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Convergence of the Sum-Product Algorithm

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Abstract —

We address the question of convergence in the sum-product algorithm. Specifically, we relate convergence of the sum-product algorithm to the existence of a weak limit for a sequence of Gibbs measures defined on the associated computation tree. Using tools from the theory of Gibbs measures we develop easily testable sufficient conditions for convergence. The failure of convergence of the sum-product algorithm implies the existence of multiple phases for the associated Gibbs specification. These results give new insight into the mechanics of the algorithm.

I. INTRODUCTION

The sum-product (also called loopy belief propagation) algorithm is an algorithm developed for computing approximate marginal statistics over graphs with cycles. This algorithm has had notable success, especially for iterative channel decoding of turbo codes and low density parity check codes. However, the behavior of the sum-product algorithm is poorly understood. In particular, it is not always known if this algorithm will converge.

Our analysis relies on the computation tree. This tree represents an unwrapping of the original graph with respect to the sum-product algorithm [4]. The initializing messages can be represented by potentials placed at the leaves of the computation tree. We can then construct a sequence of Gibbs measures defined on the infinite computation tree. If this sequence converges then the sum-product algorithm converges. Our contributions are as follows: First, we relate convergence of the sum-product algorithm to the existence of a weak limit for the sequence of measures defined on the corresponding computation tree. Second, we show that the sum-product algorithm always converges in the case when there is a unique Gibbs measure defined on the computation tree. (Conversely if the sum-product algorithm fails to converge then there exist multiple phases on the computation tree.) Third, we provide an easily testable sufficient condition to insure convergence of sum-product. Finally, we provide rates of convergence and error bounds for some special cases. Complete proofs can be found in [3].

II. BACKGROUND AND THE SUM-PRODUCT ALGORITHM

In this section we review finite Gibbs measures, the sum-product algorithm, and the associated computation tree.

Finite Gibbs Measures

Let S be a finite set of nodes. Associated with each node $i \in S$ there is a measure space $(\mathcal{X}_i, \mathcal{F}_i)$. We assume that all the $\mathcal{X}_i$ are finite. Let $(\Omega, \mathcal{F}) \triangleq (\prod_{i\in S} \mathcal{X}_i, \prod_{i\in S} \mathcal{F}_i)$ equal the product measure space. On the measure space $(\Omega, \mathcal{F})$ define an independent reference measure $\lambda = \prod_{i\in S} \lambda_i$ where each λ_i is the uniform measure on $(\mathcal{X}_i, \mathcal{F}_i)$. Let $\mathbb{S} \triangleq \{\Lambda \subset S, \Lambda \neq \emptyset\}$ be the set of all nonempty subsets of S. For $A \in \mathbb{S}$ let Ω_A and $\mathcal{F}_A$ equal the restriction of Ω and $\mathcal{F}$ to A respectively. Similarly ω_A represents the projection of $\omega \in \Omega$ onto the set Ω_A.

A *potential* is a family $\Phi = \{\Phi_A\}_{A\in\mathbb{S}}$ of functions $\Phi_A : \Omega \to I\!R$ such that each Φ_A is $\mathcal{F}_A$-measurable. For all $\Lambda \in \mathbb{S}$ and $\omega \in \Omega$ the *energy* over the set Λ is $H^{\Phi}_{\Lambda}(\omega) \triangleq \sum_{A\in\mathbb{S},\, A\cap\Lambda\neq\emptyset} \Phi_A(\omega)$. The *finite Gibbs measure* is defined as

$$\mu^{\Phi}(\omega) = \frac{e^{-H^{\Phi}_S(\omega)}\lambda(\omega)}{Z^{\Phi}_S}$$

where $Z^{\Phi}_S = \sum_{w\in\Omega} e^{-H^{\Phi}_S(\omega)}\lambda(\omega)$ is the *partition function.*

The *Markov graph* associated with a Gibbs measure with potential Φ is an undirected graph (S^{Φ}, E^{Φ}) where the vertices of the graph, S^{Φ}, are the nodes in S. The set $\{i, j\}$ is an edge in E^{Φ} if there exists a nonzero potential Φ_A with $\{i, j\} \in A$ (there are no self-loops: if $\{i, j\} \in E^{\Phi}$ then $i \neq j$.)

We define the set of directed edges associated with the edge set E^{Φ} to be $\vec{E}^{\Phi} \triangleq \{(i, j),\ (j, i)\ :\ \{i, j\} \in E^{\Phi}\}$. It will be our convention that $\{i, j\}$ represents the *undirected* edge between node i and node j whereas (i, j) represents the *directed* edge from node i to node j.

A *pairwise potential* is a potential $\Phi = \{\Phi_A\}_{A\in\mathbb{S}}$ such that if $|A| > 2$ then $\Phi_A = 0$. We limit our discussion to Gibbs measures based on potentials consisting of pairwise potentials. Without loss of generality any Gibbs measure can be represented as a Gibbs measure with pairwise potentials [4]. This new representation, though, may lead to a large state expansion.

In many inference problems one often distinguishes between two kinds of nodes: hidden and observed. Furthermore each observed nodes is assumed to be independent of all the other nodes conditioned on a particular hidden node. In this paper it will be assumed that the effects of the observations have been captured by the self-potentials: $\Phi_{\{i\}},\ i \in S$.

Sum-Product Algorithm

We are interested in computing the marginal distribution at each node of a finite Gibbs measure defined by a pairwise potential. The sum-product algorithm attempts to do this by transmitting *messages* between the nodes and computing *beliefs* at each node.

We can think of the messages and beliefs as probability measures. Specifically, the *message* for the directed edge $(i, j) \in \vec{E}^{\Phi}$ at time n is a measure : $m^n_{(i,j)} : \mathcal{X}_j \to [0, 1]$ such that $\sum_{w_j\in\mathcal{X}_j} m^n_{(i,j)}(\omega_j) = 1$. Similarly, the *belief* at node $i \in S$ at time n is a measure: $b^n_i : \mathcal{X}_i \to [0, 1]$ such that $\sum_{w_i\in\mathcal{X}_i} b^n_i(\omega_i) = 1$.

We define a generic operation η that takes a bounded, non-negative function f on a finite domain $\mathcal{X}$ and outputs its *normalization*. Specifically $\eta : f \mapsto \sum_{x\in\mathcal{X}} f(x)$.

Given a graph (S, E) and any set $A \in \mathbb{S}$ let

$$\partial A \triangleq \{j \in S \setminus A \,:\, \{i,j\} \in E \text{ for some } i \in A\}$$

be the *boundary* of the set A in the graph. We will abuse notation and use ∂i and $\partial i \setminus j$ to represent $\partial\{i\}$ and $\partial\{i\} \setminus \{j\}$ respectively.

The *sum-product algorithm* consists of the following iteration on messages. For each $(i,j) \in \vec{E}^{\Phi}$ let $m^{n+1}_{(i,j)}(\omega_j) \triangleq$

$$\eta \sum_{\omega_i \in \mathcal{X}_i} e^{-\left(\Phi_{\{i,j\}}(\omega_i,\omega_j)+\Phi_{\{i\}}(\omega_i)\right)} \prod_{k\in\partial i\setminus j} m^n_{(k,i)}(\omega_i).$$

The beliefs at time n are

$$b^n_i(\omega_i) \triangleq \eta\, e^{-\Phi_{\{i\}}(\omega_i)} \prod_{k\in\partial i} m^n_{(k,i)}(\omega_i)$$

The messages are initialized to $\{m^0_{(i,j)}(\omega_j)\}_{(i,j)\in\vec{E}^{\Phi}}$.

The sum-product algorithm is said to *converge* if there exists a unique set of messages $\{m^*_{(i,j)}\}_{(i,j)\in\vec{E}^{\Phi}}$ such that for each $(i,j) \in \vec{E}^{\Phi}$ the limit $\lim_{n\to\infty} \|m^n_{(i,j)} - m^*_{(i,j)}\|_{\text{TV}} = 0$. Where $\|\cdot\|_{\text{TV}}$ is the total variation norm. If the messages converge then clearly the beliefs converge.

Sum-Product on Finite Trees

For potentials with Markov graphs that are trees the sum-product algorithm converges to the true marginals. We state this result and give a representation of the measure μ^{Φ} in terms of the messages.

Recall a *tree* is a singly connected, undirected graph without any loops. We sometimes single out one node $s \in S$ to be called the *root*. On the tree there is a natural distance measure $d : S \times S \to \mathbb{R}^+$, where $d(i,j)$ is the number of edges on the unique path from node i to node j. Let $L^s_n \subset S$ be the set of nodes that are exactly a distance n away from the root s. We will just write L_n when the root s is obvious.

The following result is standard and can be found in, for example, [2].

Proposition II.1 *Let Φ be a pairwise potential whose Markov graph (S^{Φ}, E^{Φ}) is a tree. Then*

(1) for any set of initial messages $\{m^0_{(i,j)}\}_{(i,j)\vec{E}^{\Phi}}$ the sum-product algorithm converges to a unique set of messages $\{m^*_{(i,j)}\}_{(i,j)\vec{E}^{\Phi}}$

(2) for any connected subset $A \subset S^{\Phi}$ one has

$$\mu^{\Phi}(\omega_A) = \eta \prod_{B\subseteq A} e^{-\Phi_B(\omega_A)} \prod_{i\in\partial A} m^*_{(i,i_A)}(\omega_{i_A}) \quad (1)$$

where i_A be the unique neighbor of node $i \in \partial A$ in the set A. Hence the beliefs converge to the true marginals.

Note that equation (1) states that the marginal on any connected set of nodes in a tree can be determined by the potentials defined on the set and the messages transmitted across the set's boundary. A belief is just a marginal on one node.

The Computation Tree

We now show that n iterations of the sum-product algorithm on a given finite, pairwise potential, Gibbs measure can be represented as an exact sum-product algorithm on an associated Gibbs measure defined on a tree, specifically the *computation tree* [4].

Definition II.1 *Given a pairwise potential Φ and its graph (S^{Φ}, E^{Φ}), the associated* computation tree *of depth n with root $s \in S^{\Phi}$, denoted $(\tilde{S}^{\Phi,s}_n, \tilde{E}^{\Phi,s}_n)$, is defined as the tree that consists of all length n paths in the graph, (S^{Φ}, E^{Φ}), starting at $i_0 = s$, $(i_0, i_1, i_2, ..., i_n)$, that never backtrack. Specifically the tree consists of all length n paths where $\{i_k, i_{k+1}\} \in E^{\Phi}$ and $i_k \neq i_{k+2}$.*

Figure one shows an example of a computation tree of depth three starting at node a. In the figure we have labelled each node in the computation tree with the associated node in the original graph.

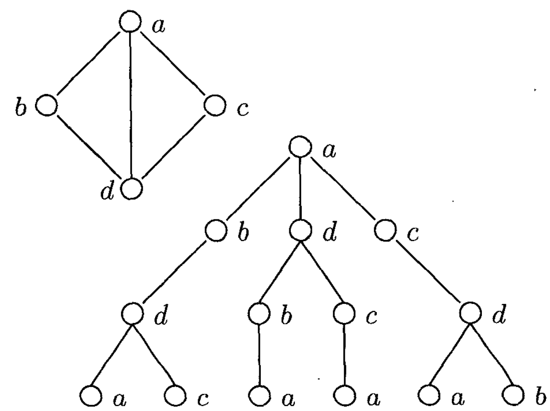

Figure 1: Example of a Computation Tree

We want to construct a Gibbs measure whose Markov graph is the computation tree. For each edge and non-leaf node we will place a potential corresponding to the potential on the original graph. We need only define the set of *boundary* self-potentials on the leaves in terms of the initializing messages. For each $(i,j) \in \vec{E}^{\Phi}$ let

$$\Phi^{i\to j}_{\{i\}}(\omega_i) = -\sum_{k\in\partial i\setminus j} \ln m^0_{(k,i)}(\omega_i).$$

Note that in the case when the initial messages are all set to the ones vector we have $\Phi^{i\to j}_{\{i\}} \equiv 0$.

Definition II.2 *Given Φ, its associated computation tree, $(\tilde{S}^{\Phi,s}_n, \tilde{E}^{\Phi,s}_n)$, and a set of boundary self-potentials $\{\Phi^{i\to j}_i\}_{(i,j)\in\vec{E}^{\Phi}}$ define the* associated *potential for the* computation tree of depth n, *denoted $\tilde{\Phi}^{n,s}$, as follows:*

(1) Let the map $\Gamma^n : \tilde{S}^{\Phi,s}_n \to S^{\Phi}$ be the map that takes node $\tilde{i}$ to its associated node i in the original graph. (As constructed in definition II.1.)

(2) For each $\{\tilde{i},\tilde{j}\} \in \tilde{E}_n^{\Phi,s}$ *let* $\tilde{\Phi}^{n,s}_{\{\tilde{i},\tilde{j}\}} = \Phi_{\{\Gamma^n(\tilde{i}),\Gamma^n(\tilde{j})\}}$

(3) For each $\tilde{i} \in \tilde{S}_n^{\Phi,s}$ *let*

$$\tilde{\Phi}^{n,s}_{\{\tilde{i}\}} = \begin{cases} \Phi_{\{\Gamma^n(\tilde{i})\}} & \text{if } \tilde{i} \in \tilde{S}_n^{\Phi,s}\backslash L_n^s \\ \Phi_{\{\Gamma^n(\tilde{i})\}} + \Phi^{\Gamma^n(\tilde{i})\to\Gamma^n(\tilde{j})}_{\{\Gamma^n(\tilde{i})\}} & \text{if } \tilde{i} \in L_n^s \text{ and} \\ \quad \text{where } \tilde{j} \text{ is the unique parent of } \tilde{i} & \end{cases}$$

Let $\mu^{\tilde{\Phi}^{n,s}}(\tilde{\omega})$ be the measured determined by $\tilde{\Phi}^{n,s}$. By construction, running the sum-product algorithm to compute the belief at the root node, s, of the computation tree of depth n is equivalent to computing the belief at node s on the original graph after n iterations of the sum-product algorithm initialized appropriately.

III. Convergence of the Sum-Product Algorithm and the Weak Limit

The sequence of measures $\{\mu^{\tilde{\Phi}^{n,s}}\}_{n\geq 1}$ has a *weak limit* if for each N there exists a measure μ_N defined on $\tilde{S}_N^{\Phi,s}$ such that for all events $A \in \mathcal{F}_{\tilde{S}_N^{\Phi,s}}$ the limit $\lim_{n\geq N, n\to\infty} \mu^{\tilde{\Phi}^{n,s}}(A) = \mu_N(A)$.

Proposition III.1 *The sum-product algorithm converges if and only if the sequence of measures* $\{\mu^{\tilde{\Phi}^{n,s}}\}_{n\geq 1}$ *has a weak limit.*

Our goal for the rest of the paper is to understand the convergence properties of the sequence of measures $\{\mu^{\tilde{\Phi}^{n,s}}\}_{n\geq 1}$. However, instead of working with a sequence of finite trees of increasing depth we will find it easier to study one infinite tree and the measures defined on it.

IV. Gibbs Measures Over a Countable Set of Sites

Constructing a Gibbs measure over a countable set of nodes can be a tricky business. Let the set of nodes S be countably infinite and redefine

$$\mathbb{S} = \{\Lambda \subset S \,:\, 0 < |\Lambda| < \infty\}$$

to be the set of all nonempty, *finite*, subsets of S. At every node $i \in S$ there is a finite measure space $(\mathcal{X}_i, \mathcal{F}_i)$. We construct, in the usual way, the product measure space $(\Omega, \mathcal{F}) = (\prod_{i\in S}\mathcal{X}_i, \prod_{i\in S}\mathcal{F}_i)$. We also extend the uniform reference measure $\lambda = \prod_{i\in S}\lambda_i$ on $(\Omega, \mathcal{F})$. As before we restrict ourselves to pairwise potentials. We assume that the number of neighbors at any node is finite and hence for any $\Lambda \in \mathbb{S}$ the energy $H^\Phi_\Lambda(\omega) = \sum_{A\in\mathbb{S},\, A\cap\Lambda\neq\emptyset} \Phi_A(\omega)$ exists.

Because there are a countably infinite number of nodes we cannot compute the partition function by summing over all nodes. But we can discuss the partition function when conditioned on a particular boundary. Define the partition function in $\Lambda \in \mathbb{S}$ for the potential Φ, boundary $\omega_{S\backslash\Lambda}$, and reference measure λ to be

$$Z^\Phi_\Lambda(\omega) \triangleq \sum_{\zeta\in\Omega_\Lambda} e^{-H^\Phi_\Lambda(\zeta,\omega_{S\backslash\Lambda})}\lambda_\Lambda(\zeta).$$

Given a potential Φ, $\omega \in \Omega$, and $\Lambda \in \mathbb{S}$. Then the measure

$$\gamma^\Phi_\Lambda(\cdot \mid \omega) \triangleq \frac{e^{-H^\Phi_\Lambda(\cdot,\, \omega_{S\backslash\Lambda})}\lambda_\Lambda(\cdot)}{Z^\Phi_\Lambda(\omega)}$$

is called the *Gibbs distribution in* Λ *with boundary* $\omega_{S\backslash\Lambda}$ *and potential* Φ. Furthermore $\gamma^\Phi = \{\gamma_\Lambda\}_{\Lambda\in\mathbb{S}}$ is called the *Gibbsian specification for* Φ.

Given a Gibbsian specification we can ask how many measures are consistent with it. Define the set of all Gibbs measures for the potential Φ by:

$$\begin{aligned} G(\gamma^\Phi) \;\triangleq\; & \{\mu \in \mathcal{P}(\Omega,\mathcal{F}) \,:\, \mu(A \mid \mathcal{F}_{\Lambda^c}) = \gamma^\Phi_\Lambda(A \mid \cdot) \\ & \mu - a.s.\ \forall A \in \mathcal{F} \text{ and } \Lambda \in \mathbb{S}\}. \end{aligned}$$

One can think of $G(\gamma^\Phi)$ as the set of all measures locally consistent with the specified potential Φ. The set $G(\gamma^\Phi)$ can either be empty, contain one measure, or contain an infinite number of measures. A potential is said to exhibit a *phase transition* if the set $|G(\gamma^\Phi)| > 1$. Phase transitions are a remarkable phenomena. For a given local specification we can have very different global behaviors. For the pairwise potential defined above it can be shown that there exists at least one Gibbs measure [1].

Limiting Gibbs Measures on Trees

We have discussed Gibbs measures defined on countable sets of nodes. We now restrict our attention to Gibbs measures defined on infinite trees. We will show that if the sum-product algorithm converges then the associated measures on the computation tree must converge to an element of $G(\gamma^\Phi)$.

How do we construct an infinite volume Gibbs measure? For Gibbs measures defined on infinite trees we will show that the measures can arise as the weak limit of a sequence of Gibbs measures with fixed boundary conditions.

Let Φ be a pairwise potential whose Markov graph (S^Φ, E^Φ) is a countably infinite tree. Let $T_n \subset S^\Phi$ be the set of nodes in the tree that are a distance of n or less from the root. Recall $L_n \subset T_n$ is the set of nodes that are exactly a distance n away from the root.

From the original potential $\Phi = \{\Phi_A\}$ we will now define a new sequence of potentials. For each $n \geq 1$ define $\Phi^{T_n} = \{\Phi^{T_n}_A\}$ by

$$\Phi^{T_n}_A = \begin{cases} \Phi_A, & \text{if } A \cap T_{n-1} \neq \emptyset; \\ \Phi^{bd,n}_A + \Phi_A, & \text{if } A \subset L_n,\ |A| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

where $\Phi^{bd,n} \triangleq \{\Phi^{bd,n}_A\}$ represents added self-potentials at the leaves of the T_n tree. If $\Phi^{bd,n}_A = 0$ for all $A \subset L_n$ then we call the boundary a *free boundary.* Note that $\gamma^{\Phi^{T_n}}_{T_n}(A \mid \omega)$ is independent of ω for $A \in \mathcal{F}_{T_n}$. For each Φ^{T_n} we can write the unique Gibbs measure as $\mu^{\Phi^{T_n}} = \lambda_{T_n^c}\gamma^{\Phi^{T_n}}_{T_n}$.

In the context of the computation tree $\mu^{\Phi^{T_n}}$ represents the measure on the infinite tree corresponding to n iterations of the sum-product algorithm when initialized with the self-potentials $\Phi^{bd,n}$. We can relate the choice of $\Phi^{bd,n}$ to the choice of boundary self-potentials, $\{\Phi^{s_i\to s_j}_{s_i}\}_{(s_i,s_j)\in\vec{E}}$, in the sum-product algorithm.

We are interested in conditions that insure the measures, $\{\mu^{\Phi^{T_n}}\}_{n\geq 1}$, converge to a limiting measure. The next proposition states that if they converge to a limiting measure then that measure must be an element of $G(\gamma^{\Phi})$. Hence examining the structure of $G(\gamma^{\Phi})$ is useful for determining convergence of the sum-product algorithm.

Proposition IV.1 *Each subsequential limit of the sequence of measures $\{\mu^{\Phi^{T_n}}\}_{n\geq 1}$ belongs to $G(\gamma^{\Phi})$.*

V. Unique Gibbs Measure Case

Here we consider the case of a unique Gibbs measure: $|G(\gamma^{\Phi})| = 1$. Clearly there can be only be one subsequential limit for the sequence of measures $\mu^{\Phi_{T_n}}$. Hence the sum-product algorithm converges. We can say something stronger though: the sum-product algorithm converges uniformly over the choice of all initializing messages. See proposition 7.11 of [1].

Proposition V.1 *If $|G(\gamma^{\Phi})| = 1$ then*

$$\lim_{n\to\infty} \gamma^{\Phi}_{T_n}(\cdot \mid \omega) = \mu^{\Phi}(\cdot) \text{ uniformly in } \omega \in \Omega.$$

Thus one can show:

Proposition V.2 *If $|G(\gamma^{\Phi})| = 1$ then for any cylinder event A we have*

$$\lim_{n\to\infty} \mu^{\Phi^{T_n}}_{T_n}(A) = \mu^{\Phi}(A)$$

uniformly over the boundary self-potentials.

Next we present Dobrushin's sufficient condition for uniqueness of the limiting Gibbs measure.

Dobrushin's Condition

A Gibbs potential can lead to many different phases if nodes that are far apart from each other do not mix fast enough with respect to the distance between them. Dobrushin proposed the following condition that insures fast mixing with distance and hence uniqueness. Let

$$C_{ij} \triangleq \sup_{\zeta,\eta\in\Omega\,:\,\zeta_{S\setminus j}=\eta_{S\setminus j}} \|\gamma_i(\cdot \mid \zeta) - \gamma_i(\cdot \mid \eta)\|.$$

This measures the variation in the conditional probability at node i when we change the value of node j. See proposition 8.7 in [1].

Proposition V.3 *If $c \triangleq \sup_{i\in S}\sum_{j\in\partial i} C_{ij} < 1$ then $|G(\gamma^{\Phi})| = 1$.*

Simon's Condition

Dobrushin's condition can be difficult to verify in practice. The following proposition introduces Simon's condition. This is an easy to check condition that implies Dobrushin's condition. See proposition 8.8 in [1].

Proposition V.4 *Let Φ be a pairwise potential. If*

$$\sup_{i\in S}\sum_{A\ni i}(|A|-1)\delta(\Phi_A) < 2)$$

then $|G(\gamma^{\Phi})| = 1$. Where $\delta(f) \triangleq \sup_x f(x) - \inf_x f(x)$.

This proposition states that the "influence" node i has on the rest of the nodes depends on two things: the number of neighbors it has and the strength of the potentials, measured by $\delta(\Phi)$, it takes part in. Note that the self-potentials do not play a part in Simon's condition.

Let us return to the issue of the sum-product algorithm. Let Φ be the potential for the finite Gibbs measure that we wish to apply the sum-product algorithm to. Let $\tilde{\Phi}$ be the corresponding potential on the computation tree. The local topology of the computation tree looks like the local topology of the original graph. Hence to show $|G(\tilde{\Phi})| = 1$ we need to show

$$\sup_{\tilde{i}\in S^{\Phi}}\sum_{A\ni\tilde{i}}(|A|-1)\delta(\tilde{\Phi}_A) = \max_{i\in S^{\Phi}}\sum_{A\ni i}(|A|-1)\delta(\Phi_A) < 2$$

Note that the maximum condition is very easy to check on finite graphs.

Rate of Convergence

Here we give a condition on the rate of convergence of the sum-product algorithm. Specifically if Dobrushin's condition is satisfied then the convergence rate is exponential in the number of iterations of the sum-product algorithm. See theorem 8.23, remark 8.26, and corollary 8.32 of [1].

Proposition V.5 *Let $\beta < -\ln c$ (where c was defined in proposition 5.3.) If $c < 1$ then*

$$|\mu^{\tilde{\Phi}^{T_n}}_{T_n}(\omega_s) - \mu^{\tilde{\Phi}}(\omega_s)| \leq \frac{1}{1-ce^{\beta}}e^{-\beta n}$$

Error Analysis

In general determining the error between the true marginal and the belief produced by the sum-product algorithm is difficult. For the case of graphs with large girths we can give a bound on the error as follows. Recall that the *girth* of a graph is the number of edges in the shortest cycle in the graph. Graphs with large girths are often studied in the analysis of low density parity check (LDPC) codes. Large girths correspond to the *locally tree-like* structure of the graphs found in LDPC codes.

Proposition V.6 *Let M be the girth of the original graph. If $c < 1$ and $\beta < -\ln c$ then*

$$|\mu^{\Phi}(\omega_s) - \mu^{\tilde{\Phi}}(\omega_s)| \leq \frac{1}{1-ce^{\beta}}e^{-\beta\frac{M}{2}}$$

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