uit drie delen: het cellichaam, de dendrieten en het axon, zie Figuur A.1. De dendrieten hebben een boomachtige vertakte structuur en zijn verbonden aan het cellichaam. Het axon is de enige uitgaande verbinding van het neuron. Aan het einde van het axon vertakt het zich en is het verbonden aan de dendrieten van de andere neuronen via synapsen. Tussen het uiteinde van elke tak en dendriet zit een smalle ruimte: de synapsspleet.

Neuronen communiceren met andere neuronen via elektrische signalen. De weg die het elektrische signaal aflegt van een neuron i naar een neuron j is als volgt, zie Figuur A.2. Eerst stroomt het signaal vanuit het cellichaam van neuron i in het axon dat verbonden is met neuron j. Dit is het *output*-signaal van neuron i. Als het signaal



Figuur A.1: *Componenten van een neuron (overgenomen uit [19])*



Figuur A.2: *De synaps (overgenomen uit [19])*

bij het einde van het axon komt, dan geeft het axon neurotransmitters af aan de synapsspleet. Dan worden de neurotransmitters door de receptoren aan de dendriet van neuron j terugvertaald naar een elektrisch signaal. Er zijn verscheidene soorten neurotransmitters. Sommige versterken het binnenkomende signaal voordat het naar de dendrieten gaat van andere neuronen, en andere verzwakken het.

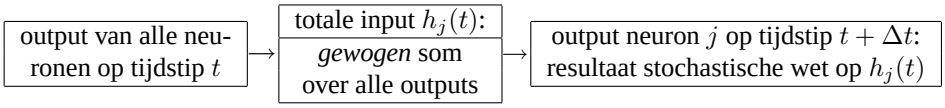
Dit uiteindelijke signaal dat door de receptoren van neuron j geproduceerd wordt, noemen we het *input*-signaal van neuron i naar neuron j. Uiteindelijk komt het signaal aan in het cellichaam van neuron j. In het cellichaam van neuron j komen alle inputs bij elkaar. De cel verwerkt de inputs (die we wiskundig modelleren door het uitvoeren van een *gewogen som*). Dat noemen we de *totale input* h_j van neuron j. Afhankelijk van de uitkomst, zendt het cellichaam een nieuw signaal uit dat gaat naar het axon van neuron j, zodat het doorgestuurd kan worden naar andere neuronen. Dit noemen we de output of de *toestand* van neuron j.

Om een bruikbaar model te maken dat gebaseerd is op deze neurale processen is het nodig om enige flinke vereenvoudigingen door te voeren. Als eerste vereenvoudiging nemen we aan dat elk neuron interactie heeft met elk ander neuron: het neurale netwerk is *volledig verbonden*. Verder nemen we aan dat elk neuron slechts twee mogelijke outputs heeft. We schrijven σ_i als we de toestand van neuron i bedoelen: $\sigma_i = +1$ als het *geëxciteerd* is en $\sigma_i = -1$ als het zich in de *rusttoestand* bevindt. Ook nemen we aan dat het signaal niet verandert wanneer het door de synapsspleet gaat. Hierdoor is de input naar neuron j die komt van neuron i gelijk aan de output σ_i van neuron i die gaat naar neuron j.

Om de dynamica van ons model te modelleren, introduceren we de tijd t. Na elke tijdperiode van Δt (met Δ zeer klein) veranderen we tegelijkertijd de output van elk neuron. Het verwerkingsproces van het cellichaam modelleren we door twee stappen:

1. Op tijdstip t vermenigvuldigen we elke binnenkomende input van de andere neuronen met een gewicht. Hierover nemen we de som over alle neuronen (behalve neuron j). Het resultaat is de totale input $h_j(t)$ op tijdstip t.

2. De output $\sigma_j(t + \Delta t)$ van neuron j op tijdstip $t + \Delta t$ is de uitkomst van een waar-



Figuur A.3: *Dynamica van het neurale netwerk model*

schijnlijkheidsverdeling over de twee mogelijke neuron toestanden. Deze verdeling komt tot stand met behulp van een stochastische wet op $h_j(t)$.

We nemen aan dat de cellichamen van de neuronen de verbindingen op een symmetrische manier behandelen: het gewicht dat neuron j aan de input van neuron i toekent, is gelijk aan het gewicht dat neuron i aan de input van neuron j toekent. In realistische neurale netwerken geldt deze interactiesymmetrie in het algemeen niet. Figuur A.3 vat de dynamica van ons model nog eens samen.

De stabiele configuraties onder deze dynamica vormen het *geheugen* van het systeem. Stabiel betekent dat als je begint met een stabiele configuratie dat het systeem alleen in configuraties terecht komt die er heel erg veel op lijken (wat neuronconfiguraties betreft). Door het kiezen van geschikte gewichten kunnen we de dynamica zo afstellen dat het geheugen wordt gevormd door een eindig aantal m vooraf geselecteerde neuron configuraties $\xi^{(m)}$, die we ook wel *patronen* noemen.

De stochastische wet hangt af van een parameter β. De inverse $T = 1/\beta$ hiervan heet *temperatuur*. Als parameter β groot is, dan heeft het neuron een sterke neiging (een grote kans) om gelijk te worden aan het teken van zijn totale input. Wanneer β oneindig nadert, dan verandert de stochastische wet in een deterministische. Als we dan een configuratie nemen die dicht genoeg bij bijvoorbeeld $\xi^{(1)}$ ligt, dan evolueert het systeem naar configuraties die gelijk zijn aan het patroon $\xi^{(1)}$. Met andere woorden: het neurale netwerk *herinnert* zich de configuratie $\xi^{(1)}$ vanuit zijn geheugen. Dit betekent dat de neuron configuratie gelijk wordt aan $\xi^{(1)}$ en dat daarna het systeem in deze configuratie blijft, zie Sectie 2.3.2.

Om de capaciteit van het geheugen te vergroten, kun je natuurlijk de generalisatie maken naar een hoger aantal q van mogelijke neuron toestanden. Maken we in bovenstaand model deze generalisatie naar $q > 2$ dan noemen we het ontstane model het Potts-Hopfield model.

In het model dat we in Hoofdstuk 3 hebben behandeld, kiezen we de gewichten in de totale inputs op een andere manier. Hiervoor moeten we eerst voor elk neuron een verzameling van p continue variabelen $\xi_i^{(p)}$ definiëren: de patronen. We nemen een willekeurig gekozen realisatie van deze variabelen. Ze hebben een Gaussische verdeling. Met de waarden van de geïntroduceerde patronen $\xi_i^{(j)}$ stellen we de gewichten voor de totale input vast. De gewichten van de totale input bepalen we door twee van deze Gaussische patronen. Het aantal mogelijke toestanden per neuron zetten we vast op drie.

Als we het aantal neuronen laten toenemen, dan gebeurt het volgende indien het aantal groot genoeg is: voor elk gegeven aantal van neuronen concentreert het geheugen zich rond zes neuron configuraties. Deze configuraties zijn aan elkaar gerelateerd door middel van een discrete symmetrie. Bij elke neuron configuratie hoort een punt in de macroscopische ruimte die wordt opgebouwd door enige macroscopische variabelen. De zes stabiele neuron configuraties zijn te verdelen in paren van lijnrecht tegenover elkaar staande punten. Als we het aantal neuronen laten toenemen, zie je altijd de discrete symmetrie terug. Als je echter kijkt naar de plaats van de zes neuron configuraties dan draaien ze rond op drie cirkels naarmate het aantal neuronen groter wordt. Dit is nader uitgewerkt in Hoofdstuk 3. In de rij van groeiende neuron aantallen vullen de optredende neuron configuraties met elkaar op een regelmatige uniforme wijze de drie cirkels op in de eerdergenoemde macroscopische ruimte.

Ferromagneten

Het andere onderwerp van het proefschrift betreft een beroemd model voor magnetische materialen: het Ising model. In het algemeen zijn er vele soorten magnetisme. De zogenaamde *paramagneten* zijn metalen die alleen gemagnetiseerd zijn, wanneer we een extern veld aanbrengen. Een ander belangrijk type metaal zijn de *ferromagneten*. Deze metalen behouden hun magnetisatie, zodra ze blootgesteld zijn aan een magnetisch veld. Als we het metaal verwarmen, dan verdwijnt uiteindelijk dit effect: het metaal gedraagt zich als een paramagneet. Voor meer uitgebreide informatie over magnetisme zie [46].

Ising modellen

Het Ising model is een model voor een ferromagneet. Om dit aannemelijk te maken, hebben we een aantal aannames nodig. We nemen aan dat de niet gepaarde elektronen in de buitenste schil *gelokaliseerd* zijn: d.w.z. sterk gebonden aan de bijbehorende atomen. Alleen deze ongepaarde elektronen zijn verantwoordelijk voor het magnetisme. In het Ising model nemen we aan dat voor elk atoom er maar een niet gepaarde elektron in de buitenste schil zich bevindt.

Elke elektron heeft een intrinsiek hoekmoment: *spin*. Deze spin is verantwoordelijk voor een magnetisch moment. Dankzij de quantummechanica kan de spin slechts twee oriëntaties t.o.v. dit magnetische moment hebben: *op* en *neer* [5]. Met enig misbruik van notatie noemen we deze oriëntaties de *waarden* van de spin. Vanwege de aanname over het aantal ongepaarde elektronen hebben we voor de totale spin per atoom ook maar twee oriëntaties. De meeste metalen hebben meer dan één ongepaard elektron in de buitenste schil. Deze metalen kunnen dan ook meer dan twee oriëntaties hebben voor de totale spin per atoom. De meeste vaste materialen hebben een kristalstructuur:

de atomen, of ionen of moleculen liggen in een regelmatig herhaald 3 dimensionaal patroon. Dit maakt een aantal spin oriëntaties qua energie voordelig.

In magnetiseerbare materialen is het metaal onderverdeeld in domeinen met een netto magnetische lading. De overgangsgebieden tussen deze domeinen noemen we *domain walls* [46]. Het Ising model heeft alleen configuraties waar twee naast elkaar gelegen elektronen gelijke of tegenovergestelde spins hebben. Als er een domain wall is, dan is de dikte ervan automatisch nul.

De interacties tussen de gelokaliseerde elektronen worden ook wel de *Weiss interacties* genoemd. In het algemeen komen twee types van interacties regelmatig voor: *naaste-buur* en *mean-field*. Voor mean-field interacties zijn de interacties tussen elk paar van atomen gelijk aan elkaar. In het Ising model beperken we ons tot naaste buur interacties. Dit houdt in dat we aannemen dat al de overige interacties tussen de elektronen nul zijn. Dit is een goede benadering voor de lanthaniden groep (een bepaalde groep van elementen). Ondanks dat het een simpel model is en voor andere magnetische metalen op zijn hoogst een ruwe benadering, is het een erg succesvol model. Het beschrijft het fenomeen van fase overgang (denk aan de overgang vloeibaar $\rightarrow$ gas) en wordt gebruikt in allerlei praktische toepassingen. Bovendien is het exact oplosbaar in 1 en 2 dimensies.

Nu geven we een wiskundige beschrijving van het model. Pak een stuk roosterpapier. Elke punt waar een verticale lijn een horizontale lijn kruist noemen we een *site*. De horizontale en verticale lijnstukken die beginnen bij een site en stoppen bij de dichtstbijzijnde kruising noemen we *bonds*. Op elke site i bevindt zich een atoom dat een netto spin heeft, met twee mogelijke oriëntaties ten opzichte van de spins van de andere atomen die we aangeven met $\sigma_i = +1$ (spin 'op') en $\sigma_i = -1$ (spin 'neer'). Zowel het atoom als de bijbehorende spinwaarden noemen we *spin*. In ons geval is de spinconfiguratie σ een array, die de spinwaarden σ_i van elke site bevat.

Tussen elke naaste-buur spinpaar is er een *interactie*

$$E_{ij}^{\text{ui}} = -\beta\sigma_i\sigma_j \equiv J\sigma_i\sigma_j \tag{A.65}$$

We noemen deze interactie ook wel de *interactie-energie* tussen de atomen op site i en j. De *energie* van een configuratie is het totaal van de interactie-energieën. De variabele β is afhankelijk van het type materiaal waar we naar aan het kijken zijn. Deze interacties bepalen de waarschijnlijkheden van de configuraties. In ferromagneten hebben de naaste-buur spins de neiging om gelijke waarden te hebben. Daarom kiezen we de interacties in het model zo, dat de kans groter is dat dit zo is: we nemen $J < 0$. Hoe hoger de energie des te lager de kans is op de configuratie. De kans op een configuratie is

$$P(\sigma) = \frac{\exp\left(\beta\sum_{i,j}\sigma_i\sigma_j\right)}{Z(\sigma)} = \frac{\exp\left(-\sum_{i,j}E_{ij}^{\text{ui}}\right)}{Z(\sigma)} \tag{A.66}$$

met $Z(\sigma)$ de som van de noemer over alle configuraties. We zien dat naarmate de temperatuur daalt, de interactie (A.65) sterker wordt. Dan is het extra waarschijnlijk

dat naaste-buur spins dezelfde spinwaarde hebben. Met (A.66) zien we meteen dat voor $T = 0$ slechts twee configuraties met minimale energie positieve kans hebben: elke spin heeft dezelfde waarde. Voor zeer hoge temperatuur heeft elke configuratie gelijke kans. Dan gedraagt het model zich als een paramagneet. De temperatuur is dus een maat van wanorde in het systeem. Voor lage temperaturen hebben de meeste spins gelijke waarden ten opzichte elkaar, voor hoge temperaturen zijn de waarden min of meer willekeurig verdeeld. In het proefschrift hebben we gekeken naar het meest interessante deelgebied van het Ising model: lage temperatuur en ferromagnetisch.

We hebben tot nu toe niet naar de omgeving gekeken. Als de energie van het systeem onafhankelijk is van zijn omgeving, zeggen we dat het systeem *vrije randvoorwaarden* heeft. Maar wat gebeurt er als de omgeving bestaat uit een ander materiaal met een vooraf gekozen spinconfiguratie? De waarden van de spins bij de rand voelen nu ook de invloed van de naaste buur spins van het andere materiaal.

In het algemeen bevat een stuk metaal een heleboel atomen. Een gram bevat al rond de 10^{23} atomen. We willen graag naar volumes kijken van die orde van grote. We meten in aantallen atomen, dus ook de volumegrootte is in de orde van 10^{23}. Eerst kijken we naar het Ising model met een groot eindig volume. Daarna proberen we om de uitdrukkingen hiervan te extrapoleren naar een 'oneindig' volume.

We laten het volume toenemen en voor elke stap nemen we de externe spinwaarden willekeurig $+1$ of -1: *willekeurige randvoorwaarden*. Ook nemen we de temperatuur laag genoeg, zodat de spins sterk de neiging hebben om gelijke
waarden aan te nemen. Het gedrag van de mate van overeenkomst van de spins blijkt af te hangen van hoe we het volume laten toenemen.

Uiteindelijk, door (A.66), zijn in alle voorkomende configuraties bijna alle spins hetzelfde. Omdat we de randvoorwaarden willekeurig hebben gekozen, heeft in de optredende configuraties desondanks de ene helft van de volumes bijna alle spins op en de andere helft bijna alle spins neer.

Als we diep in het volume ver weg van de rand kijken zien we nog steeds een effect van de randvoorwaarden. De lokale volume dichtheden van gebieden van gelijke spins zijn asymptotisch onafhankelijk van de randvoorwaarden. Maar zelfs voor hele grote volumes is er een significant effect op de spinwaarden. Als we kijken naar een heel groot volume, dan geldt met kans 1 dat ofwel alle configuraties bijna alle spins op hebben, ofwel bijna alle spins neer. De voorkeurswaarde is gelijk aan de spinwaarde die de meerderheid heeft van de externe spins van de randvoorwaarde met afstand 1 tot het volume. Omdat de temperatuur hoger is dan nul, heeft toch nog een klein deel van de spins een tegenovergestelde waarde.

Omdat we de volumes snel genoeg laten toenemen, zien we geen zogenaamde *mixtures*. Dit betekent dat het niet mogelijk is om met een positieve kans én configuraties te zien met de meeste spins op én configuraties met de meeste spins neer.

Deze onderwerpen hebben we behandeld in Hoofdstuk 4. Het was nodig om niet-triviale expansie technieken te introduceren: de *multi-scale* cluster-expansies. Onze multi-scale expansiemethode is geïnspireerd door ideeën van Fröhlich and Imbrie [35]. Het is een generalisatie van de meer bekende 'uniforme' cluster-expansie techniek. Met als doel vereenvoudiging van onze afschattingen kozen we ervoor om een andere representatie te gebruiken dan die in [35], en wel de zogenaamde *Kotecký-Preiss representatie*, die twee jaar later al werd ontwikkeld in [50].

Om bruikbare afschattingen te hebben, moesten we bepaalde *criteria* bewijzen: we hebben de convergentie nodig van sommige sommaties die gerelateerd zijn aan de expansies. Voor cluster-expansies is het cruciaal om te controleren of het Kotecký-Preiss criterium geldt. Voor onze expansies kan dit echter niet op een directe manier. Daarom introduceerden we een nieuw criterium en hebben we bewezen dat het equivalent hieraan is. Met dit nieuwe criterium kunnen we zelfs voor onze expansies bruikbare afschattingen verkrijgen. Het laatste hoofdstuk geeft meer uitleg over deze uniforme en multi-scale cluster-expansies.

Dankwoord

Bedankt FOM, voor het financiëren van mijn OIO-schap. Het is altijd goed om te weten om een flexibele werkgever te hebben, wanneer sommige dingen wat minder lekker lopen. I would like to thank the reading committee for reading my thesis. Bedankt Aernout voor de begeleiding de afgelopen jaren. Immers, geen begeleiding $\neq$ proefschrift. Hoewel het op de voorkant niet duidelijk staat, was jij mijn begeleider. Dat formele gedoe op zo'n titelpagina is eigenlijk ook maar onzin. Marinus nog bedankt voor dat ik mijn ei kwijt kon het eerste jaar bij jou. En voor het toegankelijker maken van mijn inleiding.

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