G. Slade

The Lace Expansion and its Applications

Ecole d'Eté de Probabilités de Saint-Flour XXXIV - 2004

Editor: Jean Picard

Springer

Author

Gordon Slade
Department of Mathematics
University of British Columbia
121 -1984 Mathematics Road
Vancouver, BC., V6T 1Z2
Canada
e-mail: slade@math.ubc.ca

Editor

Jean Picard
Laboratoire de Mathématiques Appliquées
UMR CNRS 6620
Université Blaise Pascal Clermont-Ferrand
63177 Aubière Cedex
France
e-mail: jean.picard@math.univ-bpclermont.fr

Cover: Blaise Pascal (1623-1662)

Library of Congress Control Number: 2006921789

Mathematics Subject Classification (2000): 60K35, 82B41, 82B43, 60G57, 05A16

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISSN Ecole d'Eté de Probabilités de St. Flour, print edition: 0721-5363
ISBN-10 3-540-31189-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-31189-8 Springer Berlin Heidelberg New York
DOI 10.1007/b128444

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the authors and SPI Publisher Services using a Springer LaTeX package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11602538 VA41/3100/SPI 5 4 3 2 1 0

Foreword

Three series of lectures were given at the 34th Probability Summer School in Saint-Flour (July 6–24, 2004), by the Professors Cerf, Lyons and Slade. We have decided to publish these courses separately. This volume contains the course of Professor Slade. We cordially thank the author for his performance at the summer school, and for the redaction of these notes.

69 participants have attended this school. 35 of them have given a short lecture. The lists of participants and of short lectures are enclosed at the end of the volume.

The Saint-Flour Probability Summer School was founded in 1971. Here are the references of Springer volumes which have been published prior to this one. All numbers refer to the *Lecture Notes in Mathematics* series, except S-50 which refers to volume 50 of the *Lecture Notes in Statistics* series.

1971: vol 307	1980: vol 929	1990: vol 1527	1998: vol 1738
1973: vol 390	1981: vol 976	1991: vol 1541	1999: vol 1781
1974: vol 480	1982: vol 1097	1992: vol 1581	2000: vol 1816
1975: vol 539	1983: vol 1117	1993: vol 1608	2001: vol 1837 & 1851
1976: vol 598	1984: vol 1180	1994: vol 1648	2002: vol 1840
1977: vol 678	1985/86/87: vol 1362 & S-50	1995: vol 1690	2003: vol 1869
1978: vol 774	1988: vol 1427	1996: vol 1665	2004: vol. 1878 & 1879
1979: vol 876	1989: vol 1464	1997: vol 1717	

Further details can be found on the summer school web site
`http://math.univ-bpclermont.fr/stflour/`

Jean Picard, Université Blaise Pascal
Chairman of the summer school

Preface

Several superficially simple mathematical models, such as the self-avoiding walk and percolation, are paradigms for the study of critical phenomena in statistical mechanics. Although these models have been studied by mathematicians for about half a century, exciting new developments continue to occur and the subject is flourishing. Much progress has been made, but it remains a major challenge for mathematical physics and probability theory to obtain a complete and mathematically rigorous understanding of the scaling theory of these models at criticality.

These lecture notes concern the lace expansion, which is a powerful tool for the analysis of the critical scaling of several models above their upper critical dimensions, namely:

- the self-avoiding walk on $\mathbb{Z}^d$ for $d > 4$,
- lattice trees and lattice animals on $\mathbb{Z}^d$ for $d > 8$,
- percolation on $\mathbb{Z}^d$ for $d > 6$,
- oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ and the contact process on $\mathbb{Z}^d$ for $d > 4$.

Results include proofs of existence of critical exponents, with mean-field values, and construction of scaling limits. Often, the scaling limit is described in terms of super-Brownian motion.

There are two distinct goals for these notes. The first goal is to provide a written accompaniment to my lectures at the XXXIV Saint–Flour International Probability School, in July 2004, and at the Pacific Institute for the Mathematical Sciences – University of British Columbia Summer School on Probability, in June 2005. The notes contain an introduction to the lace expansion and several of its applications, with sufficient background and depth to prepare a newcomer to do research using the lace expansion. Basic graduate level probability theory will be used, but no previous knowledge of the lace expansion or super-Brownian motion is assumed. The second goal is to provide a survey of the field, so that an interested reader can follow up by consulting the original literature. In pursuit of the second goal, these notes include more material than can be covered during a summer school course.

Following a brief initial chapter concerning random walk, the notes can be divided into four parts, whose contents are summarized as follows.

Part I, which concerns the self-avoiding walk, consists of Chaps. 2–6. A complete and self-contained proof is given of the convergence of the lace expansion for the nearest-neighbour model in dimensions $d \gg 4$, and for the spread-out model of self-avoiding walks which take steps of length at most L, with $L \gg 1$, in dimensions $d > 4$. The convergence proof presented here seems simpler than all previous lace expansion convergence proofs. As a consequence of convergence, it is shown that the critical exponent γ for the generating function of the number of n-step self-avoiding walks exists and is equal to 1. A survey is then given of the many extensions of this result that have been obtained using the lace expansion.

Part II, which concerns lattice trees and lattice animals, consists of Chaps. 7–8. It is shown how a minor modification of the expansion for the self-avoiding walk can be applied to give expansions for lattice trees and lattice animals, and an indication is given of the diagrammatic estimates that are necessary for proving convergence of the expansion. The relevance of the square condition is indicated, and results concerning existence of critical exponents in dimensions $d > 8$ are surveyed.

Part III, which concerns percolation, oriented percolation, and the contact process, consists of Chaps. 9–14. Detailed discussions are given of expansions for each of these models. Differential inequalities involving the triangle condition are stated (and usually proved) and are shown to imply mean-field behaviour of various critical exponents. Results concerning existence of critical exponents in dimensions $d > 6$ (for percolation) and $d > 4$ (for oriented percolation and the contact process) are surveyed.

Part IV, which concerns super-Brownian scaling limits, consists of Chaps. 15–17. Critical branching random walk with Poisson offspring distribution is analyzed in detail and used to give a self-contained construction of integrated super-Brownian excursion (ISE). The role of ISE as the scaling limit of lattice trees and of critical percolation clusters, above the upper critical dimensions, is discussed. The canonical measure of super-Brownian motion is also described, as is its role as scaling limit of critical oriented percolation clusters and the critical contact process in dimensions $d > 4$, and of lattice trees in dimensions $d > 8$.

Mathematics is not a spectator sport, and true understanding requires active participation in working out the ideas. To help facilitate this, a number of exercises for the reader appear throughout the notes. Some can be solved in a few lines, and others require more effort. I am grateful to Jeremy Flowers, Jesse Goodman, Jeffrey Hood, Sandra Kliem, Richard Liang, and Terry Soo, who collectively wrote solutions to all the exercises during the PIMS–UBC summer school.

It would not be possible to include detailed proofs of all the results discussed in these lecture notes without substantially increasing their length, and a number of important topics are only alluded to. These include: the inductive approach to the lace expansion, which is in many respects the most powerful method to prove convergence of the expansion; the "double" expansions that have been used to analyze r-point functions for $r \geq 3$; and the lace expansion on a tree, which is a method that can sometimes be used to replace a double expansion. (Two of these topics—the inductive method and double expansions—are discussed in recent lecture notes by Remco van der Hofstad [110].) Also, a complete proof of the convergence of the expansion is given only for the self-avoiding walk. This is the simplest setting for proving convergence, and convergence for the other models can be based on the ideas used in this setting. Finally, in an important new development about which it is too early to provide details, Sakai [181] has shown how to apply the lace expansion to analyze the Ising model in dimensions $d > 4$.

This work was supported in part by NSERC of Canada. Versions of the lectures were given at the University of British Columbia in Spring 2003, at EURANDOM in Fall 2003, at Saint-Flour in Summer 2004, and at PIMS/UBC in Summer 2005. The lecture notes were written primarily while I was travelling during 2003-04. I thank EURANDOM and the Thomas Stieltjes Institute, the University of Melbourne, Microsoft Research, and my hosts at these institutions, for their hospitality during visits to Eindhoven, Melbourne and Redmond.

I am grateful to the friends and colleagues with whom I have had the good fortune to work on topics related to these lecture notes. I thank Markus Heydenreich, Remco van der Hofstad, Mark Holmes, Sandra Kliem, Ed Perkins and Akira Sakai for suggesting improvements and for comments on earlier drafts of these notes. Many others have also made helpful comments of one form or another. Most of the illustrations (and all of the best ones) were produced by Bill Casselman, my colleague at the University of British Columbia and Graphics Editor of *Notices of the American Mathematical Society.*

I extend special thanks to David Brydges, whose patient teaching brought me into the subject, and to Takashi Hara and Remco van der Hofstad, who have played profound roles in the development of the ideas presented in these notes.

Vancouver, Gordon Slade
August 9, 2005

Contents

1

Simple Random Walk

The point of departure for the lace expansion is simple (ordinary) random walk, and it is helpful first to recall some elementary facts about random walk on $\mathbb{Z}^d$. This will also set some notation for later use.

1.1 Asymptotic Behaviour

Fix a finite set $\Omega \subset \mathbb{Z}^d$ that is invariant under the symmetry group of $\mathbb{Z}^d$, i.e., under permutation of coordinates or replacement of any coordinate x_i by $-x_i$. Our two basic examples are the *nearest-neighbour model*

$$\Omega = \{x \in \mathbb{Z}^d : \|x\|_1 = 1\} \tag{1.1}$$

and the *spread-out model*

$$\Omega = \{x \in \mathbb{Z}^d : 0 < \|x\|_\infty \leq L\}, \tag{1.2}$$

where L is a fixed (usually large) constant. The norms are defined, for $x = (x_1, \ldots, x_d)$, by $\|x\|_1 = \sum_{j=1}^d |x_j|$ and $\|x\|_\infty = \max_{1 \leq j \leq d} |x_j|$.

For $n \geq 1$, an n-step walk taking steps in Ω is defined to be a sequence $(\omega(0), \omega(1), \ldots, \omega(n))$ of vertices in $\mathbb{Z}^d$ such that $\omega(i) - \omega(i-1) \in \Omega$ for $i = 1, \ldots, n$. Let $\mathcal{W}_n(x, y)$ be the set of n-step walks with $\omega(0) = x$ and $\omega(n) = y$, and let $\mathcal{W}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{W}_n(0, x)$ denote the set of all n-step walks starting from the origin. Let $c_n^{(0)}(x)$ denote the cardinality of $\mathcal{W}_n(0, x)$. The superscript (0) is there to indicate that we are working with the random walk with no interaction. We allow for the degenerate case $n = 0$ by defining $\mathcal{W}_0(x, y)$ to consist of the zero-step walk (x) if $x = y$, and to be empty otherwise. Then $c_0^{(0)}(x, y) = \delta_{x,y}$. Taking into account the translation invariance, we will use the abbreviations $\mathcal{W}_n(y - x) = \mathcal{W}_n(x, y)$ and $c_n^{(0)}(y - x) = c_n^{(0)}(x, y)$.

For $n \geq 1$, by considering the possible values $y \in \Omega$ of the walk's first step, we have

$$c_n^{(0)}(x) = \sum_{y\in\Omega} c_{n-1}^{(0)}(x-y) = \sum_{y\in\mathbb{Z}^d} c_1^{(0)}(y)c_{n-1}^{(0)}(x-y). \tag{1.3}$$

Denoting the convolution of functions f and g by

$$(f*g)(x) = \sum_{y\in\mathbb{Z}^d} f(y)g(x-y), \tag{1.4}$$

(1.3) can be written as

$$c_n^{(0)}(x) = \left(c_1^{(0)} * c_{n-1}^{(0)}\right)(x). \tag{1.5}$$

The Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} f(x)e^{ik\cdot x} \qquad (k \in [-\pi,\pi]^d), \tag{1.6}$$

where $k\cdot x = \sum_{j=1}^d k_j x_j$, with inverse

$$f(x) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \hat{f}(k)e^{-ik\cdot x}. \tag{1.7}$$

The fact stated in part (a) of the following exercise makes the use of Fourier transforms very convenient.

Exercise 1.1. (a) Show that the Fourier transform of $f*g$ is $\hat{f}\hat{g}$.
(b) A closely related statement is the following. Denote the generating functions of the sequences f_n and g_n by $F(z) = \sum_{n=0}^\infty f_n z^n$ and $G(z) = \sum_{n=0}^\infty g_n z^n$, and assume these series both have positive radius of convergence. Show that the generating function $H(z)$ of the sequence $h_n = \sum_{m=0}^n f_m g_{n-m}$ is $H(z) = F(z)G(z)$.

By Exercise 1.1(a), (1.5) implies that

$$\hat{c}_n^{(0)}(k) = \hat{c}_1^{(0)}(k)\hat{c}_{n-1}^{(0)}(k). \tag{1.8}$$

Since $\hat{c}_0^{(0)}(k) = 1$, solving (1.8) by iteration gives

$$\hat{c}_n^{(0)}(k) = \hat{c}_1^{(0)}(k)^n \qquad (n \geq 0). \tag{1.9}$$

If we define the transition probability

$$D(x) = \frac{1}{|\Omega|}I[x\in\Omega] = \frac{1}{|\Omega|}c_1^{(0)}(x), \tag{1.10}$$

where $|\Omega|$ denotes the cardinality of the set Ω and I denotes the indicator function, then (1.9) can be rewritten as

$$\hat{c}_n^{(0)}(k) = |\Omega|^n \hat{D}(k)^n \qquad (n \geq 0). \tag{1.11}$$

Exercise 1.2. (a) Show that for the nearest-neighbour model,

$$\hat{D}(k) = \frac{1}{d}\sum_{j=1}^d \cos k_j, \tag{1.12}$$

and for the spread-out model

$$\hat{D}(k) = \frac{1}{|\Omega|}\left[\prod_{j=1}^d M(k_j) - 1\right], \tag{1.13}$$

where

$$M(t) = \frac{\sin[(2L+1)t/2]}{\sin(t/2)} \tag{1.14}$$

is the Dirichlet kernel.
(b) Denote the variance of D by $\sigma^2 = \sum_{x\in\mathbb{Z}^d}|x|^2 D(x)$. Show that $\sigma = 1$ for the nearest-neighbour model and that σ is asymptotic to a multiple of L as $L \to \infty$ for the spread-out model.

The number of n-step walks starting from a given vertex is of course $|\Omega|^n$, because each step can be chosen in $|\Omega|$ different ways. This fact is contained in (1.11), since the number of n-step walks starting from the origin is $\sum_{x\in\mathbb{Z}^d} c_n^{(0)}(x) = \hat{c}_n^{(0)}(0) = |\Omega|^n$, using $\hat{D}(0) = 1$.

By symmetry, $\sigma^2 = -\nabla^2\hat{D}|_{k=0}$, where $\nabla^2 = \sum_{j=1}^d \nabla_j^2$ is the Laplacian, with ∇_j denoting partial differentiation with respect to the component k_j of k. Then, by (1.11) and by the symmetry of Ω, the central limit theorem

$$\lim_{n\to\infty} \frac{\hat{c}_n^{(0)}(k/\sigma\sqrt{n})}{\hat{c}_n^{(0)}(0)} = e^{-|k|^2/2d} \tag{1.15}$$

follows, as does the fact that the mean-square displacement is given by

$$\frac{\sum_{x\in\mathbb{Z}^d}|x|^2 c_n^{(0)}(x)}{\sum_{x\in\mathbb{Z}^d} c_n^{(0)}(x)} = -\nabla^2\hat{D}^n\Big|_{k=0} = n\sigma^2. \tag{1.16}$$

Exercise 1.3. Prove (1.15) and (1.16).

The *two-point function* is defined by

$$C_z(x,y) = \sum_{n=0}^\infty \sum_{\omega\in\mathcal{W}_n(x,y)} z^n = \sum_{n=0}^\infty c_n^{(0)}(x,y)z^n. \tag{1.17}$$

The two-point function is finite for $z \in [0, 1/|\Omega|)$. For $d > 2$, it is also known to be finite for $z = 1/|\Omega|$, and for this value of z it is called the Green function. By translation invariance, we may regard the two-point function as a function of a single variable, writing $C_z(x,y) = C_z(y-x)$. By (1.11) and (1.17), its Fourier transform is

$$\hat{C}_z(k) = \sum_{n=0}^{\infty} \hat{c}_n^{(0)}(k) z^n = \frac{1}{1 - z|\Omega|\hat{D}(k)}. \tag{1.18}$$

The *susceptibility* is defined by

$$\chi(z) = \sum_{x \in \mathbb{Z}^d} C_z(0,x) = \hat{C}_z(0) = \frac{1}{1 - z|\Omega|}. \tag{1.19}$$

The *critical point* is the singularity $z_c = 1/|\Omega|$ of the susceptibility.

The inverse Fourier transform of (1.18) is

$$C_z(x) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-ik\cdot x}}{1 - z|\Omega|\hat{D}(k)}. \tag{1.20}$$

For $d > 2$,

$$C_{z_c}(x) \sim \mathrm{const}\frac{1}{|x|^{d-2}} \tag{1.21}$$

as $|x| \to \infty$, where the constant depends on d and on Ω (see [149,195], or [203] for a more general statement of this fact). The notation

$$f(x) \sim g(x) \qquad \text{denotes} \qquad \lim_{x\to\infty} f(x)/g(x) = 1, \tag{1.22}$$

and this notation will used in general for asymptotic formulas.

Exercise 1.4. Some care is needed with (1.20) when $z = z_c$, since $C_{z_c}(x)$ is not summable by (1.21) and thus its Fourier transform is problematic. Using the symmetry of Ω, prove that (1.20) does hold when $z = z_c$ for $d > 2$, and that the integral is infinite when $z = z_c$ for $d \le 2$.

Exercise 1.5. Let $f : \mathbb{Z}^d \to \mathbb{C}$. For $y \in \Omega$, define forward and backward discrete partial derivatives by $\partial_y^+ f(x) = f(x+y) - f(x)$ and $\partial_y^- f(x) = f(x) - f(x-y)$. Define the discrete Laplacian by

$$\Delta f(x) = \frac{1}{2}\frac{1}{|\Omega|}\sum_{y\in\Omega} \partial_y^- \partial_y^+ f(x) = \frac{1}{|\Omega|}\sum_{y\in\Omega} f(x+y) - f(x), \tag{1.23}$$

and let $\delta_{x,y}$ denote the Kronecker delta which takes the value 1 if $x = y$ and 0 if $x \ne y$. Show that $-\Delta C_{1/|\Omega|}(x) = \delta_{0,x}$. Thus $C_{1/|\Omega|}(x)$ is the Green function for $-\Delta$.

Exercise 1.6. Consider a simple random walk started at the origin.
(a) Let u denote the probability that the walk ever returns to the origin. The walk is *recurrent* if $u = 1$ and *transient* if $u < 1$. Let N denote the (random) number of visits to the origin, including the initial visit at time 0, and let $m = \mathbb{E}N$. Show that $m = \frac{1}{1-u}$, so the walk is recurrent if and only if $m = \infty$.
(b) Show that

$$m = \sum_{n=0}^{\infty} \mathbb{P}(\omega(n) = 0) = \int_{[-\pi,\pi]^d} \frac{1}{1 - \hat{D}(k)} \frac{\mathrm{d}^d k}{(2\pi)^d}. \tag{1.24}$$

Thus transience is characterized by the integrability of $\hat{C}_{1/|\Omega|}(k)$.
(c) For simplicity, consider the nearest-neighbour model, with Ω given by (1.1). Show that the walk is recurrent in dimensions $d \le 2$ and transient in dimensions $d > 2$.

Exercise 1.7. Let $\omega^{(1)}$ and $\omega^{(2)}$ denote two independent simple random walks started at the origin, and let

$$X = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} I[\omega^{(1)}(i) = \omega^{(2)}(j)] \tag{1.25}$$

denote the number of intersections of the two walks. Here I denotes an indicator function. Show that

$$\mathbb{E}X = \int_{[-\pi,\pi]} \frac{1}{[1 - \hat{D}(k)]^2} \frac{\mathrm{d}^d k}{(2\pi)^d}. \tag{1.26}$$

Thus $\mathbb{E}X$ is finite if and only if $\hat{C}_{1/|\Omega|}(k)$ is square integrable. Conclude, for simplicity for the nearest-neighbour model, that the expected number of intersections is finite if $d > 4$ and infinite if $d \le 4$.

The integral $(2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{C}_{z_c}(k)^2 \mathrm{d}^d k$ of (1.26) is equal, by the Parseval relation, to $\sum_{x\in\mathbb{Z}^d} C_{z_c}(x)^2$. The relevance of the condition $d > 4$ for the latter is evident from the asymptotic behaviour (1.21). However, the k-space analysis is more elementary, as it relies on the easy formulas given in (1.12) and (1.18) rather than the deeper statement (1.21). It is often much easier to use estimates in k-space than to work directly in x-space.

It is a consequence of Donsker's Theorem [24] that the scaling limit of simple random walk is Brownian motion, in all dimensions. This means that if we define a random continuous function X_n from the interval $[0,1]$ into $\mathbb{R}^d$ by setting $X_n(j/n) = \sigma^{-1} n^{-1/2}\omega(j)$ for integers $j \in [0,n]$, and interpolating linearly between consecutive vertices, then the distribution of X_n converges weakly to the Wiener measure. See Fig. 1.1.

1.2 Universality and Spread-Out Models

In these notes, we study several models that live on the integer lattice, and each has a nearest-neighbour and a spread-out version. In the nearest-neighbour model, specified by (1.1), bonds (also called edges) join pairs of

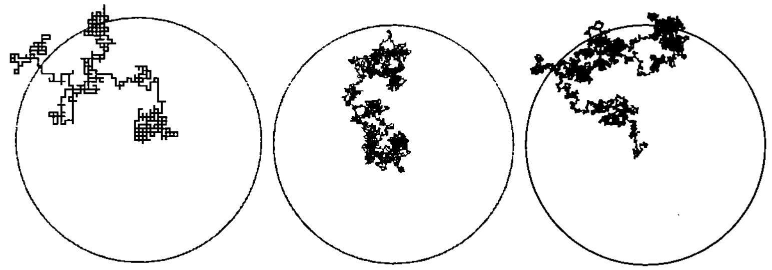


Fig. 1.1. Nearest-neighbour random walks on $\mathbb{Z}^2$ taking $n = 1,000$, 10,000 and 100,000 steps. The circles have radius $\sqrt{n}$, in units of the step size of the random walk.

vertices separated by unit Euclidean distance. In the spread-out model, specified by (1.2), bonds join pairs of vertices separated by distance between 1 and L, where L is a parameter usually taken to be large. According to the deep hypothesis of *universality*, the critical scaling of the models to be studied should be the same for the nearest-neighbour and spread-out models.

We use the spread-out model because proofs of convergence of the lace expansion require *large degree*. The degree is the cardinality of Ω. For the nearest-neighbour model the degree is $2d$, and can be taken large by increasing the dimension. The degree of the spread-out model is of order L^d for large L, and this allows for convergence proofs for the lace expansion without taking the dimension d to be large in an uncontrolled way. In the applications to be discussed, results will typically be obtained: (i) for the nearest-neighbour model for $d \geq d_0$ for some d_0 having no physical meaning, and (ii) for the spread-out model with L larger than some L_0 and d strictly greater than the upper critical dimension (4 for the self-avoiding walk, oriented percolation and the contact process; 6 for percolation; 8 for lattice trees and lattice animals). While it is of interest to prove results of type (i) with d_0 equal to the upper critical dimension plus one, failing this, results of type (ii) seem more important, as they indicate clearly the role of the upper critical dimension. Also, the fact that all large L give rise to the same scaling behaviour provides a partial proof of universality in this context. In fact, much more general spread-out models than (1.2) can be handled using the lace expansion (see, e.g., [94,120]), but we restrict attention in these notes to (1.2) for the sake of simplicity.

2

The Self-Avoiding Walk

The self-avoiding walk is a model of fundamental interest in combinatorics, probability theory, statistical physics and polymer chemistry. It is a model of random walk paths but it cannot be described in terms of transition probabilities and thus is not even a stochastic process. It is certainly non-Markovian. These features makes the subject difficult, and many of the central problems remain unsolved. See [127, 158] for extensive surveys.

The self-avoiding walk is a basic example in the theory of critical phenomena, due to its close links with models of ferromagnetism such as the Ising model. In particular, it can be understood as the $N \to 0$ limit of the N-vector model [79] (see also [158, Sect. 2.3]). In polymer chemistry, self-avoiding walks are used to model a single linear polymer molecule in a good solution [80,205]. The flexibility of the polymer is modelled by the possible configurations of a self-avoiding walk, while the self-avoidance constraint models the excluded volume effect that causes the polymer to repel itself.

In this chapter, we first give an overview of the self-avoiding walk and its predicted asymptotic behaviour. Then we define the bubble condition and show that it is a sufficient condition for a particular critical exponent (namely γ) to exist and take its mean-field value.

2.1 Asymptotic Behaviour

An n-step self-avoiding walk starting at x and ending at y is an n-step walk $(\omega(0), \omega(1), \ldots, \omega(n))$ with $\omega(0) = x$, $\omega(n) = y$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We will assume for simplicity that the walks take steps in Ω given either by (1.1) or (1.2). Let $\mathcal{S}_n(x,y)$ be the set of n-step self-avoiding walks from x to y, let $\mathcal{S}_n = \cup_{x\in\mathbb{Z}^d}\mathcal{S}_n(0,x)$ denote the set of all n-step self-avoiding walks starting from the origin, and let $\mathcal{S}(x,y) = \cup_{n=0}^{\infty}\mathcal{S}_n(x,y)$ denote the set of all self-avoiding walks of any length from x to y. Let $c_n(x,y)$ denote the cardinality of $\mathcal{S}_n(x,y)$. In particular, $c_0(x,y) = \delta_{x,y}$. We will use the abbreviations $\mathcal{S}_n(x) = \mathcal{S}_n(0,x)$, $c_n(x) = c_n(0,x)$, and $c_n = \sum_{x\in\mathbb{Z}^d} c_n(x)$.

Thus c_n counts the number of n-step self-avoiding walks that start at the origin and end anywhere.

More generally, given a walk ω, let

$$U_{st}(\omega) = \begin{cases} -1 & \text{if } \omega(s) = \omega(t) \\ 0 & \text{if } \omega(s) \neq \omega(t), \end{cases} \tag{2.1}$$

and, for $\lambda \in [0,1]$, let

$$c_n^{(\lambda)}(x) = \sum_{\omega \in \mathcal{W}_n(x)} \prod_{0 \le s < t \le n} (1 + \lambda U_{st}(\omega)). \tag{2.2}$$

For $\lambda = 0$, (2.2) is the same as the quantity $c_n^{(0)}(x)$ defined previously. For $\lambda = 1$, we have $c_n^{(1)}(x) = c_n(x)$, and we will usually omit the superscript $^{(1)}$ when $\lambda = 1$. For $0 < \lambda < 1$, (2.2) defines a much-studied model of weakly self-avoiding walks (sometimes called the Domb–Joyce model after [64]) in which walks with self intersections receive less weight than walks that are self-avoiding.

For $\lambda \in [0,1]$, let

$$c_n^{(\lambda)} = \sum_{x \in \mathbb{Z}^d} c_n^{(\lambda)}(x). \tag{2.3}$$

Since $1 + \lambda U_{st}(\omega) \le 1$ for all s, t, ω, we have

$$\prod_{0 \le s < t \le m+n} (1 + \lambda U_{st}(\omega)) \le \prod_{0 \le s < t \le m} (1 + \lambda U_{st}(\omega)) \prod_{m \le s < t \le m+n} (1 + \lambda U_{st}(\omega)), \tag{2.4}$$

from which we easily conclude that

$$c_{m+n}^{(\lambda)} \le c_m^{(\lambda)} c_n^{(\lambda)}. \tag{2.5}$$

Therefore, $\log c_n^{(\lambda)}$ is a subadditive sequence. By a standard lemma [158, Lemma 1.2.2], it follows that the limit

$$\mu_\lambda = \lim_{n \to \infty} \left(c_n^{(\lambda)}\right)^{1/n} \tag{2.6}$$

exists and that moreover

$$c_n^{(\lambda)} \ge \mu_\lambda^n. \tag{2.7}$$

When $\lambda = 1$, $\mu = \mu_1$ is known as the *connective constant.* For nearest-neighbour walks, it is easy to see that $d \le \mu \le 2d - 1$. The lower bound follows from the fact that c_n is at least as large as the number d^n of walks that take steps only in the positive coordinate directions. The upper bound follows from the fact that c_n is at most the number $2d(2d-1)^{n-1}$ of walks that never reverse a previous step. The exact value of μ is not known in general, although good rigorous numerical upper and lower bounds have been

obtained [55, 106, 171, 187]. Numerical estimates of μ are 2.638 and 4.684 for nearest-neighbour self-avoiding walks in dimensions 2 and 3 respectively—in fact there are higher precision estimates due to A.J. Guttmann and coworkers. It has been conjectured [168–170], and been confirmed by numerical evidence [70], that on the 2-dimensional hexagonal lattice $\mu = \sqrt{2+\sqrt{2}}$. It has been observed from enumeration data that on the 2-dimensional square lattice μ is very well approximated by the reciprocal of the smallest positive root of the quartic equation $581x^4 + 7x^2 - 13 = 0$ [54, 137], although no derivation or explanation of this equation has been discovered.

For the nearest-neighbour self-avoiding walk on $\mathbb{Z}^d$, the lace expansion has been used to prove that $\mu(d)$ has an asymptotic expansion to all orders in $1/d$, with integer coefficients, and that

$$\mu(d) = 2d - 1 - \frac{1}{2d} - \frac{3}{(2d)^2} - \frac{16}{(2d)^3} - \frac{102}{(2d)^4} - \frac{729}{(2d)^5} - \frac{5533}{(2d)^6} - \frac{42229}{(2d)^7} - \frac{288761}{(2d)^8} + O\left(\frac{1}{(2d)^9}\right). \tag{2.8}$$

Without using the lace expansion (which was not yet invented), the above coefficients were computed in [74], up to and including $-102(2d)^{-4}$, without a rigorous estimate for the error. About the same time, the formula $\mu(d) = 2d-1-(2d)^{-1}+O((2d)^{-2})$ was proved in [140]. In [101, 102], the lace expansion was used to prove the existence of an asymptotic expansion to all orders , and also that $\mu(d) = 2d-1-(2d)^{-1}-3(2d)^{-2}-16(2d)^{-3}-102(2d)^{-4}+O\left((2d)^{-5}\right)$. The four additional coefficients in (2.8) were obtained in [53]. It seems likely that the asymptotic expansion has radius of convergence zero, though there is no proof of this. For further $1/d$ expansion results (but without rigorous error estimates) see [77, 163, 164]. For asymptotics of the connective constant for the spread-out model, as $L \to \infty$, see [118, 176].

For $\lambda = 0$, we have seen in Chap. 1 that $c_n^{(0)} = |\Omega|^n$, and thus the number of n-step walks grows purely exponentially in n. There is overwhelming evidence to support the belief that for $\lambda \in (0,1]$, the asymptotic form of $c_n^{(\lambda)}$ is given by

$$c_n^{(\lambda)} \sim A_\lambda \mu_\lambda^n n^{\gamma - 1}. \tag{2.9}$$

Here, A_λ is a constant which, like μ_λ, depends on λ, d and Ω, but the critical exponent γ is independent of λ and Ω and is given by

$$\gamma = \begin{cases} 1 & \text{if } d = 1 \\ \frac{43}{32} & \text{if } d = 2 \\ 1.162... & \text{if } d = 3 \\ 1 \text{ with logarithmic corrections} & \text{if } d = 4 \\ 1 & \text{if } d \ge 5. \end{cases} \tag{2.10}$$

The conjectured logarithmic correction in four dimensions, predicted by the renormalization group method, is

$$c_n^{(\lambda)} \sim A_\lambda \mu_\lambda^n (\log n)^{1/4} \quad \text{if } d = 4. \tag{2.11}$$

The independence of γ on $\lambda \in (0,1]$ and Ω is referred to as universality. Similarly, the power of the logarithm in (2.11) is believed to be universal.

The exponent γ has the following probabilistic interpretation. Consider the case $\lambda = 1$, and let q_n denote the probability that two independent n-step self-avoiding walks started at the origin do not have any intersection apart from their common starting point. Since a non-intersecting pair of n-step self-avoiding walks comprises a single $2n$-step self-avoiding walk, if (2.9) holds then

$$q_n = \frac{c_{2n}}{c_n^2} \sim \frac{2^{\gamma-1}}{A_1} \frac{1}{n^{\gamma-1}}. \tag{2.12}$$

In dimensions $d > 4$, the lace expansion has been used to prove that (2.9) holds with $\gamma = 1$ for various choices of λ and Ω, including the nearest-neighbour model with $\lambda = 1$ [96–98]. Note that $\gamma = 1$ corresponds to purely exponential growth on the right hand side of (2.9), as is the case for the simple random walk. Also, there is no decay as $n \to \infty$ in (2.12) when $\gamma = 1$. Partial results for the 4-dimensional case have been obtained in [43, 44, 129] (physics references include [38, 65]). The 3-dimensional case is completely unsolved mathematically. Evidence strongly supporting the value $\gamma = \frac{43}{32}$, which was first predicted in [168–170], has been obtained in [150], by associating the 2-dimensional self-avoiding walk with $\mathrm{SLE}_{8/3}$. Numerical tests supporting the role of $\mathrm{SLE}_{8/3}$ in the description of the 2-dimensional self-avoiding walk can be found in [139]. For $d = 1$, the strictly self-avoiding nearest-neighbour model is trivial and $c_n^{(1)} = 2$ for all $n \geq 1$, so $\gamma = 1$. For the 1-dimensional strictly self-avoiding spread-out model, or for the weakly self-avoiding walk for any $\lambda \in (0,1)$, the determination of $c_n(\lambda)$ is no longer trivial, but has been analyzed in detail (see [16, 84, 116, 146]).

For $d = 2, 3, 4$, the best upper bounds on c_n (with $\lambda = 1$) are still the forty-year-old bounds

$$\mu^n \leq c_n \leq \begin{cases} \mu^n \exp[K n^{1/2}] & \text{if } d = 2 \\ \mu^n \exp[K n^{2/(2+d)} \log n] & \text{if } d = 3, 4 \end{cases} \tag{2.13}$$

for a positive constant K [88, 140] (see also [158, Chapter 3]). This is a long way from (2.9).

We can define a measure on $\mathcal{W}_n$ by

$$\mathbb{E}_n^{(\lambda)} X = \frac{1}{c_n^{(\lambda)}} \sum_{\omega \in \mathcal{W}_n} X(\omega) \prod_{0 \leq s < t \leq n} (1 + \lambda U_{st}(\omega)). \tag{2.14}$$

Exercise 2.1. For $\lambda = 1$, the above measure is the uniform measure on $\mathcal{S}_n$. A family of probability measures P_m on $\mathcal{S}_n$ is called *consistent* if $P_m(\omega) = \sum_{\rho > \omega} P_n(\rho)$ for all $n \geq m$ and for all $\omega \in \mathcal{S}_n$, where the sum is over all ρ whose first m steps agree with ω. Show that the uniform measure does not provide a consistent family.

The mean-square displacement is $\mathbb{E}_n^{(\lambda)} |\omega(n)|^2$ and it is believed that

$$\mathbb{E}_n^{(\lambda)} |\omega(n)|^2 \sim v_\lambda n^{2\nu} \tag{2.15}$$

where v_λ is a constant depending on λ, d, Ω, and where ν is universal and given by

$$\nu = \begin{cases} 1 & \text{if } d = 1 \\ \frac{3}{4} & \text{if } d = 2 \\ 0.588... & \text{if } d = 3 \\ \frac{1}{2} \text{ with logarithmic corrections} & \text{if } d = 4 \\ \frac{1}{2} & \text{if } d \geq 5. \end{cases} \tag{2.16}$$

The conjectured logarithmic correction to ν in four dimensions, predicted by the renormalization group, is

$$\mathbb{E}_n^{(\lambda)} |\omega(n)|^2 \sim v_\lambda n (\log n)^{1/4} \quad \text{if } d = 4. \tag{2.17}$$

In dimensions $d > 4$, the lace expansion has been used to prove that (2.15) holds with $\nu = 1/2$ for various choices of λ and Ω, including the nearest-neighbour model with $\lambda = 1$ [97, 98]. Partial results for $d = 4$ have been obtained in [43, 44, 129]. For $d = 2, 3, 4$, for the nearest-neighbour model with $\lambda = 1$, it is still an open problem even to prove the "obvious" bounds that the mean-square displacement is bounded below by n (cf. (1.16)) or bounded above by const $n^{2-\epsilon}$ for some $\epsilon > 0$. For $d = 1$, the ballistic behaviour $\nu = 1$ is obvious for the strictly self-avoiding nearest-neighbour model. It is not obvious that $\nu = 1$ for the 1-dimensional strictly self-avoiding spread-out model, or for the 1-dimensional weakly self-avoiding walk, but ballistic behaviour has been proved also in these cases [84, 146].

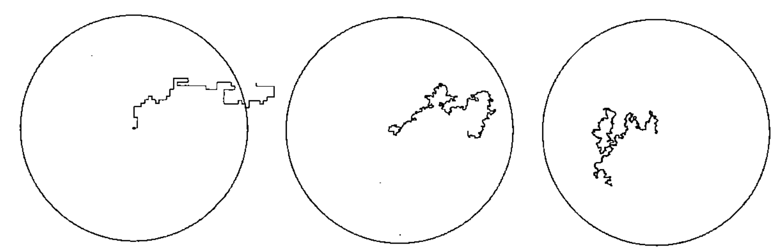


Fig. 2.1. Nearest-neighbour self-avoiding walks on $\mathbb{Z}^2$ taking $n = 100$, 1,000 and 10,000 steps, generated using the pivot algorithm [159]. The circles have radius $n^{3/4}$, in units of the step size of the self-avoiding walk.

The *two-point function* is defined by

$$G_z^{(\lambda)}(x) = \sum_{n=0}^{\infty} c_n^{(\lambda)}(x) z^n \tag{2.18}$$

and the *susceptibility* by

$$\chi^{(\lambda)}(z) = \sum_{x\in\mathbb{Z}^d} G_z^{(\lambda)}(x) = \sum_{n=0}^{\infty} c_n^{(\lambda)} z^n. \tag{2.19}$$

The latter has radius of convergence $z_c^{(\lambda)} = 1/\mu_\lambda$, by (2.6). For $\lambda = 1$, a proof that the two-point function also has this radius of convergence is given in [158, Corollary 3.2.6].

Exercise 2.2. Show that the 1-dimensional strictly self-avoiding walk ($\lambda = 1$) two-point function is given by

$$\hat{G}_z(k) = \frac{1-z^2}{1+z^2-2z\cos k}. \tag{2.20}$$

For $\lambda \in [0,1]$ and $z \in (0, z_c^{(\lambda)})$, the two-point function decays exponentially. To see this for the nearest-neighbour model, we note that $c_n^{(\lambda)}(x) = 0$ for $n < \|x\|_\infty$, and hence

$$G_z^{(\lambda)}(x) = \sum_{n=\|x\|_\infty}^{\infty} c_n^{(\lambda)}(x) z^n \le \sum_{n=\|x\|_\infty}^{\infty} c_n^{(\lambda)} z^n. \tag{2.21}$$

Since $\left(c_n^{(\lambda)}\right)^{1/n} \to \mu_\lambda$ by (2.6), for any $\epsilon > 0$ there is a positive $K_{\epsilon,\lambda}$ such that

$$c_n^{(\lambda)} \le K_{\epsilon,\lambda}(\mu_\lambda + \epsilon)^n \tag{2.22}$$

for all $n \ge 1$. Given a positive $z < z_c^{(\lambda)} = \mu_\lambda^{-1}$, we choose $\epsilon(z) > 0$ such that $\theta_{z,\lambda} = (\mu_\lambda + \epsilon(z))z < 1$. Then substitution of (2.22) into (2.21) gives

$$G_z^{(\lambda)}(x) \le C_{z,\lambda} \exp[-|\log\theta_{z,\lambda}|\,\|x\|_\infty], \tag{2.23}$$

with $C_{z,\lambda} = K_{\epsilon(z),\lambda}(1-\theta_{z,\lambda})^{-1}$. This shows the desired exponential decay of the subcritical two-point function.

Precise asymptotics of the subcritical two-point function are known in detail. This has been primarily studied for the nearest-neighbour model with $\lambda = 1$, and we assume this for the moment. First, it can be shown that for each $z \in (0, z_c)$ there is a norm $|\cdot|_z$ on $\mathbb{R}^d$ satisfying $\|u\|_\infty \le |u|_z \le \|u\|_1$ for every $u \in \mathbb{R}^d$, such that the limit

$$m(z) = \lim_{|x|_z\to\infty} \frac{-\log G_z(x)}{|x|_z} \tag{2.24}$$

exists and is finite [158, Theorem 4.1.18]. The *correlation length* is defined by

$$\xi(z) = \frac{1}{m(z)}. \tag{2.25}$$

Detailed asymptotics of the subcritical two-point function, known as Ornstein–Zernike decay, were obtained in [48,130]. It is known that $\xi(z) \to \infty$ as $z \to z_c^-$ (see, e.g., [158, Corollary 4.1.15]), and it is predicted that

$$\xi(z) \sim \text{const}\frac{1}{(1-z/z_c)^\nu} \quad \text{as } z \to z_c^-, \tag{2.26}$$

with the same exponent ν as in (2.16).

For $\lambda \in (0,1]$, it is predicted that the exponential decay of the subcritical two-point function is replaced at $z = z_c$ by

$$G_{z_c}^{(\lambda)}(x) \sim \text{const}\frac{1}{|x|^{d-2+\eta}} \quad \text{as } |x| \to \infty \tag{2.27}$$

and

$$\hat{G}_{z_c}^{(\lambda)}(k) \sim \text{const}\frac{1}{|k|^{2-\eta}} \quad \text{as } k \to 0 \tag{2.28}$$

with η given in terms of γ and ν by Fisher's relation $\gamma = (2-\eta)\nu$ (and with no logarithmic correction for $d = 4$, to leading order). Equation (2.27) has been proved (with $\eta = 0$) for the nearest-neighbour model in dimensions $d \ge 5$ [90], using the lace expansion. The k-space asymptotics are easier and are also known for the nearest-neighbour model when $d \ge 5$. Equation (2.27) has also been proved for the spread-out model with $d > 4$ and L sufficiently large [91]. In [39, 44], (2.27) is proved for a 4-dimensional hierarchical model with λ sufficiently small (again with $\eta = 0$).

It is also believed that, for all $\lambda \in (0,1]$,

$$\chi^{(\lambda)}(z) \sim \text{const}\frac{1}{(1-z/z_c)^\gamma} \quad \text{as } z \to z_c^-, \tag{2.29}$$

with a multiplicative factor $[\log(1-z/z_c)]^{1/4}$ when $d = 4$. This has been proved using the lace expansion for the nearest-neighbour model with $d > 4$ and $\lambda = 1$, with $\gamma = 1$ [97, 98]. In Sect. 5.4, we will see how to prove (2.29), with $\gamma = 1$, for the spread-out model with $d > 4$, $\lambda = 1$, and L sufficiently large, and for the nearest-neighbour model with $\lambda = 1$ and d sufficiently large.

The scaling limit, assuming it exists, is the law of the path $n^{-\nu}\omega$ in the limit $n \to \infty$ (a factor $(\log n)^{-1/4}$ should be included for $d = 4$), where ω is an n-step self-avoiding walk. The scaling limit is believed not to depend on $\lambda \in (0,1]$ in any important way. This limit is conjectured to be $\text{SLE}_{8/3}$ for $d = 2$, and the limit is not understood for $d = 3$. For $d = 4$, the scaling limit is believed to be Brownian motion, and for $d \ge 5$, the lace expansion has been used to prove that the scaling limit is Brownian motion [97, 98].

The special role of $d = 4$ for the asymptotics of the self-avoiding walk is summarized by saying that $d = 4$ is the *upper critical dimension*, and that *mean-field* behaviour applies when $d > 4$. Above $d = 4$, the self-avoiding walk has the same leading asymptotics as the simple random walk. Logarithmic corrections to simple random walk behaviour occur when $d = 4$, and different power laws appear for $d < 4$.

The critical nature of $d = 4$ can be guessed from the fact that Brownian motion is 2-dimensional. Since two 2-dimensional sets generically do not intersect in more than $4 = 2 + 2$ dimensions, above four dimensions the self-avoidance constraint does not play an important role.

2.2 Differential Inequalities and the Bubble Condition

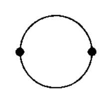
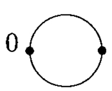
We now define the bubble condition and show that it is a sufficient condition for a particular critical exponent (namely γ) to exist and take its mean-field value. This is a useful precursor to the lace expansion. It is also a useful precursor to the study of lattice trees and percolation, where the bubble condition will be replaced by the square and triangle conditions, respectively.

For simplicity, we restrict attention in this section to the strictly self-avoiding walk with $\lambda = 1$. We fix Ω to be either (1.1) or (1.2).

The *bubble diagram* is defined by

$$B(z) = \sum_{x \in \mathbb{Z}^d} G_z(x)^2. \tag{2.30}$$

The name "bubble diagram" comes from a Feynman diagram notation in which the two-point function evaluated at vertices x and y is denoted by a line terminating at x and y. In this notation,

$B(z) = \sum_{x \in \mathbb{Z}^d} 0$  $x = 0$  ,

where in the diagram on the right it is implicit that the unlabelled vertex is summed over $\mathbb{Z}^d$. The bubble diagram can be rewritten in terms of the Fourier transform of the two-point function, using (2.30) and the Parseval relation, as

$$B(z) = \|G_z\|_2^2 = \|\hat{G}_z\|_2^2 = \int_{[-\pi,\pi]^d} \hat{G}_z(k)^2 \frac{d^d k}{(2\pi)^d}. \tag{2.31}$$

The *bubble condition* is the statement that $B(z_c) < \infty$. In other words, the bubble condition states that $\hat{G}_{z_c}(k)$ is square integrable. Recall that square integrability of $\hat{C}_{1/|\Omega|}(k)$ was important in Exercise 1.7.

In view of the definition of η in (2.27) or (2.28), it follows from (2.31) that the bubble condition is satisfied provided $\eta > (4-d)/2$. Hence the bubble condition for $d > 4$ is implied by the *infrared bound* $\eta \geq 0$. If the values for η arising from Fisher's relation and the conjectured values of γ and ν are correct, then the bubble condition will not hold in dimensions 2, 3 or 4, with the divergence of the bubble diagram being only logarithmic in four dimensions.

Throughout these notes,

$$f(z) \simeq g(z) \quad \text{denotes} \quad c^{-1} g(z) \leq f(z) \leq c g(z) \tag{2.32}$$

for some $c > 0$, uniformly in $z < z_c$. In this section, we prove a differential inequality for the susceptibility, which shows that the bubble condition implies that $\gamma = 1$ in the sense that $\chi(z) \simeq (1 - z/z_c)^{-1}$. In fact, the lower bound

$$\chi(z) \geq \frac{z_c}{z_c - z} \tag{2.33}$$

is an immediate consequence of (2.19) and the subadditivity bound $c_n \geq \mu^n = z_c^{-n}$, and holds with or without the bubble condition. It remains to prove that the complementary upper bound

$$\chi(z) \leq \text{const} \frac{1}{z_c - z} \tag{2.34}$$

is a consequence of the bubble condition. This will be shown in the following theorem. In Chap. 5, we will use the lace expansion to prove the bubble condition for the spread-out model (1.2) for $d > 4$ with L sufficiently large, and for the nearest-neighbour model with d sufficiently large.

A version of Theorem 2.3 was proved in [36]. The role of the bubble condition in proving mean-field behaviour for spin systems in dimensions $d > 4$ was developed previously, in [3, 76, 192] (see also [73]). In Sect. 5.4, it will be shown that the lace expansion actually provides a differential *equality* in place of the differential inequalities of Theorem 2.3.

Theorem 2.3. *For $0 < z < z_c$, the susceptibility obeys the differential inequalities*

$$\frac{\chi(z)^2}{B(z)} \leq \frac{d}{dz}[z\chi(z)] \leq \chi(z)^2 \tag{2.35}$$

and the inequalities

$$\frac{z_c}{z_c - z} \leq \chi(z) \leq B(z_c) \frac{2z_c - z}{z_c - z}. \tag{2.36}$$

Thus the bubble condition implies that $\chi(z) \simeq (1 - z/z_c)^{-1}$, which is to say that γ exists and equals 1.

Proof. We first prove the differential inequalities (2.35). By definition,

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_y \sum_{\omega \in \mathcal{S}(0,y)} z^{|\omega|}, \tag{2.37}$$

where $|\omega|$ denotes the number of steps in ω. For $0 < z < z_c$, term by term differentiation gives

$$Q(z) = \frac{\mathrm{d}}{\mathrm{d}z}[z\chi(z)] = \sum_y \sum_{\omega \in \mathcal{S}(0,y)} (|\omega| + 1) z^{|\omega|}, \tag{2.38}$$

where the first equality defines $Q(z)$. This can be rewritten as

$$\begin{aligned} Q(z) &= \sum_y \sum_{\omega \in \mathcal{S}(0,y)} \sum_x I[\omega(j) = x \text{ for some } j] z^{|\omega|} \\ &= \sum_{x,y} \sum_{\substack{\omega^{(1)} \in \mathcal{S}(0,x) \\ \omega^{(2)} \in \mathcal{S}(x,y)}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} = \{x\}], \end{aligned} \tag{2.39}$$

where I denotes the indicator function.

If we ignore the mutual avoidance of $\omega^{(1)}$ and $\omega^{(2)}$ in (2.39), we obtain the upper bound

$$\frac{\mathrm{d}(z\chi(z))}{\mathrm{d}z} \leq \chi(z)^2 \tag{2.40}$$

of (2.35).

To obtain a complementary bound, we rewrite $Q(z)$ by using the inclusion-exclusion relation in the form

$$I[\omega^{(1)} \cap \omega^{(2)} = \{x\}] = 1 - I[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}].$$

This gives

$$Q(z) = \chi(z)^2 - \sum_{x,y} \sum_{\substack{\omega^{(1)} \in \mathcal{S}(0,x) \\ \omega^{(2)} \in \mathcal{S}(x,y)}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}]. \tag{2.41}$$

Since $x \in \omega^{(1)} \cap \omega^{(2)}$, the indicator forces a nontrivial intersection. In the last term on the right hand side of (2.41), let $w = \omega^{(2)}(l)$ be the site of the last intersection of $\omega^{(2)}$ with $\omega^{(1)}$, where time is measured along $\omega^{(2)}$ beginning at its starting point x. Then the portion of $\omega^{(2)}$ corresponding to times greater than l must avoid all of $\omega^{(1)}$. Relaxing the restrictions that this portion of $\omega^{(2)}$ avoid both the remainder of $\omega^{(2)}$ and the part of $\omega^{(1)}$ linking w to x, and also relaxing the mutual avoidance of the two portions of $\omega^{(1)}$, gives the upper bound

$$\sum_{x,y} \sum_{\substack{\omega^{(1)} \in \mathcal{S}(0,x) \\ \omega^{(2)} \in \mathcal{S}(x,y)}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} \neq \{x\}] \leq Q(z)[B(z) - 1], \tag{2.42}$$

as illustrated in Fig. 2.2. Here the factor $B(z) - 1$ arises from the two paths joining w and x. The upper bound involves $B(z) - 1$ rather than $B(z)$ since there will be no contribution from the $x = 0$ term in (2.30).

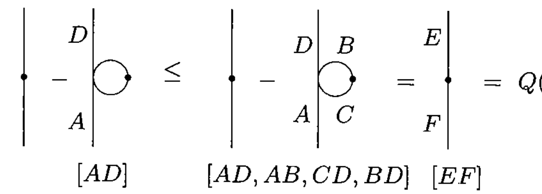


Fig. 2.2. A diagrammatic representation of the inequality $\chi(z)^2 - Q(z)[B(z) - 1] \leq Q(z)$. The lists of pairs of lines indicate interactions, in the sense that the corresponding walks must avoid each other.

Exercise 2.4. Convince yourself that (2.42) is correct.

Combining (2.41) and (2.42) gives

$$Q(z) \geq \chi(z)^2 - Q(z)[B(z) - 1]. \tag{2.43}$$

Solving for $Q(z)$ gives

$$Q(z) \geq \frac{\chi(z)^2}{B(z)}, \tag{2.44}$$

which is the lower bound of (2.35).

Next, we show that (2.35) implies (2.36). The lower bound of (2.36) has already been established in (2.33) (and also follows by integration of the upper bound of (2.35)). To obtain the upper bound of (2.36) from the lower bound of (2.35), we proceed as follows. Let $z_1 \in [0, z_c)$. The lower bound of (2.35) implies that, for $z \in [z_1, z_c)$,

$$z\left(-\frac{d\chi^{-1}}{dz}\right) \geq \frac{1}{B(z)} - \frac{1}{\chi(z)} \geq \frac{1}{B(z_c)} - \frac{1}{\chi(z_1)}, \tag{2.45}$$

where χ^{-1} denotes the reciprocal. We bound the factor of z on the left hand side by z_c and then integrate from z_1 to z_c. Using the fact that $\chi(z_c)^{-1} = 0$ by (2.33), this gives

$$z_c\chi(z_1)^{-1} \geq [B(z_c)^{-1} - \chi(z_1)^{-1}](z_c - z_1). \tag{2.46}$$

Rewriting gives the upper bound of (2.36). ∎

By (2.39), $Q(z)$ is the generating function for pairs of self-avoiding walks which do not intersect each other apart from their common starting point. It follows from Theorem 2.3 that if the bubble condition holds then

$$\frac{Q(z)}{\chi(z)^2} \simeq 1, \tag{2.47}$$

a relation related to the non-vanishing of the non-intersection probability q_n of (2.12), as $n \to \infty$, when $\gamma = 1$.

3

The Lace Expansion for the Self-Avoiding Walk

The lace expansion was derived by Brydges and Spencer in [45]. Their derivation, which is given below in Sects. 3.2–3.3, involves an expansion and resummation procedure closely related to the cluster expansions of statistical mechanics [40]. It was later noted that the lace expansion can also be seen as resulting from repeated application of the inclusion-exclusion relation [186]. For a more combinatorial description of the lace expansion, see [211]. We first discuss the inclusion-exclusion approach.

3.1 Inclusion-Exclusion

The inclusion-exclusion approach to the lace expansion is closely related to the method of proof of Theorem 2.3. In that proof, a single inclusion-exclusion was used to obtain upper and lower bounds. Here, we will derive an identity by using repeated inclusion-exclusion.

For simplicity, we restrict attention to the strictly self-avoiding walk ($\lambda = 1$). We consider a walk taking steps in a finite set Ω, so that $\omega(i+1)-\omega(i) \in \Omega$ for each i, but there is no need here for a symmetry assumption and Ω is an arbitrary finite set. As in (1.10), we write

$$D(x) = \frac{1}{|\Omega|} I[x \in \Omega]. \tag{3.1}$$

We rewrite $c_n(x)$ using the inclusion-exclusion relation. Namely, we first count all walks from 0 to x which are self-avoiding *after* the first step, and then subtract the contribution due to those which are not self-avoiding from the beginning, i.e., walks that return to the origin. Since $c_1(0,y) = 1$ for $y \in \Omega$, this gives

$$c_n(x) = (c_1 * c_{n-1})(x) - \sum_{y \in \Omega} \sum_{\omega^{(1)} \in \mathcal{S}_{n-1}(y,x)} I[0 \in \omega^{(1)}]. \tag{3.2}$$

Comparing with (1.5), it is the second term on the right hand side that makes the above equation interesting.

The inclusion-exclusion relation can now be applied to the last term of (3.2), as follows. Let s be the first (and only) time that $\omega^{(1)}(s) = 0$. Then for $y \in \Omega$,

$$\begin{aligned}\sum_{\omega^{(1)} \in \mathcal{S}_{n-1}(y,x)} & I[0 \in \omega^{(1)}] \\ &= \sum_{s=1}^{n-1} \sum_{\substack{\omega^{(2)} \in \mathcal{S}_s(y,0) \\ \omega^{(3)} \in \mathcal{S}_{n-1-s}(0,x)}} I[\omega^{(2)} \cap \omega^{(3)} = \{0\}] \qquad (3.3) \\ &= \sum_{s=1}^{n-1} \Big[c_s(y,0) c_{n-1-s}(0,x) - \sum_{\substack{\omega^{(2)} \in \mathcal{S}_s(y,0) \\ \omega^{(3)} \in \mathcal{S}_{n-1-s}(0,x)}} I[\omega^{(2)} \cap \omega^{(3)} \neq \{0\}] \Big].\end{aligned}$$

We can interpret $c_s(y,0)$ as the number of $(s+1)$-step walks which step from the origin directly to y, then return to the origin in s steps, and which have distinct vertices apart from the fact that they return to their starting point. Let $\mathcal{U}_s$ denote the set of all s-step self-avoiding loops at the origin (s-step walks which begin and end at the origin but which otherwise have distinct vertices), and let u_s be the cardinality of $\mathcal{U}_s$. Then

$$\begin{aligned}\sum_{y \in \Omega} \sum_{\omega^{(1)} \in \mathcal{S}_{n-1}(y,x)} & I[0 \in \omega^{(1)}] \\ &= \sum_{s=2}^{n} u_s c_{n-s}(x) - \sum_{s=2}^{n} \sum_{\substack{\omega^{(2)} \in \mathcal{U}_s \\ \omega^{(3)} \in \mathcal{S}_{n-s}(0,x)}} I[\omega^{(2)} \cap \omega^{(3)} \neq \{0\}]. \qquad (3.4)\end{aligned}$$

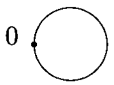
Continuing in this fashion, in the last term on the right hand side of the above equation, let $t \geq 1$ be the first time along $\omega^{(3)}$ that $\omega^{(3)}(t) \in \omega^{(2)}$, and let $v = \omega^{(3)}(t)$. Then the inclusion-exclusion relation can be applied again to remove the avoidance between the portions of $\omega^{(3)}$ before and after t, and correct for this removal by the subtraction of a term involving a further intersection. Repetition of this procedure leads to the convolution equation

$$c_n(0,x) = (|\Omega| D * c_{n-1})(x) + \sum_{m=2}^{n} (\pi_m * c_{n-m})(x), \qquad (3.5)$$

where we have used $c_1(x) = |\Omega| D(x)$, and where π_m is given by

$$\pi_m(v) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(v), \qquad (3.6)$$

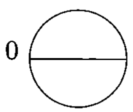
with the terms on the right hand side defined as follows. The $N = 1$ term is given by

$$\pi_m^{(1)}(v) = \delta_{0,v} u_m = \delta_{0,v} \ 0$$  ,

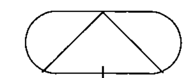
where the diagram represents u_m. The $N = 2$ term is

$$\pi_m^{(2)}(v) = \sum_{\substack{m_1, m_2, m_3 : \\ m_1 + m_2 + m_3 = m}} \sum_{\omega_1 \in \mathcal{S}_{m_1}(0,v)} \sum_{\omega_2 \in \mathcal{S}_{m_2}(v,0)} \sum_{\omega_3 \in \mathcal{S}_{m_3}(0,v)} I(\omega_1, \omega_2, \omega_3),$$

where $I(\omega_1, \omega_2, \omega_3)$ is equal to 1 if the ω_i are pairwise mutually avoiding apart from their common endpoints, and otherwise equals 0. Diagrammatically this can be represented by

$$\pi_m^{(2)}(v) = 0$$  v ,

where each line represents a sum over self-avoiding walks between the endpoints of the line, with mutual avoidance between the three pairs of lines in the diagram. Similarly

$$\pi_m^{(3)}(v) =$$  ,

0 v

where now there is mutual avoidance between some but not all pairs of lines in the diagram; a precise description requires some care. The unlabelled vertex is summed over $\mathbb{Z}^d$. A slashed diagram line is used to indicate a walk which may have zero steps, i.e., be a single site, whereas lines without a slash correspond to walks of at least one step. All the higher order terms can be expressed as diagrams in this way, and with some care it is possible to keep track of the pattern of mutual avoidance between subwalks (individual lines in the diagram) which emerges. The algebraic derivation of the expansion, described next, keeps track of this mutual avoidance automatically. Equations (3.5)–(3.6) constitute the lace expansion. No laces have appeared yet, but they will come later.

Exercise 3.1. Determine a precise expression for $\pi_m^{(3)}(v)$. What is the picture for $\pi_m^{(4)}(v)$?

3.2 Expansion

In this and the following section, we give the original derivation of the lace expansion due to Brydges and Spencer [45]. The expansion applies in a more general context than we have considered so far, and we will give a quite general derivation.

Consider walks taking steps in a finite subset $\Omega \subset \mathbb{Z}^d$. Suppose that to each walk $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ and each pair $s, t \in \{0, 1, \ldots, n\}$, we are given a complex number $\mathcal{U}_{st}(\omega)$ (for example, (2.1)).

Definition 3.2. (i) *Given an interval $I = [a, b]$ of positive integers, we refer to a pair $\{s, t\}$ $(s < t)$ of elements of I as an* edge. *To abbreviate the notation, we usually write st for $\{s, t\}$. A set of edges is called a* graph. *The set of all graphs on $[a, b]$ is denoted $\mathcal{B}[a, b]$.*
(ii) *A graph Γ is said to be* connected *if both a and b are endpoints of edges in Γ, and if in addition, for any $c \in (a, b)$, there are $s, t \in [a, b]$ such that $s < c < t$ and $st \in \Gamma$. In other words, Γ is connected if, as intervals, $\cup_{st \in \Gamma}(s, t) = (a, b)$. The set of all connected graphs on $[a, b]$ is denoted $\mathcal{G}[a, b]$.*

An apology is required for graph theorists. The above notion of connectivity is not the usual notion of path-connectivity in graph theory. Instead, the above notion relies heavily on the fact that the vertices of the graph are linearly ordered in time, and may be justified by the fact that connected graphs are those for which $\cup_{st \in \Gamma}(s, t)$ is equal to the connected interval (a, b). In any event, it is decidedly not path-connectivity. There are connected graphs that are not path-connected, and vice versa. It is convenient to have in mind the representation of graphs illustrated in Fig. 3.1.

We set $K[a, a] = 1$, and for $a < b$ we define

$$K[a, b] = \prod_{a \le s < t \le b} (1 + \mathcal{U}_{st}), \tag{3.7}$$

where the dependence on ω is left implicit. By expanding the product in (3.7), we obtain

$$K[a, b] = \sum_{\Gamma \in \mathcal{B}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}. \tag{3.8}$$

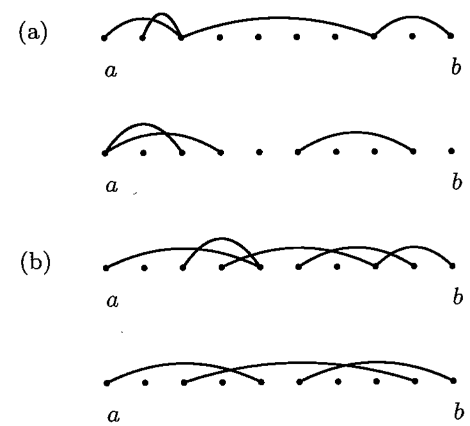


Fig. 3.1. Graphs in which an edge st is represented by an arc joining s and t. The graphs in (a) are not connected, whereas the graphs in (b) are connected.

Note that $\mathcal{B}[a, b]$ contains the graph with no edges, so our convention that $K[a, a] = 1$ is consistent with the standard convention that an empty product is equal to 1.

Exercise 3.3. Prove (3.8).

We set $J[a, a] = 1$, and for $a < b$ we define a quantity analogous to $K[a, b]$, but with the sum over graphs restricted to connected graphs:

$$J[a, b] = \sum_{\Gamma \in \mathcal{G}[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}. \tag{3.9}$$

Lemma 3.4. *For any $a < b$,*

$$K[a, b] = K[a + 1, b] + \sum_{j=a+1}^{b} J[a, j] K[j, b]. \tag{3.10}$$

Proof. The contribution to the sum on the right hand side of (3.8) due to all graphs Γ for which a is not in an edge is exactly $K[a + 1, b]$. To resum the contribution due to the remaining graphs, we proceed as follows. If Γ does contain an edge containing a, let $j(\Gamma)$ be the largest value of j such that the set of edges in Γ with both ends in the interval $[a, j]$ forms a connected graph on $[a, j]$. Then the sum over Γ factorizes into sums over connected graphs on $[a, j]$ and arbitrary graphs on $[j, b]$, and resummation of the latter gives

$$K[a, b] = K[a + 1, b] + \sum_{j=a+1}^{b} \sum_{\Gamma \in \mathcal{G}[a,j]} \prod_{st \in \Gamma} \mathcal{U}_{st} \, K[j, b], \tag{3.11}$$

which with (3.9) proves the lemma. ∎

Let

$$c_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} K[0, n] = \sum_{\omega \in \mathcal{W}_n(x)} \prod_{0 \le s < t \le n} (1 + \mathcal{U}_{st}(\omega)), \tag{3.12}$$

a generalization of (2.2). It is simplest if we assume that $\mathcal{U}_{st}(\omega)$ is invariant under spatial translation of ω, and under an equal shift of each of s, t and the time parameter of ω, and we make this assumption. Note that (2.1) obeys the assumption. We substitute (3.10) into (3.12). A key point is that in the last term of (3.10) the portion of the walk from time j onwards is independent of the portion up to time j. Let

$$\pi_m(x) = \sum_{\omega \in \mathcal{W}_m(0,x)} J[0, m]. \tag{3.13}$$

Then for $n \ge 1$, we obtain

$$c_n(x) = (|\Omega| D * c_{n-1})(x) + \sum_{m=1}^{n} (\pi_m * c_{n-m})(x), \tag{3.14}$$

as in (3.5).[1] To obtain a more useful representation of π_m than (3.13), we perform a resummation of (3.13) using the notion of laces.

3.3 Laces and Resummation

Definition 3.5. *A* lace *is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a,b]$ is denoted by $\mathcal{L}[a,b]$, and the set of laces on $[a,b]$ which consist of exactly N edges is denoted $\mathcal{L}^{(N)}[a,b]$.*

We write $L \in \mathcal{L}^{(N)}[a,b]$ as $L = \{s_1t_1, \ldots, s_Nt_N\}$, with $s_l < t_l$ for each l. The fact that L is a lace is equivalent to a certain ordering of the s_l and t_l. For $N = 1$, we simply have $a = s_1 < t_1 = b$. For $N \geq 2$, $L \in \mathcal{L}^{(N)}[a,b]$ if and only if

$$a = s_1 < s_2, \quad s_{l+1} < t_l \leq s_{l+2} \quad (l = 1, \ldots, N-2), \quad s_N < t_{N-1} < t_N = b \tag{3.15}$$

(for $N = 2$ the vacuous middle inequalities play no role); see Fig. 3.2. Thus L divides $[a,b]$ into $2N-1$ subintervals:

$$[s_1, s_2],\ [s_2, t_1],\ [t_1, s_3],\ [s_3, t_2],\ \ldots\ ,[s_N, t_{N-1}],\ [t_{N-1}, t_N]. \tag{3.16}$$

Of these, intervals number 3, 5, ..., $(2N-3)$ can have zero length for $N \geq 3$, whereas all others have length at least 1.

Exercise 3.6. Prove that (3.15) characterizes laces.

Given a connected graph $\Gamma \in \mathcal{G}[a,b]$, the following prescription associates to Γ a unique lace $\mathsf{L}_\Gamma \subset \Gamma$: The lace L_Γ consists of edges $s_1t_1, s_2t_2, \ldots$, with $t_1, s_1, t_2, s_2, \ldots$ determined, in that order, by

$$t_1 = \max\{t : at \in \Gamma\}, \quad s_1 = a,$$

$$t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \quad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}.$$

The procedure terminates when $t_{i+1} = b$. Given a lace L, the set of all edges $st \notin L$ such that $\mathsf{L}_{L\cup\{st\}} = L$ is denoted $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be *compatible* with L. Fig. 3.3 illustrates these definitions.

Exercise 3.7. Show that $\mathsf{L}_\Gamma = L$ if and only if L is a lace, $L \subset \Gamma$, and $\Gamma \setminus L \subset \mathcal{C}(L)$.

[1] For $m = 1$, there is a single connected graph $\{01\}$, and when $\mathcal{U}_{st}$ is given by (2.1) we have $\pi_1(x) = \sum_{\omega \in \mathcal{W}_1(0,x)} U_{01}(\omega) = 0$, since it is always the case that $\omega(0) \neq \omega(1)$. Thus the sum over m in (3.14) can be started at $m = 2$ in this case.

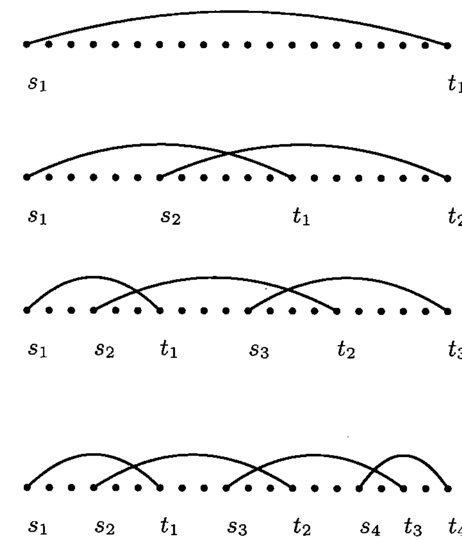


Fig. 3.2. Laces in $\mathcal{L}^{(N)}[a,b]$ for $N = 1,2,3,4$, with $s_1 = a$ and $t_N = b$.

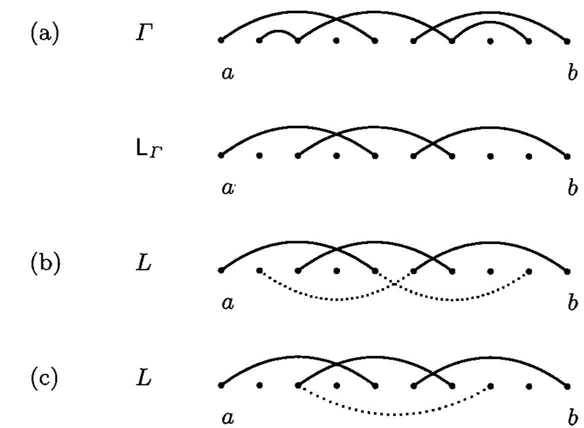


Fig. 3.3. (a) A connected graph Γ and its associated lace $L = \mathsf{L}_\Gamma$. (b) The dotted edges are compatible with the lace L. (c) The dotted edge is not compatible with the lace L.

The sum over connected graphs in (3.9) can be performed by first summing over all laces and then, given a lace, summing over all connected graphs associated to that lace by the above prescription. This gives

$$J[a,b] = \sum_{L \in \mathcal{L}[a,b]} \prod_{st \in L} \mathcal{U}_{st} \sum_{\Gamma : \mathsf{L}_\Gamma = L} \prod_{s't' \in \Gamma \setminus L} \mathcal{U}_{s't'}. \tag{3.17}$$

But, writing $\Gamma' = \Gamma \setminus L$, it follows from Exercise 3.7 that

$$\sum_{\Gamma:\mathsf{L}_\Gamma = L} \prod_{s't' \in \Gamma \setminus L} \mathcal{U}_{s't'} = \sum_{\Gamma' \subset \mathcal{C}(L)} \prod_{s't' \in \Gamma'} \mathcal{U}_{s't'} = \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}). \tag{3.18}$$

Therefore,

$$J[a,b] = \sum_{L \in \mathcal{L}[a,b]} \prod_{st \in L} \mathcal{U}_{st} \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}). \tag{3.19}$$

Inserting this in (3.13) gives

$$\pi_m(x) = \sum_{\omega \in \mathcal{W}_m(0,x)} \sum_{L \in \mathcal{L}[0,m]} \prod_{st \in L} \mathcal{U}_{st} \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}). \tag{3.20}$$

For $a < b$ we define $J^{(N)}[a,b]$ to be the contribution to (3.17) from laces consisting of exactly N bonds:

$$J^{(N)}[a,b] = \sum_{L \in \mathcal{L}^{(N)}[a,b]} \prod_{st \in L} \mathcal{U}_{st} \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}). \tag{3.21}$$

For the special case in which $\mathcal{U}_{st}$ is given by (2.1), each term in the above sum is either 0 or $(-1)^N$. By (3.17) and (3.21),

$$J[a,b] = \sum_{N=1}^{\infty} J^{(N)}[a,b]. \tag{3.22}$$

The sum over N in (3.22) is a finite sum, since the sum in (3.21) is empty for $N > b - a$ and hence $J^{(N)}[a,b] = 0$ if $N > b - a$.

Now we define

$$\begin{aligned} \pi_m^{(N)}(x) &= (-1)^N \sum_{\omega \in \mathcal{W}_m(x)} J^{(N)}[0,m] \\ &= \sum_{\omega \in \mathcal{W}_m(x)} \sum_{L \in \mathcal{L}^{(N)}[0,m]} \prod_{st \in L} (-\mathcal{U}_{st}) \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}). \end{aligned} \tag{3.23}$$

The factor $(-1)^N$ on the right hand side of (3.23) has been inserted to arrange that

$$\pi_m^{(N)}(x) \geq 0 \quad \text{for all } N, m, x \tag{3.24}$$

when $\mathcal{U}_{st}$ is given by U_{st} of (2.1). By (3.13), (3.22) and (3.23),

$$\pi_m(x) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(x). \tag{3.25}$$

For the special case in which $\mathcal{U}_{st}$ is given by (2.1), walks making a nonzero contribution to (3.23) are constrained to have the topology indicated in Fig. 3.4. In the figure, for $\prod_{s't' \in \mathcal{C}(L)} (1 + U_{s't'}) \neq 0$, each of the $2N-1$ subwalks must be a self-avoiding walk, and in addition there must be mutual avoidance between some (but not all) of the subwalks. The number of loops (faces excluding the "outside" face) in a diagram is equal to the number of edges in the corresponding lace. The lines which are slashed correspond to subwalks which may consist of zero steps, but the others correspond to subwalks consisting of at least one step. This gives an interpretation of $\pi_m^{(N)}$ identical to that obtained in Sect. 3.1, but here there is the advantage that explicit formulas keep track of the mutual avoidance between subwalks.

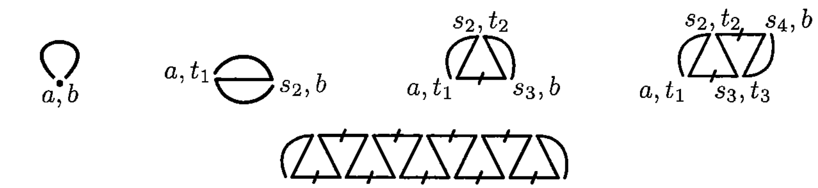


Fig. 3.4. Self-intersections required for a walk ω with $\prod_{st \in L} U_{st}(\omega) \neq 0$, with U_{st} given by (2.1), for the laces with $N = 1, 2, 3, 4$ bonds depicted in Fig. 3.2. The picture for $N = 11$ is also shown.

It is sometimes convenient to modify the definitions of "connected graph" and "lace," and we will do so in Sect. 8.1. A more general theory of laces is developed and applied in [124, 126], for the analysis of networks of mutually-avoiding self-avoiding walks. See also [125] for an application of the more general theory to lattice trees.

3.4 Transformations

Equation (3.14) involves convolution in both space and time. It has been studied in this form in [29], via fixed point methods.

It is tempting to use transformations to eliminate one or both of these convolutions. We can eliminate the convolution in space if we take the Fourier transform (1.6). For $n \geq 1$, this gives

$$\hat{c}_n(k) = |\Omega| \hat{D}(k) \hat{c}_{n-1}(k) + \sum_{m=1}^{n} \hat{\pi}_m(k) \hat{c}_{n-m}(k). \tag{3.26}$$

Conditions are given in [120] which ensure that solutions of (3.26) have Gaussian asymptotics, via an analysis based on induction on n.

We may instead prefer to eliminate the convolution in time, by going to generating functions. Using (2.18) and (3.14), this gives

$$\begin{aligned} G_z(x) &= \delta_{0,x} + \sum_{n=1}^{\infty} c_n(x) z^n \\ &= \delta_{0,x} + z|\Omega|(D * G_z)(x) + (\Pi_z * G_z)(x), \end{aligned} \tag{3.27}$$

where

$$\Pi_z(x) = \sum_{m=1}^{\infty} \pi_m(x) z^m. \tag{3.28}$$

Equation (3.27) has been studied in [90, 91].

Finally, we may prefer to eliminate both convolutions by using both the Fourier transform and generating functions. Taking the Fourier transform of (3.27) gives

$$\hat{G}_z(k) = 1 + z|\Omega|\hat{D}(k)\hat{G}_z(k) + \hat{\Pi}_z(k)\hat{G}_z(k), \tag{3.29}$$

which can be solved to give

$$\hat{G}_z(k) = \frac{1}{1 - z|\Omega|\hat{D}(k) - \hat{\Pi}_z(k)}. \tag{3.30}$$

Equation (3.30) has been the point of departure for several studies of the self-avoiding walk, and we will work with (3.30) in Chap. 5.

Exercise 3.8. The memory-2 walk is the walk with $\mathcal{U}_{st} = U_{st}$ if $t - s \leq 2$, and otherwise $\mathcal{U}_{st} = 0$. This is a random walk with no immediate reversals. Suppose that $0 \notin \Omega \subset \mathbb{Z}^d$ is finite and invariant under the symmetries of the lattice.

(a) What is the value of $\hat{c}_n(0)$, the number of n-step memory-2 walks? (Calculation is not required.)

(b) Prove that for the memory-2 walk, for $m \geq 2$,

$$\pi_m(x) = \begin{cases} -|\Omega|\delta_{x,0} & \text{if } m \text{ is even} \\ I[x \in \Omega] & \text{if } m \text{ is odd.} \end{cases}$$

(c) Suppose that $|\Omega| > 2$. Show that the mean-square displacement for the memory-two walk is given by

$$\sigma^2 \left[\left(\frac{1+\delta}{1-\delta} \right) n - \frac{2\delta(1-\delta^n)}{(1-\delta)^2} \right] \sim \left(\frac{\sigma^2 |\Omega|}{|\Omega| - 2} \right) n,$$

where $\sigma^2 = \sum_x |x|^2 D(x)$ is the variance of D and $\delta = (|\Omega| - 1)^{-1}$. One approach[2] is to use (3.26) to compute $\nabla^2 \hat{c}_n(0)$. This problem goes back a long way [18, 63, 72].

(d) Show that for the memory-two walk,

$$\hat{G}_z(k) = \frac{1 - z^2}{1 + (|\Omega| - 1)z^2 - z|\Omega|\hat{D}(k)}$$

[2] Verification of the formula by induction seems an unsatisfactory solution, since it requires prior knowledge of the formula.

(compare Exercise 2.2 for $d = 1$). This formula was used to compute the mean-square displacement via contour integration in [158, Sect. 5.3].

The memory-τ walk is the walk with $\mathcal{U}_{st} = U_{st}$ if $t - s \leq \tau$, and otherwise $\mathcal{U}_{st} = 0$. Finite-memory walks played an important role in the original analysis of the lace expansion in [45], but will not concern us further here. For a study of generating functions of the number of memory-τ walks, for $\tau \leq 8$, see [171].

4

Diagrammatic Estimates for the Self-Avoiding Walk

The difficulty in analyzing the lace expansion is to understand the function $\pi_m(x)$, or one of its transforms. In this chapter, we will prove estimates for the Fourier transform $\hat{\Pi}_z(k)$ of the generating function $\Pi_z(x) = \sum_{m=1}^{\infty} \pi_m(x) z^m$. Related estimates of one sort or another have been used in every analysis of the lace expansion for the self-avoiding walk. Throughout this chapter, we use the notation of Sect. 3.2, and we take $\mathcal{U}_{st}$ to be given by (2.1), i.e.,

$$\mathcal{U}_{st} = U_{st} = \begin{cases} -1 \text{ if } \omega(s) = \omega(t) \\ \ \ 0 \text{ if } \omega(s) \neq \omega(t). \end{cases} \tag{4.1}$$

We also assume that our walks take steps in a finite set Ω which is invariant under the symmetries of $\mathbb{Z}^d$, namely permutation of coordinates and replacement of any coordinate x_i by $-x_i$.

We will obtain estimates for $\sum_{x\in\mathbb{Z}^d} \Pi_z^{(N)}(x)$, which is an upper bound for $|\hat{\Pi}_z^{(N)}(k)|$, and for $\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\Pi_z^{(N)}(x)$, which is an upper bound for $\hat{\Pi}_z^{(N)}(0) - \hat{\Pi}_z^{(N)}(k)$. To motivate the latter, let $\hat{F}_z(k) = 1/\hat{G}_z(k)$, and note from (3.30) that

$$\begin{aligned} \hat{G}_z(k) &= \frac{1}{\hat{F}_z(0) + [\hat{F}_z(k) - \hat{F}_z(0)]} \\ &= \frac{1}{\hat{F}_z(0) + z|\Omega|[1-\hat{D}(k)] + [\hat{\Pi}_z(0) - \hat{\Pi}_z(k)]}. \end{aligned} \tag{4.2}$$

Our estimate for $\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\Pi_z^{(N)}(x)$ will ultimately allow us to compare the terms $[\hat{\Pi}_z(0) - \hat{\Pi}_z(k)]$ and $z|\Omega|[1-\hat{D}(k)]$ in the denominator.

4.1 The Diagrammatic Estimates

Recall from (3.23) and (3.25) that $\pi_m(x) = \sum_{N=1}^{\infty}(-1)^N \pi_m^{(N)}(x)$, with

$$\pi_m^{(N)}(x) = \sum_{\omega\in\mathcal{W}_m(x)} \sum_{L\in\mathcal{L}^{(N)}[0,m]} \prod_{st\in L}(-U_{st}) \prod_{s't'\in\mathcal{C}(L)} (1+U_{s't'}). \tag{4.3}$$

For $z \geq 0$, we define the non-negative generating function

$$\Pi_z^{(N)}(x) = \sum_{m=2}^{\infty} \pi_m^{(N)}(x) z^m. \tag{4.4}$$

In the above series, we omit the term involving $\pi_1^{(N)}(x)$ because it is always zero, since the only lace on $[0,1]$ is $L = \{01\}$, and $U_{01} = 0$ since a walk cannot be at the same place at consecutive times. By (3.28), we have

$$\Pi_z(x) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(N)}(x). \tag{4.5}$$

Let

$$H_z(x) = G_z(x) - \delta_{0,x} = \sum_{n=1}^{\infty} c_n(x) z^n. \tag{4.6}$$

The following theorem gives bounds on $\Pi_z^{(N)}$ in terms of norms of G_z and H_z.

Theorem 4.1. *For all $z \geq 0$,*

$$\sum_{x \in \mathbb{Z}^d} \Pi_z^{(1)}(x) \leq z|\Omega| \, \|H_z\|_\infty \tag{4.7}$$

and

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \Pi_z^{(1)}(x) = 0. \tag{4.8}$$

For $z \geq 0$ and $N \geq 2$,

$$\sum_{x \in \mathbb{Z}^d} \Pi_z^{(N)}(x) \leq \|H_z\|_\infty \|H_z * G_z\|_\infty^{N-1}, \tag{4.9}$$

and

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] \Pi_z^{(N)}(x) \leq (N+1)\lfloor N/2 \rfloor \|[1 - \cos(k \cdot x)] H_z(x)\|_\infty \|H_z * G_z\|_\infty^{N-1}. \tag{4.10}$$

We refer to the bounds of Theorem 4.1 as diagrammatic estimates, as they are inspired by the diagrams of Fig. 3.4. Moreover, the diagrams themselves provide a pictorial representation of the bounds, and have the dual interpretation of depicting both walk trajectories that contribute to $\Pi_z^{(N)}(x)$ as well as upper bounds on these quantities. See Fig. 4.1.[1]

[1] Fig. 4.1 shows a slight improvement of (4.9) (as $H_z \leq G_z$) and it is possible to prove the improvement, but we will only prove (4.9), which suffices for our needs.

Fig. 4.1. Depiction of the estimate $\sum_x \Pi_z^{(4)}(x) \leq \|H_z * H_z\|_\infty \|H_z * G_z\|_\infty^2 \|H_z\|_\infty$ by decomposition of the diagram for $\Pi_z^{(4)}$.

In applying Theorem 4.1, we will use the estimate

$$\|H_z * G_z\|_\infty = \|H_z + (H_z * H_z)\|_\infty \leq \|H_z\|_\infty + \|H_z\|_2^2, \tag{4.11}$$

using the triangle and Cauchy–Schwarz inequalities in the last step. Since $G_z(0) = 1$, it follows from (4.6) and (2.30) that

$$\|H_z\|_2^2 = \|G_z\|_2^2 - 1 = B(z) - 1. \tag{4.12}$$

To control the sum over N in (4.5) using (4.9), our method in Chap. 5 will require, in particular, that $B(z_c) - 1$ be small. This is a restrictive form of the bubble condition of Sect. 2.2.

4.2 Proof of the Diagrammatic Estimates

In this section, we prove Theorem 4.1.

4.2.1 Proof of (4.7)–(4.8)

The estimates (4.7)–(4.8) are easy, and we prove them first. Since the unique lace on $[0,m]$ consisting of a single bond is simply the bond $0m$, it follows from (3.23) that

$$\pi_m^{(1)}(x) = \delta_{0,x} \sum_{\omega \in \mathcal{W}_m(0,0)} \prod_{s't' \in \mathcal{C}(0m)} (1 + U_{s't'}). \tag{4.13}$$

There is no 1-step walk from 0 to 0, so this is nonzero only for $m \geq 2$. Since $\mathcal{C}(0m) \supset \mathcal{B}[0, m-1]$ for $m \geq 2$, it follows from (4.1) that

$$0 \leq \pi_m^{(1)}(x) \leq \delta_{0,x} \sum_{\omega \in \mathcal{W}_m(0,0)} K[0, m-1] = \delta_{0,x} \sum_{y \in \Omega} c_{m-1}(y). \tag{4.14}$$

Therefore, after multiplying by z^m and summing over $m \geq 2$, we obtain

$$0 \leq \Pi_z^{(1)}(x) \leq \delta_{0,x} z \sum_{y \in \Omega} H_z(y), \tag{4.15}$$

which immediately implies (4.7)–(4.8).

4.2.2 The Diagrams

In preparation for the proof of (4.9)–(4.10), we now prove a preliminary estimate on $\Pi_z^{(N)}(x)$.

For $N \geq 2$, we define $p_m^{(N)}(x,y)$ inductively as follows. Let

$$p_m^{(2)}(x,y) = \sum_{0<i<j\leq m} c_i(x)c_{j-i}(x)c_{m-j}(y) \quad (m \geq 2), \tag{4.16}$$

$$a_m(u,v,x,y) = \delta_{v,x}\sum_{l=1}^{m} c_l(u-v)c_{m-l}(y-u) \quad (m \geq 1). \tag{4.17}$$

For $m = 0, 1$, we set $p_m^{(2)}(x,y) = 0$, and we set $a_0(u,v,x,y) = 0$. For $N \geq 3$, let

$$p_m^{(N)}(x,y) = \sum_{u,v\in\mathbb{Z}^d}\sum_{i=0}^{m} p_i^{(N-1)}(u,v)a_{m-i}(u,v,x,y) \quad (m \geq 2). \tag{4.18}$$

For $N \geq 2$, we also define the generating functions

$$P_z^{(N)}(x,y) = \sum_{m=2}^{\infty} p_m^{(N)}(x,y)z^m, \tag{4.19}$$

$$\begin{aligned} A_z(u,v,x,y) &= \sum_{m=1}^{\infty} a_m(u,v,x,y)z^m \\ &= \delta_{v,x}H_z(u-v)G_z(y-u). \end{aligned} \tag{4.20}$$

It follows from (4.18) that, for $N \geq 3$,

$$P_z^{(N)}(x,y) = \sum_{u,v\in\mathbb{Z}^d} P_z^{(N-1)}(u,v)A_z(u,v,x,y). \tag{4.21}$$

The diagrammatic representations for $P_z^{(N)}(x,y)$ shown in Fig. 4.2 are closely related to the diagrams appearing in Fig. 3.4.

Proposition 4.2. *For $N \geq 2$, $m \geq 2$, and $z \geq 0$,*

$$\pi_m^{(N)}(x) \leq p_m^{(N)}(x,x) \tag{4.22}$$

and

$$\Pi_z^{(N)}(x) \leq P_z^{(N)}(x,x). \tag{4.23}$$

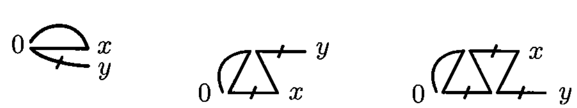


Fig. 4.2. $P_z^{(N)}(x,y)$ for $N = 2, 3, 4$.

Proof. We prove the first inequality, as the second then follows immediately from (4.4).

For $N \geq 2$ we write a lace $L \in \mathcal{L}^{(N-1)}[0,j]$ as $\{s_1t_1,\ldots,s_{N-1}t_{N-1}\}$, with $s_1 = 0$ and $t_{N-1} = j$. For $N \geq 2$, we define

$$\begin{aligned} &J_x^{(N)}[0,m] \\ &= \sum_{j=0}^{m} K[j,m] \sum_{L\in\mathcal{L}^{(N-1)}[0,j]} \sum_{i=t_{N-2}}^{j-1} \delta_{x,\omega(i)} \prod_{st\in L}(-U_{st}) \prod_{s't'\in\mathcal{C}(L)}(1+U_{s't'}), \end{aligned} \tag{4.24}$$

where we set $t_0 = 1$ when $N = 2$. We first show that, for every ω and every $N \geq 2$,

$$0 \leq (-1)^N J^{(N)}[0,m] \leq J_{\omega(m)}^{(N)}[0,m]. \tag{4.25}$$

The first inequality is immediate, and we concentrate on the second. For this, comparing with (3.21), L in (4.24) corresponds to $L\backslash\{s_Nt_N\}$ in (3.21), and i and j of (4.24) correspond to s_N and t_{N-1} of the lace L in (3.21). The set of compatible edges in (3.21) contains $\mathcal{C}(s_1t_1,\ldots,s_{N-1}t_{N-1}) \cup \mathcal{B}[t_{N-1},m]$, and omitting factors in the product over $s't'$ in (3.21) can only increase the product. When $x = \omega(m)$, the factor $\delta_{x,\omega(i)}$ is $-U_{i,m} = -U_{s_Nt_N}$. This leads to (4.25).

For $N \geq 2$, we define

$$\pi_m^{(N)}(x,y) = \sum_{\omega\in\mathcal{W}_m(0,y)} J_x^{(N)}[0,m]. \tag{4.26}$$

It follows from (3.23) and (4.25) that

$$\pi_m^{(N)}(x) \leq \pi_m^{(N)}(x,x). \tag{4.27}$$

We will show that

$$\pi_m^{(N)}(x,y) \leq p_m^{(N)}(x,y) \qquad (N \geq 2), \tag{4.28}$$

which then gives the proposition.

The proof of (4.28) is by induction on N. We begin the induction with the case $N = 2$. In this case, the sum over L in (4.24) consists of the single term $L = \{0j\}$. Since $\mathcal{C}(0j) \supset \mathcal{B}[0,i] \cup \mathcal{B}[i,j]$ for $0 < i < j$, using symmetry we obtain

$$\begin{aligned} \pi_m^{(2)}(x,y) &\leq \sum_{0<i<j<m}\sum_{\omega\in\mathcal{W}_m(0,y)} K[0,i]\delta_{x,\omega(i)}K[i,j]\delta_{0,\omega(j)}K[j,m] \\ &= \sum_{0<i<j<m} c_i(x)c_{j-i}(x)c_{m-j}(y) \\ &\leq p_m^{(2)}(x,y). \end{aligned} \tag{4.29}$$

To advance the induction, we fix $N \geq 3$ and assume that (4.28) holds for $N-1$. We replace the factor $-U_{s_{N-1}t_{N-1}}$ in the first product of (4.24) by

$$-U_{s_{N-1}t_{N-1}} = \delta_{\omega(s_{N-1}),\omega(t_{N-1})} = \sum_{u \in \mathbb{Z}^d} \delta_{\omega(s_{N-1}),u}\delta_{u,\omega(t_{N-1})}. \tag{4.30}$$

Given $L \in \mathcal{L}^{(N-1)}[0,j]$, let $L' = L \backslash \{s_{N-1}t_{N-1}\}$. For $t_{N-2} \leq i < j = t_{N-1}$, we then have $\mathcal{C}(L) \supset \mathcal{C}(L') \cup \mathcal{B}[t_{N-2}, i] \cup \mathcal{B}[i,j]$. Using (4.30), we conclude from (4.24) that

$$J_x^{(N)}[0,m] \leq \sum_u \sum_{0 \leq i < j \leq m} J_u^{(N-1)}[0,i]K[i,j]\delta_{\omega(i),x}K[j,m]\delta_{u,\omega(j)}. \tag{4.31}$$

Therefore, recalling (4.17) and using the induction hypothesis,

$$\begin{aligned}\pi_m^{(N)}(x,y) &\leq \sum_u \sum_{0 \leq i < j \leq m} \pi_i^{(N-1)}(u,x)c_{j-i}(u-x)c_{m-j}(y-u) \\ &\leq \sum_{u,v} \sum_{i=0}^{m} p_i^{(N-1)}(u,v)a_{m-i}(u,v,x,y) \\ &= p_m^{(N)}(x,y). \end{aligned} \tag{4.32}$$

This completes the proof. ■

Exercise 4.3. Convince yourself that (4.31) holds.

4.2.3 Proof of (4.9)–(4.10)

We prove two lemmas, and combine them with Proposition 4.2 to obtain (4.9)–(4.10).

We define the operators

$$(\mathcal{M}_z f)(x) = H_z(x)f(x), \tag{4.33}$$
$$(\mathcal{H}_z f)(x) = (H_z * f)(x), \tag{4.34}$$
$$(\mathcal{H}'_z f)(x) = (G_z * f)(x). \tag{4.35}$$

Lemma 4.4. *For $N \geq 2$ and $z \geq 0$,*

$$\sum_x P_z^{(N)}(x, x+y) = \left[(\mathcal{H}'_z\mathcal{M}_z)^{N-1}H_z\right](y). \tag{4.36}$$

Proof. The proof is by induction on N. For $N=2$, we conclude from (4.16) that

$$\begin{aligned}\sum_x P_z^{(2)}(x,x+y) &= \sum_x H_z(x)^2 G_z(x+y) \\ &= \sum_x H_z(x)^2 G_z(y-x) = [(\mathcal{H}'_z\mathcal{M}_z)H_z](y), \end{aligned} \tag{4.37}$$

using $H_z(-x) = H_z(x)$ in the second step.

To advance the induction, we assume that (4.36) holds for $N-1$. By (4.20)–(4.21),

$$\begin{aligned}\sum_x P_z^{(N)}(x,x+y) &= \sum_{x,u,v} P_z^{(N-1)}(u,v)A_z(u,v,x,x+y) \\ &= \sum_{u,v} P_z^{(N-1)}(u,v)H_z(u-v)G_z(v-u+y) \\ &= \sum_w \left(\sum_u P_z^{(N-1)}(u,u+w)\right) H_z(-w)G_z(w+y) \\ &= \sum_w \left(\sum_u P_z^{(N-1)}(u,u+w)\right) H_z(w)G_z(y-w), \end{aligned} \tag{4.38}$$

where in the last step we replaced w by $-w$ and used the fact that the first factor is unchanged by this replacement (see Exercise 4.5 below). Writing $F(w)$ for the first factor, the above is equal to

$$(G_z * H_z F)(y) = (\mathcal{H}'_z \mathcal{M}_z F)(y). \tag{4.39}$$

We then apply the induction hypothesis to complete the proof. ■

Exercise 4.5. Prove the identity $\sum_u P_z^{(N)}(u,u+w) = \sum_u P_z^{(N)}(u,u-w)$ used in (4.38).

The combination of Proposition 4.2 with Lemma 4.4 gives

$$\sum_x \Pi_z^{(N)}(x) \leq \sum_x P_z^{(N)}(x,x) = \left[(\mathcal{H}'_z\mathcal{M}_z)^{N-1}H_z\right](0). \tag{4.40}$$

Note that for $N=2$ the upper bound can be replaced by $[(\mathcal{H}_z\mathcal{M}_z)^{N-1}H_z](0)$, which is equal to the above right hand side in this special case. The right hand side of (4.40) can be estimated using the following lemma.

Lemma 4.6. *Given non-negative even functions $f_0, f_1, \ldots, f_{2M}$ on $\mathbb{Z}^d$, define $\mathcal{H}_j$ and $\mathcal{M}_j$ to be respectively the operations of convolution with f_{2j} and multiplication by f_{2j-1}, for $j = 1, \ldots, M$. Then for any $k \in \{0, \ldots, 2M\}$,*

$$\|\mathcal{H}_M\mathcal{M}_M \cdots \mathcal{H}_1\mathcal{M}_1 f_0\|_\infty \leq \|f_k\|_\infty \prod \|f_j * f_{j'}\|_\infty, \tag{4.41}$$

where the product is over disjoint consecutive pairs j, j' taken from the set $\{0, \ldots, 2M\} \setminus \{k\}$ (e.g., for $k=3$ and $M=3$, the product has factors with j, j' equal to $0,1$; $2,4$; $5,6$).

Proof. The proof is by induction on M. The desired result for $M = 1$ is a consequence of the elementary estimates

$$\sum_y f_2(x-y)f_1(y)f_0(y) \leq \begin{cases} \|f_0\|_\infty \|f_1 * f_2\|_\infty \\ \|f_1\|_\infty \|f_0 * f_2\|_\infty \\ \|f_2\|_\infty \|f_0 * f_1\|_\infty, \end{cases} \tag{4.42}$$

where for the last of these inequalities we used the fact that $\sum_y f_1(y)f_0(y) = (f_0 * f_1)(0)$ for even f_0. To advance the induction, we assume that (4.41) holds for $1, \ldots, M-1$. We write the function inside the norm on the left hand side of (4.41) as $\mathcal{H}_M \mathcal{M}_M F_{M-1}$, with $F_l = \mathcal{H}_l \mathcal{M}_l \cdots \mathcal{H}_1 \mathcal{M}_1 f_0$, and estimate its infinity norm using the result for $M = 1$. If we associate the infinity norm to F_{M-1}, an estimate of the form (4.41) follows from the induction hypothesis, for any $k \leq M-1$. It remains to show that the infinity norm can also be associated to f_{2M} or f_{2M-1}.

We show this for the latter, and the former is similar. Applying the $M = 1$ case to $\mathcal{H}_M \mathcal{M}_M F_{M-1}$ gives an upper bound $\|f_{2M-1}\|_\infty \|f_{2M} * F_{M-1}\|_\infty$. Let $\widetilde{\mathcal{H}}_{M-1}$ denote convolution by $f_{2M} * f_{2M-2}$, so that

$$f_{2M} * F_{M-1} = \widetilde{\mathcal{H}}_{M-1} \mathcal{M}_{M-1} F_{M-2} \tag{4.43}$$

(with $F_0 = f_0$). We apply the induction hypothesis to estimate the infinity norm of the right hand side, associating the infinity norm to $\widetilde{\mathcal{H}}_{M-1}$. This gives the desired estimate. ∎

Exercise 4.7. Give the details omitted at the end of the above proof, for the case in which the infinity norm is associated to f_{2M}.

Proof of (4.9)–(4.10). The bound (4.9) follows from (4.40) and Lemma 4.6.

It remains to prove (4.10). Fix $N \geq 2$. Our goal is to estimate

$$\sum_x [1 - \cos(k \cdot x)] \Pi_z^{(N)}(x) = \sum_{m=2}^{\infty} z^m \sum_x [1 - \cos(k \cdot x)] \pi_m^{(N)}(x). \tag{4.44}$$

To do so, we investigate how the argument leading to (4.40) is modified by the factor $[1 - \cos(k \cdot x)]$.

Because of the factor $\prod_{ij \in L}(-U_{ij})$ occurring in the definition of $\pi_m^{(N)}(x)$ (see (4.3)), a nonzero contribution occurs only for those ω for which $\omega(i) = \omega(j)$ for each edge $ij \in L$. Let I_j denote the j^{th} time interval listed in (3.16) ($j = 1, \ldots, 2N-1$), and let y_j denote the displacement performed on I_j by a walk ω contributing to $\pi_m^{(N)}(x)$. These displacements y_j correspond to the subwalk displacements in Figs. 3.4 and 4.2. The constraints that $\omega(i) = \omega(j)$ for all $ij \in L$, together with the subinterval structure (3.16), impose the constraints

$$y_1 + y_2 = 0, \qquad \sum_{j=2p}^{2p+2} y_j = 0 \quad (p = 1, \ldots, N-2), \qquad y_{2N-2} + y_{2N-1} = 0. \tag{4.45}$$

It can also be seen from this (see also Fig. 3.4) that the total displacement x is given by

$$x = \sum_{i=1}^{\lfloor N/2 \rfloor} y_{4i-1} = \sum_{i=1}^{\lceil N/2 \rceil} y_{4i-3} = -\sum_{i=1}^{N-1} y_{2i} \tag{4.46}$$

(we need only the first equality).

Let $t = \sum_{j=1}^{J} t_j$. Taking the real part of the telescoping sum

$$1 - e^{it} = \sum_{j=1}^{J} [1 - e^{it_j}] e^{i \sum_{m=1}^{j-1} t_m} \tag{4.47}$$

leads to the bound

$$1 - \cos t \leq \sum_{j=1}^{J} [1 - \cos t_j] + \sum_{j=1}^{J} \sin t_j \sin\Big(\sum_{m=1}^{j-1} t_m\Big). \tag{4.48}$$

It is a consequence of the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$ that $|\sin(x+y)| \leq |\sin x| + |\sin y|$. Applying this recursively gives

$$1 - \cos t \leq \sum_{j=1}^{J} [1 - \cos t_j] + \sum_{j=1}^{J} \sum_{m=1}^{j-1} |\sin t_j| |\sin t_m|. \tag{4.49}$$

In the last term we use $|ab| \leq (a^2 + b^2)/2$, and then $1 - \cos^2 a \leq 2[1 - \cos a]$, to obtain

$$\begin{aligned} 1 - \cos t &\leq \sum_{j=1}^{J} [1 - \cos t_j] + \frac{1}{2} \sum_{j=1}^{J} \sum_{m=1}^{j-1} [\sin^2 t_j + \sin^2 t_m] \\ &\leq \sum_{j=1}^{J} [1 - \cos t_j] + J \sum_{j=1}^{J} \sin^2 t_j \\ &= \sum_{j=1}^{J} [1 - \cos t_j] + J \sum_{j=1}^{J} [1 - \cos^2 t_j] \\ &\leq (2J+1) \sum_{j=1}^{J} [1 - \cos t_j]. \end{aligned} \tag{4.50}$$

We use the decomposition of x given by the first equality of (4.46), and apply (4.50) with $t = k \cdot x = \sum_{j=1}^{\lfloor N/2 \rfloor} k \cdot y_{4j-1}$, to obtain

$$1 - \cos(k \cdot x) \leq (N+1) \sum_{j=1}^{\lfloor N/2 \rfloor} [1 - \cos(k \cdot y_{4j-1})]. \qquad (4.51)$$

The modification of the upper bound (4.40) due to the factor $[1 - \cos(k \cdot y_{4j-1})]$ is simply to replace one of the factors H_z or G_z occurring in the right hand side by $[1 - \cos(k \cdot y_{4j-1})]H_z(y_{4j-1})$. Then we apply Lemma 4.6, associating the infinity norm to this particular factor, to obtain the desired estimate (4.10). ■

5

Convergence for the Self-Avoiding Walk

In this chapter, we prove convergence of the lace expansion for the nearest-neighbour model in sufficiently high dimensions, and for sufficiently spread-out models in dimensions $d > 4$. As part of the proof, we will show that the critical bubble diagram $B(z_c)$ is finite in these cases, and hence, by Theorem 2.3, that the critical exponent γ exists and equals 1. This is restated in the following theorem.

Theorem 5.1. *The bubble condition $B(z_c) < \infty$ for the self-avoiding walk holds for the nearest-neighbour model in dimensions $d \geq d_0$, and for the spread-out model with $L \geq L_0(d)$ in dimensions $d > 4$, for some constants d_0 and $L_0(d)$. Thus the critical exponent γ exists and equals 1, in the sense that $\chi(z) \simeq (1 - z/z_c)^{-1}$ as $z \to z_c^-$.*

Remark 5.2. The conclusion of Theorem 5.1 can easily be improved to an asymptotic formula $\chi(z) \sim A(1 - z/z_c)^{-1}$ as $z \to z_c^-$. See Exercise 5.19 below.

Recall that $H_z(x) = G_z(x) - \delta_{0,x} = \sum_{n=1}^{\infty} c_n(x) z^n$, so that $\|H_z\|_2^2 = \|G_z\|_2^2 - 1 = B(z) - 1$. We will prove not just that the critical bubble diagram $B(z_c)$ is finite, but that in fact $\|H_{z_c}\|_2^2 = B(z_c) - 1$ is small, under the hypotheses of Theorem 5.1. As a preliminary, we first analyze some related issues for simple random walk.

5.1 Random-Walk Estimates

By the Parseval relation, $\|H_{z_c}\|_2^2 = \|\hat{H}_{z_c}\|_2^2 = \|\hat{G}_{z_c} - 1\|_2^2$. By (1.18), the random walk analogue of the latter is the integral

$$\int_{[-\pi,\pi]^d} \left| \frac{1}{[1 - \hat{D}(k)]} - 1 \right|^2 \frac{d^d k}{(2\pi)^d} = \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^2} \frac{d^d k}{(2\pi)^d}. \qquad (5.1)$$

The following proposition shows that this integral is small under the hypotheses of Theorem 5.1. (We have already encountered a closely related integral in Exercise 1.7.)

Proposition 5.3. *Let $d > 4$. Then*

$$\int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2} \frac{\mathrm{d}^d k}{(2\pi)^d} \le \beta, \tag{5.2}$$

with $\beta = K(d-4)^{-1}$ (K a universal constant) for the nearest-neighbour model, and with $\beta = KL^{-d}$ (K dependent on d) for the spread-out model.

Proof. This is a calculus problem. For the nearest-neighbour model, a proof can be found in [158, Lemma A.3]. For the spread-out model, there is a proof in [158, Lemma A.5] but with a β which is larger by a factor $(\log L)^{d/2}$. We show here how the improvement can be achieved for the spread-out model.

It is shown in [120] that there are positive constants η, c_1 (independent of L) such that for all $k \in [-\pi,\pi]^d$,

$$1-\hat{D}(k) \ge c_1 L^2 |k|^2 \qquad (\|k\|_\infty \le L^{-1}), \tag{5.3}$$
$$1-\hat{D}(k) > \eta \qquad (\|k\|_\infty \ge L^{-1}). \tag{5.4}$$

The integral $(2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{D}(k)^2 \mathrm{d}^d k$ is equal to $(D*D)(0)$, which is the probability of return to the origin after two steps, namely $|\Omega|^{-1}$. For $j \ge 4$ even, it follows from (5.3)–(5.4) that

$$\begin{aligned}\int_{[-\pi,\pi]^d} \hat{D}(k)^j \frac{\mathrm{d}^d k}{(2\pi)^d} &\le \int_{\mathbb{R}^d} \mathrm{e}^{-c_1 j L^2 |k|^2} \frac{\mathrm{d}^d k}{(2\pi)^d} \\ &\quad + \int_{[-\pi,\pi]^d} \hat{D}(k)^2 (1-\eta)^{j-2} \frac{\mathrm{d}^d k}{(2\pi)^d} \\ &\le \mathrm{const} L^{-d} j^{-d/2} + |\Omega|^{-1}(1-\eta)^{j-2} \\ &\le \mathrm{const} L^{-d} j^{-d/2}.\end{aligned} \tag{5.5}$$

For $j \ge 3$ odd,

$$\begin{aligned}(2\pi)^{-d}\int_{[-\pi,\pi]^d} |\hat{D}(k)|^j \mathrm{d}^d k &\le (2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{D}(k)^{j-1} \mathrm{d}^d k \\ &\le \mathrm{const} L^{-d} j^{-d/2},\end{aligned} \tag{5.6}$$

applying the estimate for j even in the last step. Now we expand $[1-\hat{D}(k)]^{-2}$ in (5.2) as $\sum_{j=1}^\infty j\hat{D}(k)^{j-1}$, and use the above estimates to see that the left hand side of (5.2) is bounded above by a multiple of L^{-d}, assuming $d > 4$. ■

Exercise 5.4. Prove (5.2) for the nearest-neighbour model, with $\beta = K(d-4)^{-1}$.

The following lemma notes some useful implications of (5.2). The left hand sides of (5.8)–(5.9) are respectively the random walk analogues of $\|G_{z_c}\|_2^2$ and of $\|[1-\cos(k\cdot x)]G_{z_c}(x)\|_\infty = \|[1-\cos(k\cdot x)]H_{z_c}(x)\|_\infty$ (cf. (4.10)).

Lemma 5.5. *If (5.2) holds, then for $z \in [0, 1/|\Omega|]$,*

$$\sup_{x\in\mathbb{Z}^d} D(x) \le \beta, \tag{5.7}$$
$$\|C_z\|_2^2 \le 1+3\beta, \tag{5.8}$$
$$\|[1-\cos(k\cdot x)]C_z(x)\|_\infty \le 5(1+3\beta)[1-\hat{D}(k)]. \tag{5.9}$$

We first prove (5.7)–(5.8).

Proof of (5.7)–(5.8). The left hand side of (5.7) is simply $|\Omega|^{-1}$. Since the left hand side of (5.2) is at least $(2\pi)^{-d}\int_{[-\pi,\pi]^d}\hat{D}(k)^2\mathrm{d}^d k = |\Omega|^{-1}$, the bound (5.7) follows.

For (5.8) (and also (5.9)), it suffices to consider $z = 1/|\Omega|$. We use the Parseval relation to rewrite the left hand side as $\|\hat{C}_{1/|\Omega|}\|_2^2$. By (1.18), this equals

$$\begin{aligned}&\int_{[-\pi,\pi]^d} \frac{1}{[1-\hat{D}(k)]^2}\frac{\mathrm{d}^d k}{(2\pi)^d} \\ &= \int_{[-\pi,\pi]^d}\left(1 + 2\frac{\hat{D}(k)}{[1-\hat{D}(k)]} + \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2}\right)\frac{\mathrm{d}^d k}{(2\pi)^d} \\ &\le \int_{[-\pi,\pi]^d}\left(1 + 3\frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^2}\right)\frac{\mathrm{d}^d k}{(2\pi)^d},\end{aligned} \tag{5.10}$$

by Exercise 5.6. The right hand side is at most $1+3\beta$, assuming (5.2). ■

Exercise 5.6. Prove the inequality (5.10) by comparing the integrals of $\hat{D}[1-\hat{D}]^{-1}$ and $\hat{D}^2[1-\hat{D}]^{-2}$.

Before proving (5.9), we develop some useful preliminaries. We first note that

$$\sum_x \cos(k\cdot x)C_z(x)\mathrm{e}^{\mathrm{i}l\cdot x} = \frac{1}{2}[\hat{C}_z(l+k) + \hat{C}_z(l-k)]. \tag{5.11}$$

Therefore, applying the general fact that $\|f\|_\infty \le \|\hat{f}\|_1$, we obtain

$$\|[1-\cos(k\cdot x)]C_z(x)\|_\infty \le \|\hat{C}_z(l) - \frac{1}{2}[\hat{C}_z(l+k) + \hat{C}_z(l-k)]\|_1, \tag{5.12}$$

where the L^1 norm involves integration with respect to l, with k fixed. The expression $\hat{C}_z(l) - \frac{1}{2}(\hat{C}_z(l+k) + \hat{C}_z(l-k))$ is closely related to a sort of second derivative of $\hat{C}_z(l)$, and in general we make the abbreviation

$$-\frac{1}{2}\Delta_k\hat{A}(l) = \hat{A}(l) - \frac{1}{2}(\hat{A}(l+k) + \hat{A}(l-k)). \tag{5.13}$$

In this notation, (5.12) reads

$$\|[1-\cos(k\cdot x)C_z(x)\|_\infty \le \frac{1}{2}\|\Delta_k\hat{C}_z(l)\|_1, \tag{5.14}$$

where the integration on the right hand side is with respect to l.

Lemma 5.7. *Suppose that $a(-x) = a(x)$ for all $x \in \mathbb{Z}^d$, and let*

$$\hat{A}(k) = \frac{1}{1-\hat{a}(k)}. \tag{5.15}$$

Then for all $k, l \in [-\pi,\pi]^d$,

$$\frac{1}{2}|\Delta_k\hat{A}(l)| \le \frac{1}{2}[\hat{A}(l-k)+\hat{A}(l+k)]\hat{A}(l)[\hat{a}^{\mathrm{av}}(0)-\hat{a}^{\mathrm{av}}(k)] \tag{5.16}$$
$$+ 4\hat{A}(l-k)\hat{A}(l)\hat{A}(l+k)[\hat{a}^{\mathrm{av}}(0)-\hat{a}^{\mathrm{av}}(k)][\hat{a}^{\mathrm{av}}(0)-\hat{a}^{\mathrm{av}}(l)],$$

where $a^{\mathrm{av}}(x) = |a(x)|$.

Proof of (5.9). We use Lemma 5.7 to estimate the right hand side of (5.14), with $\hat{a}(k) = \hat{D}(k)$ and $\hat{A}(k) = \hat{C}_{1/|\Omega|}(k)$. Writing the latter simply as $\hat{C}(k)$, this gives

$$\frac{1}{2}|\Delta_k\hat{C}(l)| \le [1-\hat{D}(k)]\left(\frac{1}{2}[\hat{C}(l-k)+\hat{C}(l+k)]\hat{C}(l) + 4\hat{C}(l-k)\hat{C}(l+k)\right). \tag{5.17}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\frac{1}{2}\|\Delta_k\hat{C}\|_1 \le [1-\hat{D}(k)]5\|\hat{C}\|_2^2, \tag{5.18}$$

and (5.9) follows from (5.8). ■

Proof of Lemma 5.7. Since a is even, $\hat{a}(l) = \sum_x a(x)\cos(l\cdot x)$. For such an a, we define

$$\hat{a}^{\cos}(l,k) = \sum_x a(x)\cos(l\cdot x)\cos(k\cdot x) = \frac{1}{2}[\hat{a}(l-k)+\hat{a}(l+k)], \tag{5.19}$$

$$\hat{a}^{\sin}(l,k) = \sum_x a(x)\sin(l\cdot x)\sin(k\cdot x) = \frac{1}{2}[\hat{a}(l-k)-\hat{a}(l+k)]. \tag{5.20}$$

We first show that, for all $k, l \in [-\pi,\pi]^d$,

$$-\frac{1}{2}\Delta_k\hat{A}(l) = \frac{1}{2}[\hat{A}(l-k)+\hat{A}(l+k)]\hat{A}(l)[\hat{a}(l)-\hat{a}^{\cos}(l,k)]$$
$$-\hat{A}(l-k)\hat{A}(l)\hat{A}(l+k)\hat{a}^{\sin}(l,k)^2. \tag{5.21}$$

Let $\hat{a}_\pm = \hat{a}(l\pm k)$ and write $\hat{a} = \hat{a}(l)$. Direct computation using (5.13) gives

$$-\frac{1}{2}\Delta_k\hat{A}(l) \tag{5.22}$$
$$= \frac{1}{2}\hat{A}(l)\hat{A}(l+k)\hat{A}(l-k)\Big[[2\hat{a}-\hat{a}_+-\hat{a}_-] + [2\hat{a}_+\hat{a}_- - \hat{a}\hat{a}_- - \hat{a}\hat{a}_+]\Big]$$
$$= \hat{A}(l)\hat{A}(l+k)\hat{A}(l-k)\Big[[\hat{a}(l)-\hat{a}^{\cos}(l,k)] + [\hat{a}_+\hat{a}_- - \hat{a}(l)\hat{a}^{\cos}(l,k)]\Big],$$

using (5.19) in the last step. By definition, and using the identity $\cos(u+v) = \cos u\cos v - \sin u \sin v$,

$$\hat{a}_+\hat{a}_- = \sum_{x,y} a(x)a(y)\cos((l+k)\cdot x)\cos((l-k)\cdot y)$$
$$= \hat{a}^{\cos}(l,k)^2 - \hat{a}^{\sin}(l,k)^2. \tag{5.23}$$

Substitution of (5.23) in (5.22) gives

$$-\frac{1}{2}\Delta_k\hat{A}(l) = \hat{A}(l-k)\hat{A}(l)\hat{A}(l+k)\left[[\hat{a}(l)-\hat{a}^{\cos}(l,k)][1-\hat{a}^{\cos}(l,k)] - \hat{a}^{\sin}(l,k)^2\right]. \tag{5.24}$$

Finally, we use (5.19) to rewrite $1-\hat{a}^{\cos}(l,k)$ and obtain (5.21).

Now we use (5.21) to prove (5.16). First, we note that

$$|\hat{a}(l)-\hat{a}^{\cos}(l,k)| \le \sum_x [1-\cos(k\cdot x)]\,|\cos(l\cdot x)|\,|a(x)|$$
$$\le \hat{a}^{\mathrm{av}}(0)-\hat{a}^{\mathrm{av}}(k). \tag{5.25}$$

Also, by (5.20) and the Cauchy-Schwarz inequality,

$$\hat{a}^{\sin}(k,l)^2 \le \left(\sum_x |a(x)|\sin^2(k\cdot x)\right)\left(\sum_x |a(x)|\sin^2(l\cdot x)\right). \tag{5.26}$$

With the elementary estimate $\sin^2 t = 1-\cos^2 t \le 2[1-\cos t]$, this gives

$$\hat{a}^{\sin}(k,l)^2 \le \sum_x |a(x)|[1-\cos^2(k\cdot x)]\sum_y |a(y)|[1-\cos^2(l\cdot y)]$$
$$\le 4[\hat{a}^{\mathrm{av}}(0)-\hat{a}^{\mathrm{av}}(k)][\hat{a}^{\mathrm{av}}(0)-\hat{a}^{\mathrm{av}}(l)]. \tag{5.27}$$

The desired estimate (5.16) then follows from (5.21), (5.25) and (5.27). ■

5.2 Convergence of the Expansion

In this section we prove convergence of the lace expansion, assuming (5.2), and also prove Theorem 5.1. Convergence will be proved in the process of proving the following theorem, which shows that if the critical simple random walk bubble diagram is sufficiently small, then the critical self-avoiding walk bubble diagram is also small. (In both diagrams, the trivial term 1 is omitted to obtain a small quantity.)

Theorem 5.8. *There is a $\beta_0 > 0$ and a constant c such that if (5.2) holds with $\beta \le \beta_0$, then $B(z_c) - 1$ is less than $c\beta$.*

Proof of Theorem 5.1. This is an immediate consequence of Proposition 5.3, Theorem 5.8, and Theorem 2.3. ∎

We will prove Theorem 5.8 in the remainder of Chap. 5. The proof is inspired by the method of [32]. It is possible to go beyond Theorem 5.8 in several respects, and this will be discussed in Chap. 6. In particular, critical exponents of the nearest-neighbour strictly self-avoiding walk in dimensions $d \ge 5$ are computed in [97,98].

It is not obvious, at first, how to approach the issue of convergence of the lace expansion. The conclusion of Theorem 5.8 is that $\|H_{z_c}\|_2$ is small. On the other hand, recall from (4.11) that

$$\|H_z * G_z\|_\infty \le \|H_z\|_\infty + \|H_z\|_2^2. \tag{5.28}$$

To use this to perform the sum over N in Theorem 4.1 to estimate Π_z, we already need to know that $\|H_z\|_2^2$ is small uniformly in $z < z_c$. The following elementary lemma will be used to allow us to pick ourselves up by our bootstraps.

Lemma 5.9. *Let $a < b$, let f be a continuous function on the interval $[z_1, z_2)$, and assume that $f(z_1) \le a$. Suppose for each $z \in (z_1, z_2)$ that if $f(z) \le b$ then in fact $f(z) \le a$. Then $f(z) \le a$ for all $z \in [z_1, z_2)$.*

Proof. By hypothesis, $f(z)$ cannot lie strictly between a and b for any $z \in (z_1, z_2)$. Since $f(z_1) \le a$, it follows by continuity that $f(z) \le a$ for all $z \in [z_1, z_2)$. ∎

For $z \in [0, z_c)$, we define $p(z) \in [0, 1/|\Omega|)$ by

$$\hat{G}_z(0) = \chi(z) = \frac{1}{1 - p(z)|\Omega|} = \hat{C}_{p(z)}(0), \tag{5.29}$$

which is equivalent to

$$p(z)|\Omega| = 1 - \frac{1}{\chi(z)}. \tag{5.30}$$

Our choice of f is motivated, in part, by the intuition that $\hat{G}_z(k)$ and $\hat{C}_{p(z)}(k)$ are comparable in size. We also expect $\frac{1}{2}\Delta_k \hat{G}_z(l)$ and $\frac{1}{2}\Delta_k \hat{C}_{p(z)}(l)$ to be comparable. However, rather than comparing the latter directly, we will compare $\frac{1}{2}\Delta_k \hat{G}_z(l)$ with

$$U_{p(z)}(k,l) = 16\hat{C}_{p(z)}(k)^{-1}\Big(\hat{C}_{p(z)}(l-k)\hat{C}_{p(z)}(l) + \hat{C}_{p(z)}(l+k)\hat{C}_{p(z)}(l) + \hat{C}_{p(z)}(l-k)\hat{C}_{p(z)}(l+k)\Big), \tag{5.31}$$

which can be seen using (5.16) to be an upper bound for $\frac{1}{2}|\Delta_k \hat{C}_{p(z)}(l)|$.

We will apply Lemma 5.9 with $z_1 = 0$, $z_2 = z_c$, $b = 4$, $a = 1 + \text{const}\beta$ (the constant being determined in the course of the proof), and

$$f(z) = \max\{f_1(z), f_2(z), f_3(z)\}, \tag{5.32}$$

where

$$f_1(z) = z|\Omega|, \quad f_2(z) = \sup_{k \in [-\pi,\pi]^d} \frac{|\hat{G}_z(k)|}{\hat{C}_{p(z)}(k)}, \tag{5.33}$$

$$f_3(z) = \sup_{k,l \in [-\pi,\pi]^d} \frac{\frac{1}{2}|\Delta_k \hat{G}_z(l)|}{U_{p(z)}(k,l)}. \tag{5.34}$$

Note that the factor $\hat{C}_{p(z)}(k)^{-1}$ in the denominator of f_3 becomes arbitrarily small when $k = 0$ and $z \to z_c^-$. We will verify in Lemmas 5.12, 5.14 and 5.16 that the hypotheses of Lemma 5.9 hold when β is sufficiently small. From this, we can conclude that $f(z) \le a = 1 + \text{const}\beta$ uniformly in $z \in [0, z_c)$.

Proof of Theorem 5.8. We will show below in Lemma 5.10 that it follows from $f(z) \le a$ (which we will conclude as noted above) that $\|H_z\|_2^2 \le c_a\beta$, where c_a is the constant of Lemma 5.10 when $f(z) \le K = a$. This proves $\|H_z\|_2^2 \le c_a\beta$ uniformly in $z < z_c$. By the monotone convergence theorem, this implies that

$$\|H_{z_c}\|_2^2 = \lim_{z \to z_c^-} \|H_z\|_2^2 \le c_a\beta, \tag{5.35}$$

which proves Theorem 5.8 since $B(z_c) - 1 = \|H_{z_c}\|_2^2$ by (4.12). ∎

Note that the inequality $f_2(z) \le a$ implies the *infrared bound*

$$\hat{G}_z(k) \le a\hat{C}_{p(z)}(k) \tag{5.36}$$

(we will actually prove in (5.53)–(5.60) that $\hat{G}_z(k)/\hat{C}_{p(z)}(k) = 1 + O(\beta)$, which implies, in particular, that $\hat{G}_z(k) \ge 0$, permitting removal of the absolute value on the left hand side of (5.36)).

Before going into the details, the basic strategy is as follows. First, it is straightforward to verify the two hypotheses on f in Lemma 5.9 that f is continuous and that $f(0) \le a$, and the main work goes into verifying that $f(z) \le b$ implies that $f(z) \le a$. For this, we use the assumption $f(z) \le b$ to compare norms of H_z with norms of $C_{p(z)}$, and use (5.2) and Lemma 5.5 to see that the latter are small. We then apply Theorem 4.1 to conclude that $\hat{\Pi}_z(k)$ is as small as we like, assuming that β is sufficiently small. Importantly, this can be done even for a poor (large) value of b, because the effect of taking β small compensates for the lack of sharpness in the bound $f(z) \le b$. This implies that $G_z(k)$ is close to a simple random walk quantity, and from this we will be able to conclude the sharper bound $f(z) \le a$. The details are carried out below.

Lemma 5.10. *Fix $z \in (0, z_c)$, assume that f of (5.32) obeys $f(z) \le K$, and assume (5.2). Then there is a constant c_K, independent of z, such that*

$$\|[1 - \cos(k \cdot x)]H_z\|_\infty \le c_K(1+\beta)\hat{C}_{p(z)}(k)^{-1}, \tag{5.37}$$

$$\|H_z\|_2^2 \le c_K\beta, \qquad \|H_z\|_\infty \le c_K\beta. \tag{5.38}$$

Proof. As in (5.14),

$$\begin{aligned}\|[1 - \cos(k \cdot x)]H_z\|_\infty &= \|[1 - \cos(k \cdot x)]G_z\|_\infty \\ &\le \frac{1}{2}\|\Delta_k \hat{G}_z\|_1.\end{aligned} \tag{5.39}$$

It then follows from $f_3(z) \le K$, the Cauchy–Schwarz inequality, and (5.8) that

$$\begin{aligned}\|[1 - \cos(k \cdot x)]H_z\|_\infty &\le 16K\hat{C}_{p(z)}(k)^{-1}3\|\hat{C}_{p(z)}\|_2^2 \\ &\le 48(1+3\beta)K\hat{C}_{p(z)}(k)^{-1},\end{aligned} \tag{5.40}$$

which gives (5.37).

Next, we estimate $\|H_z\|_2^2$. We first use subadditivity and $f_1(z) \le K$ to obtain

$$H_z(x) \le z|\Omega|(D * G_z)(x) \le K(D * G_z)(x). \tag{5.41}$$

Using $f_2(z) \le K$, the Parseval relation, and (5.2), this implies that

$$\begin{aligned}\|H_z\|_2^2 &\le K^2\|D * G_z\|_2^2 = K^2\|\hat{D}\hat{G}_z\|_2^2 \\ &\le K^4\|\hat{D}\hat{C}_{p(z)}\|_2^2 = K^4\|D * C_{p(z)}\|_2^2 \\ &\le K^4\|D * C_{1/|\Omega|}\|_2^2 = K^4\|\hat{D}[1-\hat{D}]^{-1}\|_2^2 \le K^4\beta.\end{aligned} \tag{5.42}$$

This proves the first bound of (5.38).

Iteration of (5.41) gives $H_z(x) \le KD(x) + K^2(D * D * G_z)(x)$. Therefore,

$$\begin{aligned}\|H_z\|_\infty &\le K\|D\|_\infty + K^2\|\hat{D}^2\hat{G}_z\|_1 \\ &\le K\beta + K^3\|\hat{D}^2\hat{C}_{p(z)}\|_1 \\ &= K\beta + K^3(D * D * C_{p(z)})(0) \\ &\le K\beta + K^3(D * D * C_{p(z)} * C_{p(z)})(0) \\ &\le K\beta + K^3(D * D * C_{1/|\Omega|} * C_{1/|\Omega|})(0) \\ &\le K\beta + K^3\beta,\end{aligned} \tag{5.43}$$

using (5.7) in the second inequality, and the inverse Fourier transform and (5.2) in the last. ■

Remark. The bounds of Lemma 5.10 can be combined with Theorem 4.1 to give bounds on $\Pi^{(N)}$, and hence on Π. This is the content of the following lemma. Note that once we have verified that the hypotheses of Lemma 5.9 all hold, we can conclude that $f(z) \le a = 1 + \text{const}\beta$, thereby verifying the hypothesis $f(z) \le K$ of Lemma 5.11 with $K = a$. After the fact, this then gives unconditional bounds on Π_z (of course assuming (5.2)). These bounds are then of lasting importance (see, e.g., Exercises 5.17–5.18).

Lemma 5.11. *Fix $z \in (0, z_c)$, assume that f of (5.32) obeys $f(z) \le K$, and assume that (5.2) holds. There is a constant $\bar{c}_K$, independent of z, such that if β is sufficiently small (independent of z), then*

$$\sum_{x \in \mathbb{Z}^d} |\Pi_z(x)| \le \bar{c}_K\beta, \tag{5.44}$$

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)]|\Pi_z(x)| \le \bar{c}_K\beta\hat{C}_{p(z)}(k)^{-1}. \tag{5.45}$$

Proof. It follows from Theorem 4.1, Lemma 5.10, and the estimate (5.28) that there is a constant c'_K such that

$$\sum_{x \in \mathbb{Z}^d} \Pi_z^{(N)}(x) \le (c'_K\beta)^N \tag{5.46}$$

for all $N \ge 1$, and

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)]\Pi_z^{(N)}(x) \begin{cases} = 0 & \text{if } N = 1 \\ \le \hat{C}_{p(z)}(k)^{-1}N^2(c'_K\beta)^{N-1} & \text{if } N \ge 2. \end{cases} \tag{5.47}$$

The bounds (5.44)–(5.45) then follow immediately. ■

We now confirm that f of (5.32) obeys the hypotheses of Lemma 5.9, with $z_1 = 0$, $z_2 = z_c$, $b = 4$ and $a = 1 + \text{const}\beta$ (the particular value 4 for b is not an essential choice). We first verify that $f(0) = 1$, which is of course less than a.

Lemma 5.12. *The function f of (5.32) obeys $f(0) = 1$.*

Proof. By definition, $f_1(0) = 0$. Also, $p(0) = 0$ by (5.30) and hence $f_2(0) = 1$. Finally, $f_3(0) = 0$. ■

To prove the continuity of f, we will use the following elementary lemma.

Lemma 5.13. *Let $(f_\alpha)_{\alpha \in A}$ be an equicontinuous family of functions on an interval $[t_1, t_2]$, and suppose that $\sup_{\alpha \in A} f_\alpha(t) < \infty$ for each $t \in [t_1, t_2]$. Then $\sup_{\alpha \in A} f_\alpha$ is continuous on $[t_1, t_2]$.*

Proof. Let $\bar{f} = \sup_{\alpha \in A} f_\alpha$, and let $\epsilon > 0$ be given. The statement that $(f_\alpha)_{\alpha \in A}$ is equicontinuous means that there is a $\delta > 0$ such that $|f_\alpha(s) - f_\alpha(t)| < \epsilon/2$ whenever $|s - t| < \delta$, uniformly in $\alpha \in A$. Fix s, t with $|s - t| < \delta$, and assume without loss of generality that $\bar{f}(s) \geq \bar{f}(t)$. Choose α' such that $0 \leq \bar{f}(s) - f_{\alpha'}(s) < \epsilon/2$. Then

$$\begin{aligned} 0 &\leq \bar{f}(s) - \bar{f}(t) \leq \bar{f}(s) - f_{\alpha'}(t) \\ &\leq [\bar{f}(s) - f_{\alpha'}(s)] + |f_{\alpha'}(s) - f_{\alpha'}(t)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{5.48}$$

Therefore, $\bar{f}$ is continuous. ■

Lemma 5.14. *The function f of (5.32) is continuous on the interval $[0, z_c)$.*

Proof. It suffices to show that each of f_1, f_2, f_3 is continuous on $[0, z_c)$. The function f_1 is linear, so it is certainly continuous.

For f_2, it suffices to show that f_2 is continuous in $[0, r]$ for every $r < z_c$. By Lemma 5.13, it suffices to show that $|\hat{G}_z(k)|/\hat{C}_{p(z)}(k)$ is equicontinuous in $z \in [0, r]$. Here α is k. Since $(|f_\alpha|)_{\alpha \in A}$ is an equicontinuous family whenever $(f_\alpha)_{\alpha \in A}$ is, it suffices to obtain a bound on the derivative

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{\hat{G}_z(k)}{\hat{C}_{p(z)}(k)} = \frac{1}{\hat{C}_{p(z)}(k)^2} \left[\hat{C}_{p(z)}(k) \frac{\mathrm{d}\hat{G}_z(k)}{\mathrm{d}z} - \hat{G}_z(k) \left. \frac{\mathrm{d}\hat{C}_p(k)}{\mathrm{d}p} \right|_{p=p(z)} \frac{\mathrm{d}p(z)}{\mathrm{d}z} \right], \tag{5.49}$$

uniformly in k and in $z \in [0, r]$. This follows from the bounds

$$\frac{1}{2} \leq \frac{1}{1 - p(z)|\Omega|\hat{D}(k)} = \hat{C}_{p(z)}(k) \leq \hat{C}_{p(z)}(0) = \chi(z) \leq \chi(r), \tag{5.50}$$

$|\hat{G}_z(k)| \leq \chi(r)$, $|\frac{\mathrm{d}\hat{G}_z(k)}{\mathrm{d}z}| \leq \chi'(r)$, $|\frac{\mathrm{d}\hat{C}_p(k)}{\mathrm{d}p}| \leq |\Omega|\chi(r)^2$, and $\frac{\mathrm{d}p(z)}{\mathrm{d}z} \leq |\Omega|^{-1}\chi'(r)$.

For f_3, it again suffices to show continuity in $[0, r]$ for every $r < z_c$. Again we show equicontinuity on $[0, r]$ for every $r < z_c$, by obtaining a uniform bound on the derivative with respect to z, and this follows as before. ■

Exercise 5.15. Fill in the missing details in the continuity proof of f_3.

Finally, we verify that f obeys the substantial hypothesis of Lemma 5.9.

Lemma 5.16. *Fix $z \in (0, z_c)$ and suppose that $f(z) \leq 4$. If (5.2) holds with β sufficiently small (independent of z), then it is in fact the case that $f(z) \leq 1 + c\beta$ for some $c > 0$ independent of z.*

Proof. For $f_1(z)$, we simply note that $\chi(z) > 0$ and hence, by (3.30),

$$\chi(z)^{-1} = 1 - z|\Omega| - \hat{\Pi}_z(0) > 0. \tag{5.51}$$

Therefore, by Lemma 5.11,

$$f_1(z) = z|\Omega| < 1 - \hat{\Pi}_z(0) \leq 1 + \bar{c}_4\beta \tag{5.52}$$

if β is sufficiently small.

For f_2, we first write $\hat{F}_z(k) = 1/\hat{G}_z(k)$, so that

$$\frac{\hat{G}_z(k)}{\hat{C}_{p(z)}(k)} = \frac{1 - p(z)|\Omega|\hat{D}(k)}{\hat{F}_z(k)} = 1 + \frac{1 - p(z)|\Omega|\hat{D}(k) - \hat{F}_z(k)}{\hat{F}_z(k)}. \tag{5.53}$$

We will show that the last term on the right hand side is $O(\beta)$, which implies that $f_2(z) = 1 + O(\beta)$.

We first obtain bounds on the numerator of the last term in (5.53), and afterwards consider the denominator. By (5.30) and (3.30), $p(z)|\Omega| = 1 - \hat{F}_z(0) = z|\Omega| + \hat{\Pi}_z(0)$, and thus the numerator of the last term in (5.53) is

$$1 - p(z)|\Omega|\hat{D}(k) - \hat{F}_z(k) = \hat{\Pi}_z(0)[1 - \hat{D}(k)] - [\hat{\Pi}_z(0) - \hat{\Pi}_z(k)]. \tag{5.54}$$

This is bounded above by $4\bar{c}_4\beta$, by (5.44). Additionally, by (5.44)–(5.45), it is also bounded above by

$$\bar{c}_4\beta[1 - \hat{D}(k)] + \bar{c}_4\beta[1 - p(z)|\Omega|\hat{D}(k)]. \tag{5.55}$$

Since

$$[1 - \hat{D}(k)]\hat{C}_{p(z)}(k) = 1 + \hat{D}(k)\frac{p(z)|\Omega| - 1}{1 - p(z)|\Omega|\hat{D}(k)} \leq 2, \tag{5.56}$$

the numerator of (5.53) is bounded by

$$3\bar{c}_4\beta[1 - p(z)|\Omega|\hat{D}(k)] \leq 3\bar{c}_4\beta\left[\hat{F}_z(0) + [1 - \hat{D}(k)]\right]. \tag{5.57}$$

The denominator of (5.53) is

$$\begin{aligned} \hat{F}_z(k) &= \hat{F}_z(0) + [\hat{F}_z(k) - \hat{F}_z(0)] \\ &= \hat{F}_z(0) + z|\Omega|[1 - \hat{D}(k)] + [\hat{\Pi}_z(0) - \hat{\Pi}_z(k)]. \end{aligned} \tag{5.58}$$

For $z \le 1/2|\Omega|$, we use $\hat{F}_z(0) \ge \hat{C}_z(0)^{-1} \ge \frac{1}{2}$, $1 - \hat{D}(k) \ge 0$, and (5.44) to see that

$$\hat{F}_z(k) \ge \hat{F}_z(0) - 2\bar{c}_4\beta \ge \frac{1}{2} - 2\bar{c}_4\beta. \tag{5.59}$$

For $1/2|\Omega| \le z < z_c$, we use $\hat{F}_z(0) > 0$, (5.45) and $1 - p(z)|\Omega|\hat{D}(k) = 1 - (1 - \hat{F}_z(0))\hat{D}(k) \le 1 - \hat{D}(k) + \hat{F}_z(0)$ to obtain

$$\begin{aligned}\hat{F}_z(k) &\ge \hat{F}_z(0) + \frac{1}{2}[1 - \hat{D}(k)] - \bar{c}_4\beta[1 - p(z)|\Omega|\hat{D}(k)] \\ &\ge \left[\frac{1}{2} - \bar{c}_4\beta\right]\left[\hat{F}_z(0) + [1 - \hat{D}(k)]\right]. \end{aligned} \tag{5.60}$$

In either case, combining these inequalities with the bounds obtained above for the numerator of (5.53) gives $f_2(z) = 1 + O(\beta)$.

Finally, we consider f_3. We write

$$\hat{g}_z(k) = z|\Omega|\hat{D}(k) + \hat{\Pi}_z(k), \tag{5.61}$$

so that

$$\hat{G}_z(k) = \frac{1}{1 - \hat{g}_z(k)}. \tag{5.62}$$

Note that $g_z(x) = g_z(-x)$, so we can apply Lemma 5.7 to obtain

$$\begin{aligned}\frac{1}{2}|\Delta_k\hat{G}_z(l)| \le \frac{1}{2}&[\hat{G}_z(l-k) + \hat{G}_z(l+k)]\hat{G}_z(l)[\hat{g}_z^{\mathrm{av}}(0) - \hat{g}_z^{\mathrm{av}}(k)] \\ &+4\hat{G}_z(l-k)\hat{G}_z(l)\hat{G}_z(l+k)[\hat{g}_z^{\mathrm{av}}(0) - \hat{g}_z^{\mathrm{av}}(k)][\hat{g}_z^{\mathrm{av}}(0) - \hat{g}_z^{\mathrm{av}}(l)]. \end{aligned} \tag{5.63}$$

Using $f_2(z) \le 1 + O(\beta)$, we can bound each factor of $\hat{G}_z$ above by $[1 + O(\beta)]\hat{C}_{p(z)}$. Also,

$$\begin{aligned}\hat{g}_z^{\mathrm{av}}(0) - \hat{g}_z^{\mathrm{av}}(k) &\le \sum_x [1 - \cos(k\cdot x)][z|\Omega|D(x) + |\Pi_z(x)|] \\ &\le z|\Omega|[1 - \hat{D}(k)] + \bar{c}_4\beta\hat{C}_{p(z)}(k)^{-1} \\ &\le [2 + O(\beta)]\hat{C}_{p(z)}(k)^{-1}, \end{aligned} \tag{5.64}$$

using (5.45) for the second inequality, and $f_1(z) \le 1 + O(\beta)$ and (5.56) for the third. Combining these bounds gives $f_3(z) \le 1 + O(\beta)$.

This completes the proof that $f(z) \le 1 + O(\beta)$. ■

This completes the proof that f obeys the hypotheses of Lemma 5.9, and also completes the proof of Theorem 5.8.

Exercise 5.17. (a) Give a monotonicity argument to conclude that the factor $\hat{C}_{p(z)}(k)^{-1}$ in (5.45) can be replaced by $1 - \hat{D}(k)$.
(b) Use the result of (a) to prove that $\sum_x |x|^2|\Pi_z(x)|$ is bounded above by $O(\beta\sigma^2)$ uniformly in $z < z_c$, where $\sigma^2 = \sum_x |x|^2 D(x)$.
(c) The *correlation length of order 2*, $\xi_2(z)$, is defined by

$$\xi_2(z)^2 = \frac{1}{\chi(z)}\sum_x |x|^2 G_z(x). \tag{5.65}$$

Prove that $\xi_2(z) \simeq (1 - z/z_c)^{-1/2}$.

As a final observation, we note that by (5.46) and dominated convergence, $\hat{\Pi}_{z_c}(k)$ is finite and is equal to the limit of $\hat{\Pi}_z(k)$ as z approaches z_c from the left. Since $\chi(z)$ diverges to infinity in this limit, it follows from (3.30) that

$$1 - z_c|\Omega| - \hat{\Pi}_{z_c}(0) = 0, \tag{5.66}$$

and hence

$$z_c = \frac{1}{|\Omega|}\left(1 - \hat{\Pi}_{z_c}(0)\right) = \frac{1}{|\Omega|} + O\left(\frac{1}{|\Omega|^2}\right), \tag{5.67}$$

where we have used $|\hat{\Pi}_{z_c}(0)| \le O(\beta) = O(|\Omega|^{-1})$ (and we assume $d > 4$ for the spread-out model). For the spread-out model in dimensions $d > 4$, an extension of (5.67) can be found in [118].[1] See [176] for related results for the spread-out model in dimensions $d \le 4$. For the nearest-neighbour model, (5.67) is the first step in the proof of the asymptotic formula (2.8) for $\mu = 1/z_c$. The following exercise pushes (5.67) a bit further.

Exercise 5.18. Consider the nearest-neighbour model.
(a) Fix an integer $m \ge 1$. Show that $\|[1 - \hat{D}]^{-m}\|_1$ is nonincreasing in $d > 2m$. Hint: $A^{-m} = \Gamma(m)^{-1}\int_0^\infty u^{m-1}e^{-uA}du$.
(b) Let $H_z^{(j)}(x) = \sum_{m=j}^\infty c_m(x)z^m$. Show that $\|H_{z_c}^{(j)}\|_\infty \le O(d^{-j/2})$, where the constant may depend on j. To do so, it is helpful first to show that $\|\hat{D}^{2j}\|_1 \le c_j(2d)^{-j}$ for some constant c_j depending on $j = 1, 2, \ldots$.
(c) Prove that

$$\hat{\Pi}_{z_c}^{(1)}(0) = \frac{1}{2d} + \frac{3}{(2d)^2} + O\left(\frac{1}{(2d)^3}\right). \tag{5.68}$$

(d) Prove that

$$\hat{\Pi}_{z_c}^{(2)}(0) = \frac{1}{(2d)^2} + O\left(\frac{1}{(2d)^3}\right). \tag{5.69}$$

(e) Conclude that the connective constant $\mu = z_c^{-1}$ obeys

$$\mu = 2d - 1 - \frac{1}{2d} + O\left(\frac{1}{(2d)^2}\right). \tag{5.70}$$

The strategy in this exercise is based on that used in [53, 122, 123], and is simpler than that used in [101]. Equation (5.70) was first proved in [140], using completely different methods.

[1] The results of [118] are expressed in terms of p_c defined by $p_c = z_c|\Omega|$.

5.3 Finite Bubble vs Small Bubble

According to Theorem 2.3, the susceptibility obeys the mean-field behaviour $\chi(z) \simeq (1 - z/z_c)^{-1}$ if the critical bubble diagram $B(z_c)$ is finite. On the other hand, convergence of the lace expansion has been proved only when $B(z_c) - 1$ is *small.* This leads to the restrictions that the dimension be large for the nearest-neighbour model, or that L be large for the spread-out model in dimensions $d > 4$, in our use of Proposition 5.3 to drive the convergence proof.

For the nearest-neighbour model in dimension $d = 5$, it was shown in [97, 98] that $B(z_c) - 1 \leq 0.493$. This is small, although not very small. With considerable effort, and with a computer-assisted proof, convergence of the lace expansion was proved in [97, 98] for the nearest-neighbour model in dimensions $d \geq 5$.

It would be of great interest to find a proof of the bubble condition that would be applicable in situations where the bubble diagram could be large, rather than relying on it being small.

5.4 Differential Equality and the Bubble Condition

It is instructive now to revisit Theorem 2.3, which used inclusion-exclusion to give upper and lower bounds on the derivative of $z\chi(z)$. As in the proof of Lemma 5.16, we write

$$\hat{F}_z(0) = \frac{1}{\chi(z)} = 1 - z|\Omega| - \hat{\Pi}_z(0). \tag{5.71}$$

Then direct calculation gives

$$\frac{\mathrm{d}[z\chi(z)]}{\mathrm{d}z} = \left(\hat{F}_z(0) - z\frac{\mathrm{d}\hat{F}_z(0)}{\mathrm{d}z}\right)\frac{1}{\hat{F}_z(0)^2} = V(z)\chi(z)^2, \tag{5.72}$$

with

$$V(z) = 1 + z\frac{\mathrm{d}\hat{\Pi}_z(0)}{\mathrm{d}z} - \hat{\Pi}_z(0). \tag{5.73}$$

The identity (5.72) gives an identity in place of the inequalities of (2.35), and corresponds to inclusion-exclusion carried out to all orders.

It is significant that $V(z_c)$ is finite, under the basic assumption of Chap. 5 that (5.2) is sufficiently small. We have already seen in Sect. 5.2 that $\hat{\Pi}_{z_c}(0)$ is finite. To see that $V(z_c)$ is finite, we must verify that the derivative $\mathrm{d}\hat{\Pi}_{z_c}(0)/\mathrm{d}z$ is also finite. Here is a sketch of a proof of this last fact.

It suffices to obtain a bound on

$$\sum_{m=1}^{\infty} m\hat{\pi}_m^{(N)}(0)z_c^{m-1} \tag{5.74}$$

which is summable in N. As in the proof of (4.10), we associate to $\hat{\pi}_m^{(N)}(0)$ a diagram consisting of $2N - 1$ subwalks, whose lengths $m_1, \ldots, m_{2N-1}$ sum to m. We decompose the factor m in (5.74) as $m = \sum_{j=1}^{2N-1} m_j$ and obtain a sum of $2N - 1$ terms. In the j^{th} term, there is a factor m_j associated to the j^{th} line. We apply Lemma 4.6 to estimate the j^{th} term, associating the infinity norm to the special line. Then we use the bound

$$\begin{aligned} \|\sum_{m=1}^{\infty} mc_m(x)z_c^{m-1}\|_\infty &= \|\mathrm{d}H_{z_c}(x)/\mathrm{d}z\|_\infty \\ &\leq \|H_{z_c} * G_{z_c}\|_\infty \leq \|H_{z_c}\|_\infty + \|H_{z_c}\|_2^2, \end{aligned} \tag{5.75}$$

and draw the desired conclusion. The first inequality of (5.75) follows as in the upper bound of (2.35), and the second inequality is (4.11).

This shows that, as $z \to z_c^-$,

$$\frac{\mathrm{d}[z\chi(z)]}{\mathrm{d}z} \sim V(z_c)\chi(z)^2. \tag{5.76}$$

The left hand side is equal to the generating function of two mutually-avoiding self-avoiding walks starting from the origin. The asymptotic formula (5.76) shows that this generating function behaves in the same way as the generating function for two *independent* self-avoiding walks, up to a vertex factor $V(z_c)$ which takes into account the local effect of the mutual avoidance.

Exercise 5.19. Prove that when β of (5.2) is sufficiently small, the susceptibility obeys the asymptotic formula

$$\chi(z) \sim A(1 - z/z_c)^{-1} \text{ as } z \to z_c^-, \tag{5.77}$$

with $A = z_c^{-1}[|\Omega| + \frac{\mathrm{d}}{\mathrm{d}z}\hat{\Pi}_{z_c}(0)]^{-1}$. This improves the conclusion of Theorem 5.1 to an asymptotic formula, and also avoids any appeal to Theorem 2.3.