A STROLL THROUGH THE GARDEN OF GRAPH ZETA FUNCTIONS

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Part 1. A Quick Look at Various Zeta Functions

The goal of this book is to guide the reader in a stroll through the garden of zeta functions of graphs. The subject arose in the late part of the last century modelled after zetas found in the other gardens.

Number theory involves many zetas starting with Riemann's - a necessary ingredient in the study of the distribution of prime numbers. Other zetas of interest to number theorists include the Dedekind zeta function of an algebraic number field and the analog for function fields. Many Riemann hypotheses have been formulated and a few proved. The statistics of the complex zeros of zeta have been connected with the statistics of the eigenvalues of random Hermitian matrices (the GUE distribution of quantum chaos). Artin L-functions are also a kind of zeta associated to a representation of a Galois group of number or function fields. We will find graph analogs of all of these.

Differential geometry has its own zeta - the Selberg zeta function which is used to study the distribution of lengths of prime geodesics in compact or arithmetic Riemann surfaces. There is a third zeta function known as the Ruelle zeta function which is associated to dynamical systems. We will look at these zetas briefly in the introduction. The graph theory zetas are related to these zetas too.

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FIGURE 1. in the garden of zetas

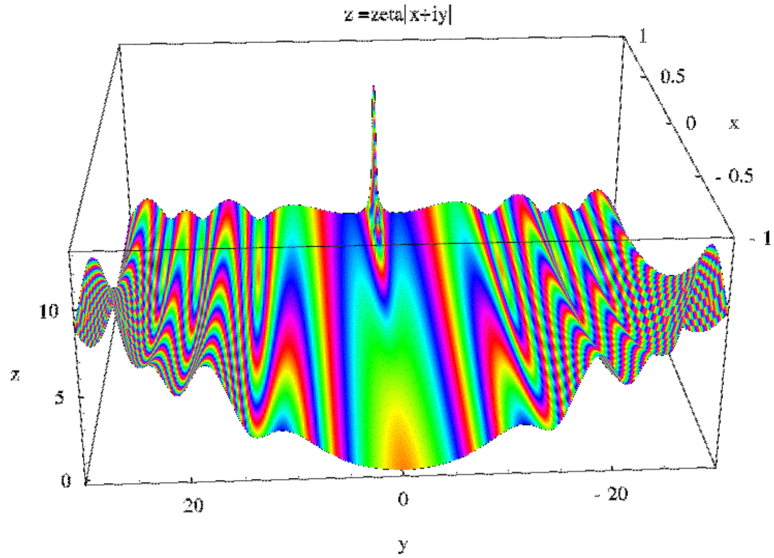

FIGURE 2. Graph of $z = |\zeta(x+iy)|$ showing the pole at $x+iy=1$ and the complex zeros nearest the real axis (all of which are on the line $\mathrm{Re}(s) = \frac{1}{2}$, of course.

In this section we give a brief glimpse of four sorts of zeta functions to motivate the rest of the book. Much of the first section is not necessary for the rest of the book. Feel free to skip all but Section 2 on the Ihara zeta function.

Most of this book arises from joint work with Harold Stark.

Thanks are due to the many people who listened to my lectures on this book and helped with the research; e.g., Matthew Horton, Derek Newland at U.C.S.D...... I would especially like to thank the people who encouraged me by attending my Ulam Seminar at the University of Colorado, Boulder; especially Lynne Walling, David Grant, Su-ion Ih, Vinod Radhakrishnan, Erika Frugoni, Jonathan Kish, and Mike Daniel.

I would also like to thank the Newton Institute in Cambridge, England for its hospitality while I worked on this manuscript.

1. Riemann's zeta function

There are many popular books about the Riemann zeta and many "serious" ones as well. Serious references for this topic include H. Davenport [24], H. Edwards [27], Iwaniec and Kowalski [45], S. J. Miller and R. Takloo-Bighash [61], and S. J. Patterson [67]. I googled "zeta functions" today and got around 135,000 hits. The most extensive website was www.aimath.org.

The theory of zeta functions was developed by many people but Riemann's work in 1859 was certainly the most important. The concept was generalized for the purposes of number theorists by Dedekind, Dirichlet, Hecke, Takagi, Artin and others. Here we will concentrate on the original, namely, Riemann's zeta function. The definition is as follows.

Riemann's zeta function for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The infinite product here is called an Euler product. In 1859 Riemann extended the definition of zeta to an analytic function in the whole complex plane except for a simple pole at $s = 1$. He also showed that there is an unexpected symmetry known as the functional equation relating the value of zeta at s and the value at $1 - s$. It says

$$\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \Lambda(1-s). \tag{1.1}$$

The **Riemann hypothesis** (or **RH**) says that the non-real zeros of $\zeta(s)$ (equivalently those with $0 < \mathrm{Re}(s) < 1$) are on the line $\mathrm{Re}(s) = \frac{1}{s}$. It is equivalent to giving an explicit error term in the prime number theorem stated below.

The Riemann hypothesis is now checked to 10^{13}-th zero. (October 12th 2004), by Xavier Gourdon with the help of Patrick Demichel. See Ed Pegg Jr.'s website for an article called the Ten Trillion Zeta Zeros:

$$http://www.maa.org/editorial/mathgames.$$

You win \$1 million if you have a proof of the Riemann hypothesis. See the Clay Math. Institute website:

www.claymath.org.

A. Odlyzko has studied the spacings of the zeros and found that they appear to be the spacings of eigenvalues of a random Hermitian matrix (GUE). See Figure 21 and the paper on his website

www.dtc.umn.edu/~odlyzko/doc/zeta.htm.

This sort of computation can also be found on the website of X. Gourdon mentioned above.

If one knows the Hadamard product formula for zeta (from a graduate complex analysis course) as well as the Euler product formula (1.1) above, one obtains exact formulas displaying a relationship between primes and zeros of zeta. Such reasoning ultimately led Hadamard and de la Vallee Poussin to prove the prime number theorem about 50 years after Riemann's paper. The **prime number theorem** says

$$\#\{p = \text{prime} \mid p \le x\} \sim \frac{x}{\log x}, \text{ as } x \to \infty.$$

The explicit formulas of prime number theory give the connection between zeros of zeta and primes. The Riemann hypothesis is equivalent to giving an error term in the prime number theorem.

Figure 2 is a graph of $z = |\zeta(x + iy)|$ drawn by Mathematica. The cover of *The Mathematical Intelligencer* (Vol. 8, No. 4, 1986) shows a similar graph with the pole at $x + iy = 1$ and the first 6 zeros, which are on the line $x = 1/2$, of course. The picture was made by D. Asimov and S. Wagon to accompany their article on the evidence for the Riemann hypothesis. The Mathematica people will sell you a huge poster of the Riemann zeta function.

Exercise 1. *Use Mathematica to do a contour plot of the Riemann zeta function in the same region as that of Figure 2.*

Hints. *Mathematica has a command to give you the Riemann zeta function. It is Zeta[s]. Using this, we made Figure 2 with the following statement.*

```
Plot3D[{ Abs[ Zeta[x + I y] ],Hue[Abs[ Zeta[x + I y] ]]}, {x,-1, 1}, {y, -30, 30},
PlotPoints\[Rule]400,Mesh -> False,PlotRange->All, AxesLabel->{"x","y"," z"},
PlotLabel->"z=|zeta(x+iy)|",ViewPoint->{-2.776, -0.132, 1.765}];
```

There was also an extra line for the font.

```
$TextStyle={FontFamily\[Rule]"Times",FontSize\[Rule]7}
```

You will need to know the command to get a contour plot. This gave the poster for the website of the 2006 women & math program at IAS.

Another topic in the work on Riemann's zeta function is the explicit formulas saying sums over the zeros of zeta are equal to sums over the primes. A reference is Murty [63].

Many other kinds of zeta functions have been investigated since Riemann. In number theory there is the Dedekind zeta function of an algebraic number field K, such as $K = \mathbb{Q}(\sqrt{2})$, for example. This zeta is an infinite product over prime ideals $\mathfrak{p}$ in O_K, the ring of algebraic integers of K. For our example, $O_K = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. The terms in the product are $\left(1 - N\mathfrak{p}^{-s}\right)^{-1}$, where $N\mathfrak{p} = \#(O_K/\mathfrak{p})$. Riemann's work can be extended to this zeta function and it can be used to prove the prime ideal theorem. The RH is unproved but conjectured to be true for the Dedekind zeta function, except for some modifications concerning a possible zero near 1 on the real axis called a "Siegel zero." A reference for this zeta is Lang [53].

There are also function field zeta functions where K is replaced by a finite algebraic extension of $\mathbb{F}_q(x)$, the rational functions of one variable over a finite field $\mathbb{F}_q$. The RH has been proved for this zeta which is a rational function of $u = q^{-s}$. See Rosen [71]. See Figures 3, 4, 5, and 6 for summaries of the facts about zeta and L-functions for $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{d})$. We will find graph theory analogs of many of these things.

Another generalization of Riemann's zeta function is the Dirichlet L-function associated to a multiplicative character χ defined on the group of integers $a(\text{mod}\, m)$ with a relatively prime to m. This function is thought of as a function on the integers which is 0 unless a and m have no common divisors. Then one has the **Dirichlet L-function**, for $\text{Re}\, s > 1$ defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This L-function also has an Euler product, analytic continuation, functional equation, Riemann hypothesis (the Extended Riemann Hypothesis or ERH). This function can be used to prove the Dirichlet theorem saying that there are infinitely many primes in an arithmetic progression of the form $a, a+d, a+2d, a+3d, ..., a+kd, ...,$ assuming that a and d are relatively prime.

Yet another sort of zeta is the Epstein's zeta function attached to a quadratic form

$$Q[x] = \sum_{i,j=1}^{n} q_{ij} x_i x_j.$$

We assume that the q_{ij} are real and that Q is positive definite, meaning that $Q[x] > 0$. Then **Epstein zeta function** is defined for complex s with $\text{Re}\, s > \frac{n}{2}$ by:

$$Z(Q, s) = \sum_{a \in \mathbb{Z}^n - 0} Q[a]^{-s}.$$

As for the Riemann zeta, there is an analytic continuation to all $s \in \mathbb{C}$ with a pole at $s = \frac{n}{2}$. And there is a functional equation relating $Z(Q, s)$ and $Z(Q, n-s)$. Even when $n = 2$ the analog of the Riemann hypothesis may be false for the Epstein zeta function. See Terras [93] for more information on this zeta function.

If $Q[x] \in \mathbb{Z}$, for all $x \in \mathbb{Z}^n$, then defining $N_m(Q) = |\{x \in \mathbb{Z}^n | \, Q[x] = m\}|$, we see that $Z(Q, s) = \sum_{m \geq 1} N_m m^{-s}$, assuming $\text{Re}\, s > \frac{n}{2}$. Similarly one can defrine zeta functions attached to many lists of numbers like $N_m(Q)$, in particular, to the Fourier coefficients of modular forms. Classically modular forms are holomorphic functions on the upper half plane having an invariance property under groups of fractional linear transformations like the modular group $SL(2, \mathbb{Z})$ consisting of 2×2 matrices with integer entries and determinant 1. See S. J. Miller and R. Takloo-Bighash [61], Sarnak [75], or Terras [93] for more information. Now the idea of modula forms has been vastly generalized and even plays a role in Andrew Wiles proof of Fermat's last theorem.

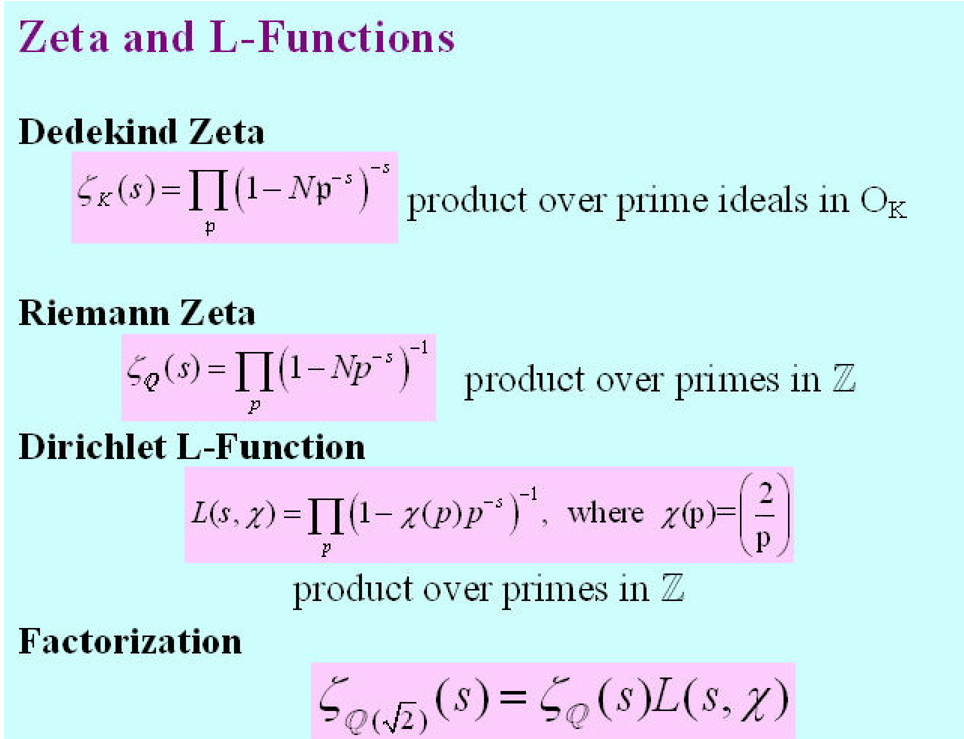

FIGURE 3. A summary of facts about zeta and L-functions associated to the number fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$. See Figure 6 for the definition of the Legendre symbol $\left(\frac{2}{p}\right)$.

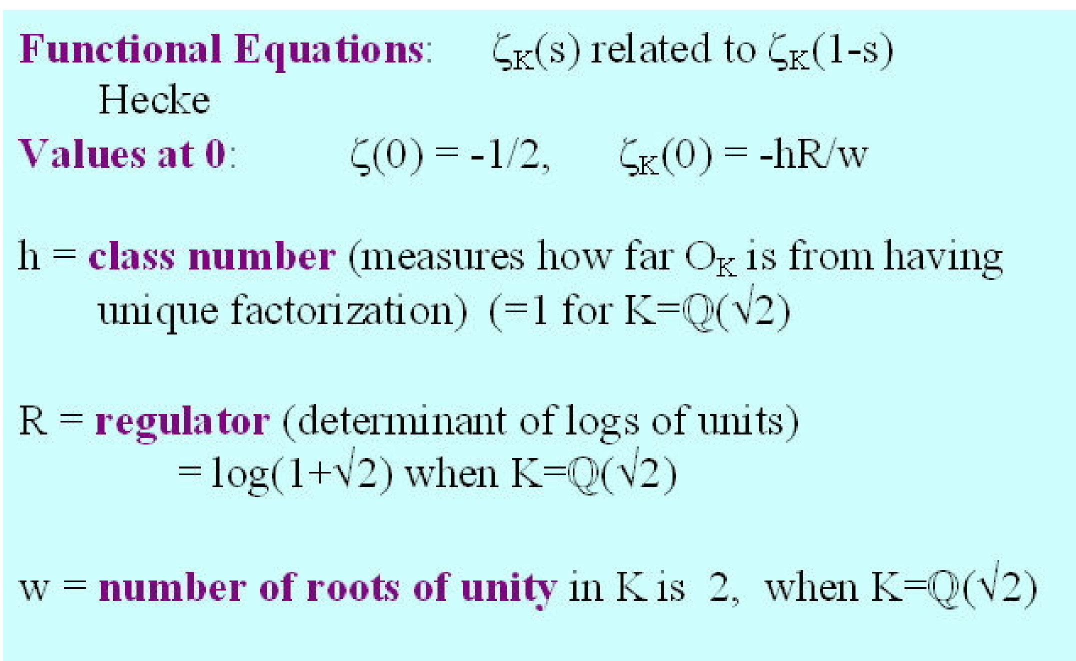

FIGURE 4. What the zeta and L-functions say about the number fields

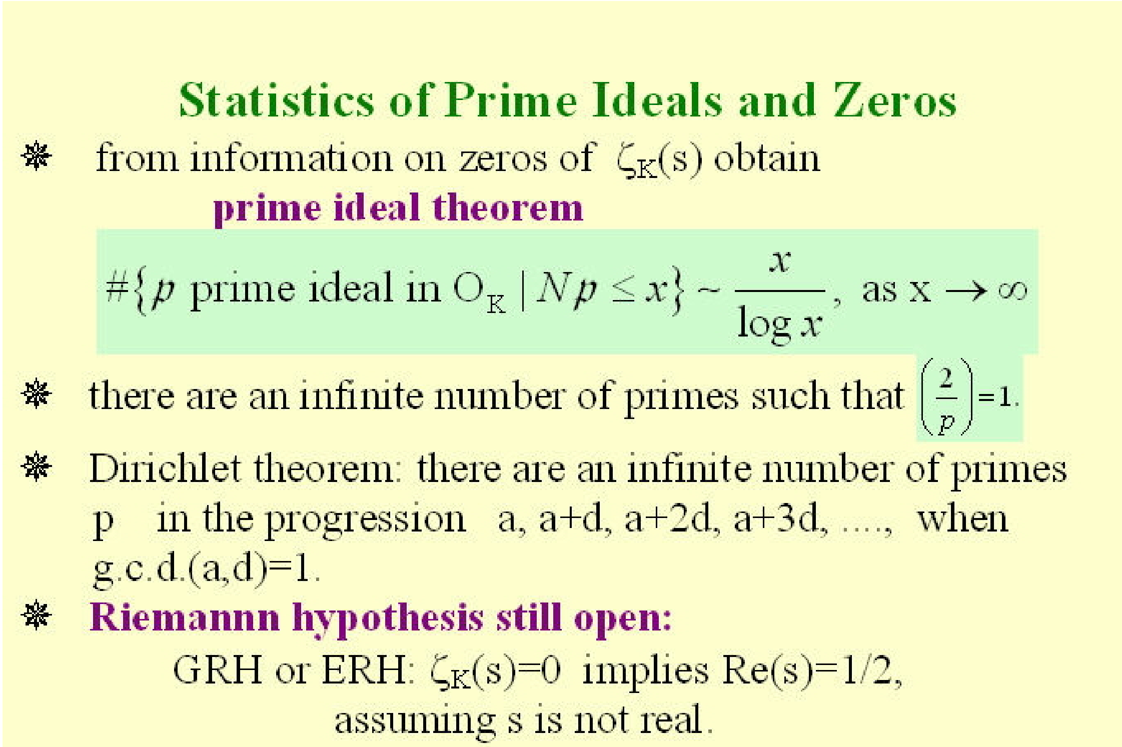

FIGURE 5. Statistics of prime ideals

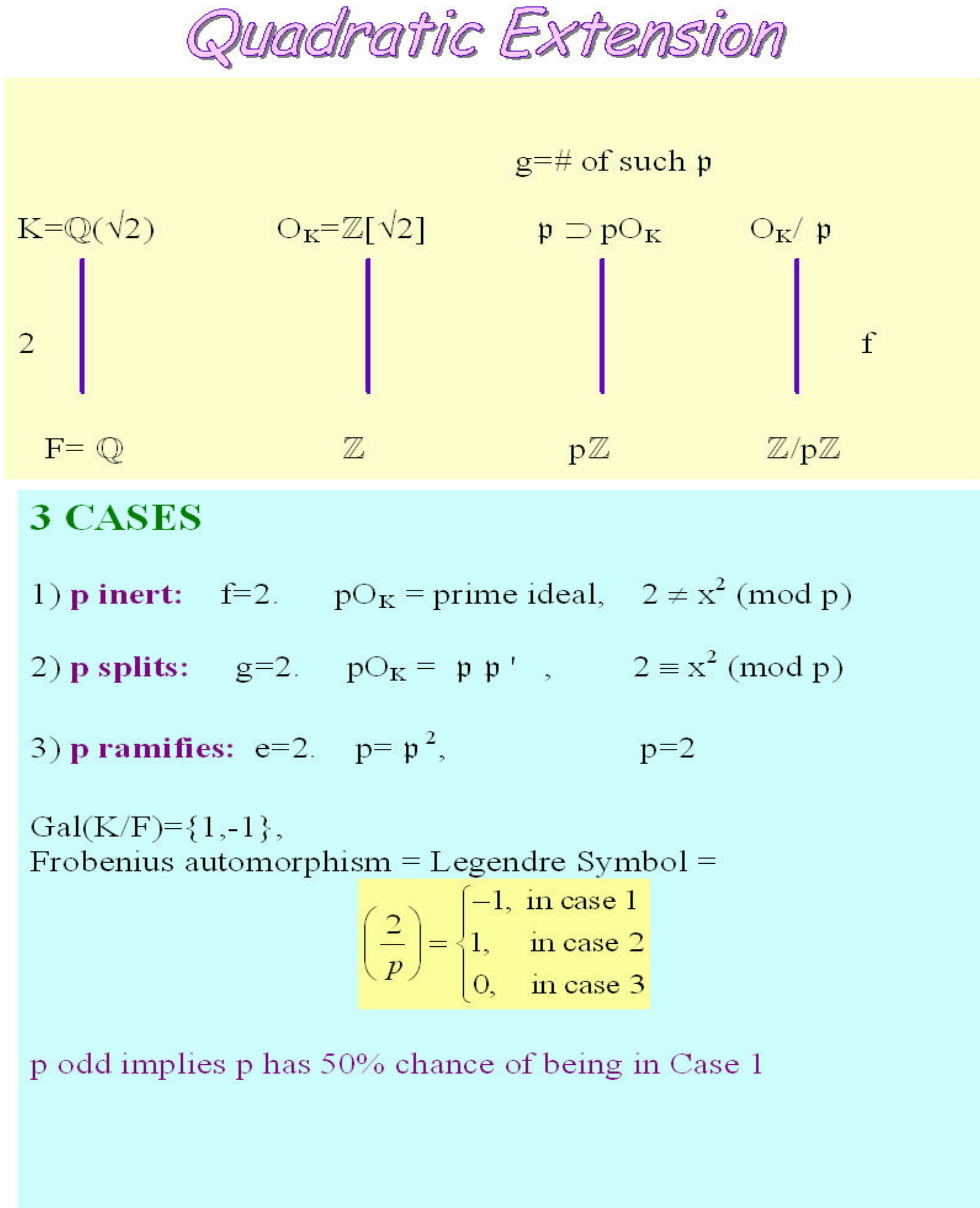

FIGURE 6. **Splitting of primes in quadratic extensions.** At the top, the 4 blue lines represent the number field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, then the corresponding rings of integers, then prime ideals, and finally the finite residue fields. Here f is the degree of the extension of finite residue fields, g is the number of primes of O_K containing the prime p of $\mathbb{Z}$, and e is the ramification exponent. We have $2 = efg$ in the case under discussion with $K = \mathbb{Q}(\sqrt{2})$.

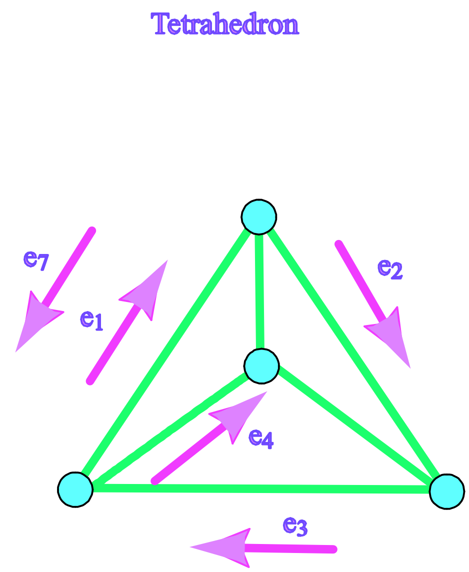

FIGURE 7. The tetrahedron or K4 graph with some of the edges oriented and labelled.

2. IHARA'S ZETA FUNCTION

References for graph theory include Biggs [11], Bollobás [15], Chung [18], and Cvetković et al [23]. Our graphs will be finite, connected and undirected. It will usually be assumed that they contain no degree 1 vertices (called "leaves" or "hair" or "danglers"). They may not be **regular** meaning that every vertex has the same **degree** (number of edges coming out of the vertex). A graph is **k-regular** if every vertex has degree k. Our graphs may also have loops and multiple edges.

Definition 1. *Let V denote the vertex set of our graph X with n=$|V|$. The* ***adjacency matrix*** *A of X is an $n \times n$ matrix with i,j entry*

$$a_{ij} = \begin{cases} \text{number of undirected edges connecting vertex } i \text{ to vertex } j, & \text{if } i \neq j\,; \\ 2 \times \textit{number of loops at vertex } i, & \textit{if } i = j\,. \end{cases}$$

In order to define the Ihara zeta function, we need to define a prime in a finite connected undirected graph X with edge set E having $m = |E|$ elements. To do this, we first orient the edges of our graph arbitrarily and **label the edges** as follows

$$e_1, ..., e_m, e_{m+1} = e_1^{-1}, ..., e_{2m} = e_m^{-1}. \tag{2.1}$$

Here $m = |E|$ is the number of unoriented edges of X and e_j^{-1} is the edge e_j with the opposite orientation. See Figure 7 for an example.

Primes in X, Some Definitions.

A path or walk $C = a_1 \cdots a_s$, where a_j is an oriented edge of X, is said to have a **backtrack** if $a_{j+1} = a_j^{-1}$, for some $j = 1, ..., s-1$. A path $C = a_1 \cdots a_s$ is said to have a **tail** if $a_s = a_1^{-1}$. The length of $C = a_1 \cdots a_s$ is $s = \nu(C)$. A **closed path** means the starting vertex is the same as the terminal vertex. The closed path $C = a_1 \cdots a_s$ is called a **primitive or prime path** if it has no backtrack or tail and $C \neq D^f$, for $f > 1$. For the path $C = a_1 \cdots a_s$, the **equivalence class** $[C]$ means the following

$$[C] = \{a_1 \cdots a_s,\ a_2 \cdots a_s a_1,\ \ldots,\ a_s a_1 \cdots a_{s-1}\}.$$

That is, we call two prime paths **equivalent** if we get one from the other by changing the starting point. The class $[C]$ can be identified with a conjugacy class in the fundamental group of X. A **prime** in the graph X is an equivalence class $[C]$ of prime paths. So, for example in Figure 7, a prime is $[e_1e_2e_3] = \{e_1e_2e_3, e_2e_3e_1, e_3e_1e_2\}$. The **length of the path** C is $\nu(C) = s$, the number of edges in C.

As long as the graph (assuming it is connected with no degree 1 vertices) is not a cycle or a single vertex, there will be infinitely many primes (as the rank of the fundamental group is greater than 1).

Exercise 2. *Prove the last statement.*

However, there are great differences between primes in the integers and primes in graphs. For example, given 2 primes $[P_1]$ and $[P_2]$ such that $[P_1] \neq [P_2^{\pm 1}]$, then we have another prime $[P_3] = [P_1P_2]$. Here $[P_1P_2] = [P_2P_1]$. Assuming we have

3 primes $[P_1], [P_2], [P_3]$ such that $[P_i] \neq [P_j^{\pm 1}]$, for $i \neq j$, then $[P_1 P_2 P_3] \neq [P_1 P_3 P_2]$. In particular, for the graph theory version of things, one does not have unique factorization into primes.

Definition 2. *The **Ihara zeta function** for a finite connected graph (without degree 1 vertices) is defined to be the following function of the complex number u, with $|u|$ sufficiently small:*

$$\zeta_X(u) = \prod_{[P]} \left(1 - u^{\nu(P)}\right)^{-1},$$

where the product is over all primes $[P]$ in X. Recall that $\nu(P)$ denotes the length of P.

In the product defining the Ihara zeta function, we distinguish the prime $[P]$ from $[P^{-1}]$, which is the path traversed in the opposite direction. Generally the product is infinite. We will see later how small $|u|$ must be for the product to converge. There is one case, however, when the product is finite. Normally we will exclude this case.

Example 1. ***Cycle Graph.*** *Let X be a cycle graph with n vertices. Then*

$$\zeta_X(u) = (1 - u^n)^{-2}.$$

As a power series in the complex variable u, the Ihara zeta function has non-negative coefficients. Thus, by a classic theorem of Landau, both the series and the product defining $\zeta_X(u)$ will converge absolutely in a circle $|u| < R_X$ with a singularity (pole of order 1 for connected X) at $u = R_X$. See Apostol [3], p. 237 for Landau's theorem.

Definition 3. *R_X is the **radius of the largest circle of convergence** of the Ihara zeta function.*

In fact, R_X is rather small. When X is a $(q+1)$-regular graph, $R_X = 1/q$. We will say more about the size of R_X for irregular graphs later. Amazingly the Ihara zeta function is the reciprocal of a polynomial by Theorem 1 below.

The Fundamental Group of a Graph.

Our formula for the Ihara zeta function involves the **fundamental group** $\Gamma = \pi_1(X, v)$ of the graph X. Later we will even define the path zeta which is more clearly attached to this group.

The fundamental group of a topological space such as our graph X has elements which are closed directed paths starting and ending at a fixed basepoint $v \in X$. Two paths are equivalent iff one can be continuously deformed into the other (i.e., one is homotopic to the other within X, while still starting and ending at v). The product of 2 paths a, b means first go around a then b.

It turns out (by the Seifert-von Kampen theorem, for example) that the fundamental group of graph X is a free group on r generators, where r is the number of edges left out of a spanning tree for X. Let us try to explain this a bit. More information can be found on the web; e.g., the algebraic topology book of Allen Hatcher, Chapter 1: www.math.cornell.edu/~hatcher. You could also look at Massey [59], p. 198, or Gross and Tucker [32].

What is a **free group** G on at set S of r generators? The group G is the set of words obtained by forming finite strings or words $a_1 \cdots a_t$ of symbols $a_j \in S$ modulo an equivalence relation. Two words $a_1 a_2 \cdots a_t$ and $a_1 a_2 \cdots b b^{-1} \cdots a_t$, $b \in S$ are called equivalent. The product of words $a_1 \cdots a_r$ and $b_1 \cdots b_s$ is $a_1 \cdots a_r b_1 \cdots b_s$. The result is a group.

What is a spanning tree T in a graph X? First we say that a graph T is a **tree** if it is a connected graph without any closed backtrackless paths of length > 1. A **spanning tree** T for graph X means a tree which is a subgraph of X containing all the vertices of X.

From the graph X we construct a new graph $X^{\#}$ by shrinking a spanning tree T of X to a point. The new graph will be a bouquet of r loops as in Figure 8. It turns out that the fundamental group of X is the same as that of $X^{\#}$. Why? The quotient map $X \to X/T$ is what algebraic topologists call a "homotopy equivalence." This means that intuitively you can continuously deform one graph into the other. For more information, see Allen Hatcher, Chapter 0: www.math.cornell.edu/~hatcher.

So what is the fundamental group of the bouquet of r loops in Figure 8? We claim it is clearly the free group on r generators. The generators are the loops! The elements are the words in these loops modulo the equivalence relation defined above for words in a free group.

Exercise 3. *Show that $r - 1 = |E| - |V|$.*

We have a 1-1 correspondence between conjugacy classes $\{C\}$ in $\Gamma = \pi_1(X, v)$ and equivalence classes of backtrackless, tailless cycles $[C]$ (i.e., closed paths) in X. If a closed path C starting and ending at point v gives rise to a conjugacy class $\{C\}$ in Γ, we may take C in its homotopy class so that C has no backtracking. We then remove the tail from C so as to get a tailless cycle C^*. The 1-1 correspondence referred to comes from the fact that the conjugacy class of C in Γ corresponds to the equivalence class of the backtrackless, tailless cycle C^*. It can be shown (**Exercise**) that the change of C in its conjugacy class corresponds to a change of C^* in its equivalence class. In the other direction of the correspondence, given C^*, we grow

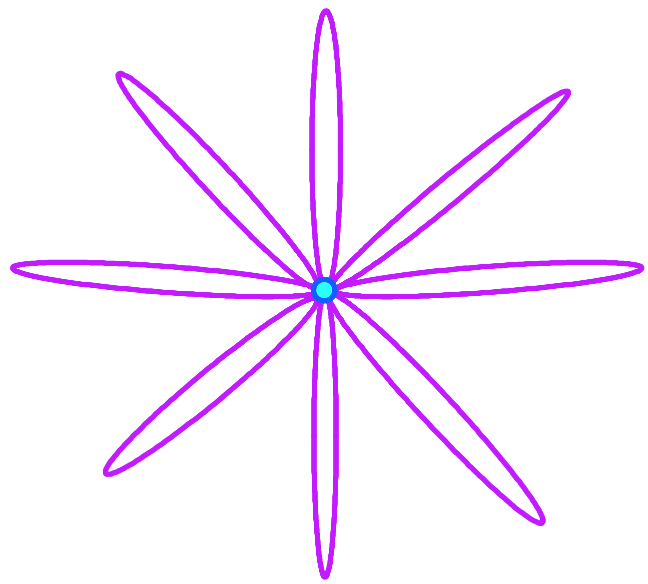

FIGURE 8. A bouquet of loops.

a tail so as to reach v, thus getting a path C which determines an element of Γ. A different tail simply conjugates C. Another way of thinking about the correspondence is that the elements of the equivalence class of C^* are precisely the closed cycles of minimal length which are freely homotopic to C. **Freely homotopic** means the base point v is not fixed.

The fundamental group Γ is a free group of rank r. Thus, for $\gamma \neq 1$ in Γ, the **centralizer** $C_\Gamma(\gamma) = \{\delta \in \Gamma | \gamma\delta = \delta\gamma\}$ is a cyclic subgroup of Γ. Under the 1-1 correspondence between classes $[C]$ of backtrackless, tailless cycles in Γ, prime cycles P correspond to conjugacy classes $\{P\}$ in Γ such that the centralizer $C_\Gamma(P)$ is generated by P. Such conjugacy classes $\{P\}$ are called **primitive.**

We remark here that we will not always be consistent in our notation. We will want to use capital Latin letters for paths in a graph. We will want to use small Greek letters for elements of the fundamental group (or a Galois group acting on a graph covering), but sometimes conflicts will arise and consistency will seem impossible.

Exercise 4. *Prove that the centralizer of $\gamma \neq 1$ in the fundamental group Γ is cyclic.*

Algebraic topology (see the references above) tells us that there is a "universal covering tree" T (meaning that it is without cycles and is a covering of the original graph X as in Definition 5) below. See Figure 9 for a picture of the 4-regular tree which is the universal cover of any 4-regular graph.

There is an action of the fundamental group Γ on T such that we can identify T/Γ with X. You can also view the tree in the $(p+1)$-regular case as coming from p-adic matrix groups. See Serre [79], Trimble [97], and the last chapter of Terras [92] as well as [41] and [96].

If V is the set of vertices of graph X and E the set of edges, then the **Euler characteristic** of X is $\chi(X) = |V| - |E| = 1 - r$, where r is the rank of the fundamental group Γ of X.

One moral of the preceding considerations is that we can rewrite the product in Definition 2 in the language of the fundamental group of X as

$$\zeta_X(u) = \prod_{\{C\}} \left(1 - u^{\nu(C)}\right)^{-1},$$

where the product is over primitive conjugacy classes in the fundamental group Γ of X. A primitive conjugacy class $\{C\}$ in Γ means that C generates the centralizer of C in Γ. Here $\nu(C)$ means the length of the element C^* in the equivalence class of tailless, backtrackless paths corresponding to $\{C\}$. Alternatively $\nu(C)$ is the minimal length of all cycles freely homotopic to C. As above, we distinguish between C and C^{-1} in this product.

Theorem 1. ***(Ihara theorem generalized by Bass, Hashimoto, etc.).*** *Let A be the adjacency matrix of X and Q the diagonal matrix with jth diagonal entry q_j such that $q_j + 1$ is the degree of the jth vertex of X. Suppose that r is the rank of the fundamental group of X; $r - 1 = |E| - |V|$. Then*

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1} \det\left(I - Au + Qu^2\right).$$

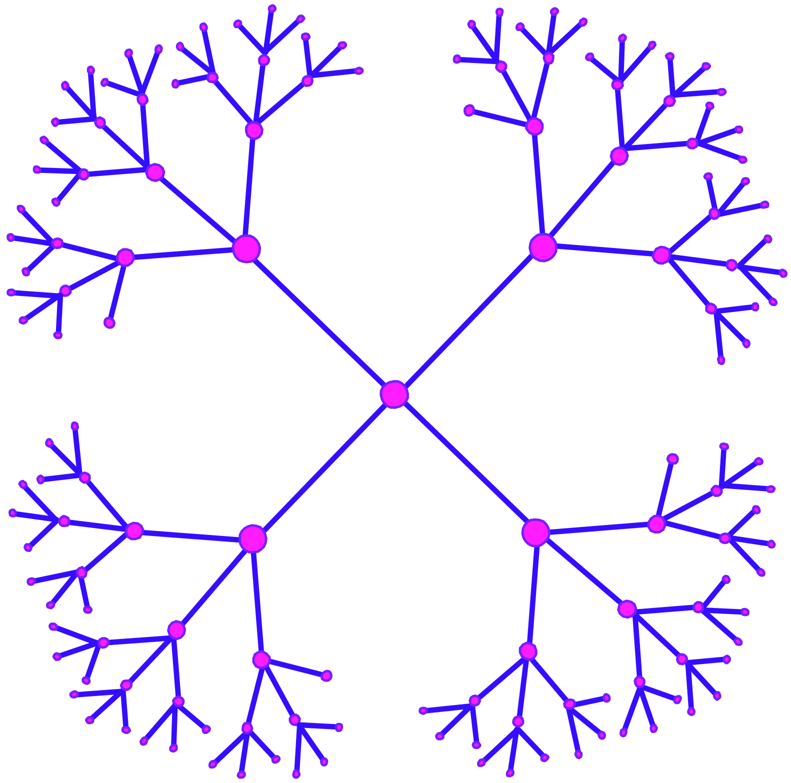

FIGURE 9. Part of the 4-regular tree.

Exercise 5. *Show that* $r - 1 = \frac{1}{2}Tr(Q - I)$.

There is an elementary proof of the preceding theorem using the method of Bass [9]. We will present it in Part 3. In [92] we presented another proof for k-regular graphs using the Selberg trace formula on the k-regular tree. For k-regular graphs, the Ihara zeta function has much in common with the Riemann zeta function.

Suppose X is a $q + 1$ regular graph. Then the Ihara zeta function has functional equations relating the value at u with the value at $1/(qu)$. Setting $u = q^{-s}$, one finds that the functional equation relates the value at s with that at $1 - s$, just as for Riemann's zeta function. See Proposition 3.

When the graph X is $(q + 1)$-regular, there is also an analogue of the Riemann hypothesis. It turns out to hold if and only if the graph is Ramanujan as defined by Lubotzky, Phillips and Sarnak in [56].

Definition 4. *A connected (q+1)-regular graph X is* ***Ramanujan,*** *iff, when*

$$\mu = \max\{|\lambda| \,|\lambda \in Spectrum(A), |\lambda| \neq q+1\}$$

then $\mu \leq 2\sqrt{q}$.

Some graphs are Ramanujan and some are not. In the 1980s, Margulies and independently Lubotzky, Phillips and Sarnak [56] found a construction of infinite families of Ramanujan graphs of fixed degree equal to $1 + p^e$, where p is a prime. They used the Ramanujan conjecture (now proved by Deligne) to show that the graphs were Ramanujan. Such graphs are of interest to computer scientists because they provide efficient communication as they have good expansion properties. See Guiliana Davidoff et al [25], Lubotzky [54], Sarnak [75], or Terras [92] for more information. See Joel Friedman's website (www.math.ubc.ca/~jf) for a paper proving that a random regular graph is almost Ramanujan. A survey on expander graphs and their applications is that of Hoory, Linial and Wigderson [39].

Example 2. *The* ***tetrahedron graph*** K_4 *in Figure 7 has Ihara zeta function*

$$\zeta_{K_4}(u)^{-1} = \left(1-u^2\right)^2 (1-u)(1-2u)\left(1+u+2u^2\right)^3.$$

The 5 poles of this zeta function are located at the points $-1, \frac{1}{2}, 1, \frac{-1\pm\sqrt{-7}}{4}$. *The absolute value of the complex pole is* $\frac{1}{\sqrt{2}} \cong$ 0.707 11 *The closest pole to the origin is* $\frac{1}{2} = \frac{1}{q} = R_{K_4}$.

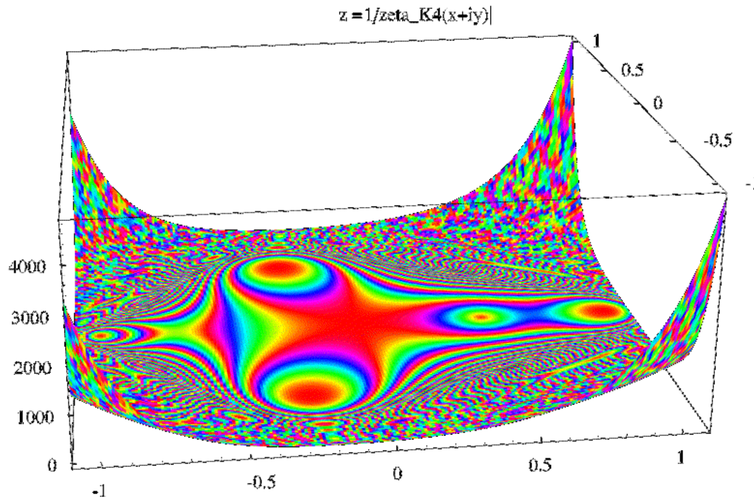

FIGURE 10. The graph is a contour map of the reciprocal of the Ihara zeta function $z = 1/|\zeta_{K_4}(x+iy)|$ drawn by Mathematica. You can see the 5 roots (not counting multiplicity).

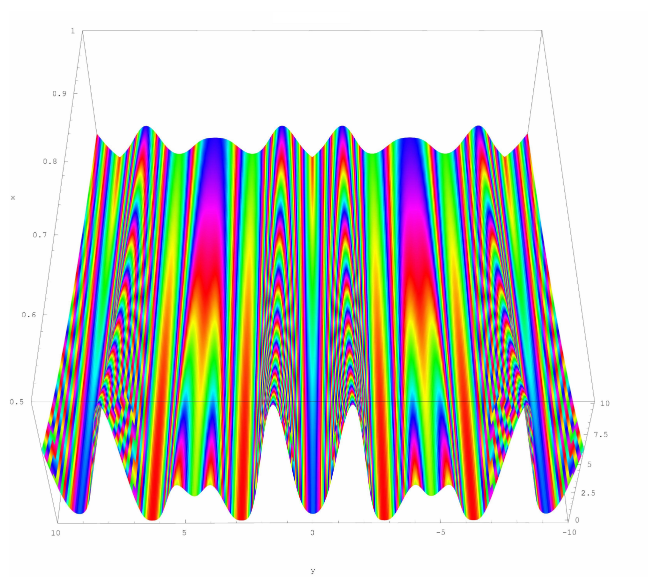

FIGURE 11. A contour map for $z = 1/|\zeta_{K_4}(2^{-(x+iy)})|$, drawn by Mathematica.

Of course, K_4 is a Ramanujan graph.

Next we illustrate what the Ihara zeta counts. The nth coefficient of the generating function below is the number of length n closed paths in X (without backtracking or tails). So there are 8 primes of length 3 in X, for example. See Lemma 2 in the section on the graph theory prime number theorem below.. We find using the Ihara theorem 1 and Mathematica, for example, that

$$u\frac{d}{du}\log\zeta_{K_4}(u) = 24u^3 + 24u^4 + 96u^6 + 168u^7 + 168u^8 + 528u^9 + 1200u^{10} + 1848u^{11} + O(u^{12}).$$

Figure 10 is a contour map of $z = \left|\zeta_{K_4}(x+iy)\right|^{-1}$, while Figure 11 is a contour map of $z = \left|\zeta_{K_4}(2^{-(x+iy)})\right|^{-1}$. The second graph is more like that for the Riemann zeta function which was Figure 2.

Example 3. *Let $X = K_4 - e$ be the graph obtained from K_4 by deleting an edge e. See Figure 12. Then*

$$\zeta_X(u)^{-1} = \left(1-u^2\right)(1-u)\left(1+u^2\right)\left(1+u+2u^2\right)\left(1-u^2-2u^3\right).$$

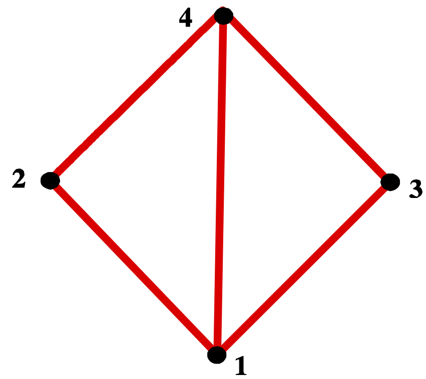

FIGURE 12. The graph obtained from the tetrahedron by deleting an edge.

There are 9 roots: $\pm 1, \pm i, \frac{-1\pm\sqrt{-7}}{4}$, *3 roots of the cubic,* s_1, s_2, s_3 *with* $|s_1| \cong 0.657\,30, |s_2| = |s_3| \cong 0.872\,18$. *So for this example,* $R_X \cong 0.657\,30$.

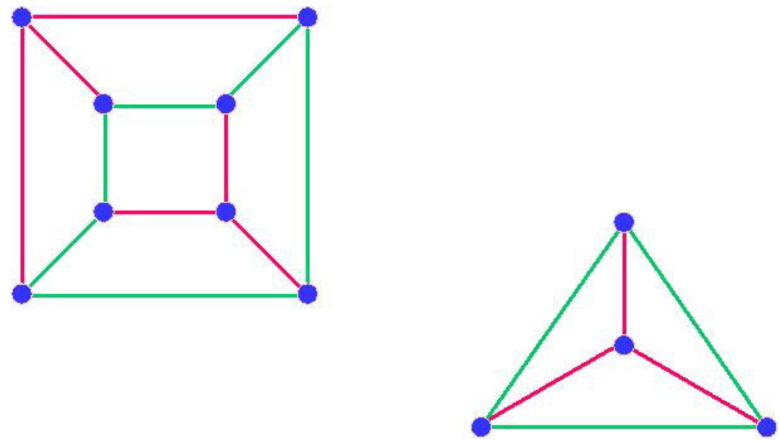

FIGURE 13. The cube is a quadratic covering of the tetrahedron. A spanning tree for the tetrahedron is indicated in red. Two copies of this tree are seen in the cube.

It is not obvious how to get the Ihara zeta function for the graph with an edge deleted from that for the original graph. We will have more to say about this topic when we have discussed the edge zeta functions.

Exercise 6. *Compute the Ihara zeta functions of your favorite graphs; e.g., the cube, the dodecahedron, the buckyball.*

Next we consider an unramified finite covering graph Y of our finite graph X. This is analogous to an extension of algebraic number fields. We assume that both X and Y are connected. A discussion of covering graphs can be found in Massey [59].

Definition 5. *If the graph has no multiple edges and loops we can say that the graph Y is an* ***unramified covering*** *of the graph X if we have a covering map $\pi : Y \to X$ which is an onto graph mapping (i.e., taking adjacent vertices to adjacent vertices) such that for every $x \in X$ and for every $y \in \pi^{-1}(x)$, the collection of points adjacent to $y \in Y$ is mapped 1-1 onto the collection of points adjacent to $x \in X$.*

The factorization of the Ihara zeta function of the quadratic covering in the example below is analogous to what happens for Dedekind zeta function of quadratic extensions of number fields. In a later section, we will show that the entire theory of Dedekind zeta functions (and Artin L-functions) has a graph theory analog.

Example 4. ***Unramified Quadratic Covering of K_4*** *Consider Figure 13. The cube Y is obtained by drawing two copies of a spanning tree (with red edges) for the tetrahedron $X = K_4$ and then drawing the rest of the edges of the cover to go between sheets of the cover. We find that $\zeta_Y(u)^{-1} = L(u, \rho, Y/X)^{-1}\zeta_X(u)^{-1}$, where*

$$L(u, \rho, Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$$

Exercise 7. *Draw the analogs of Figure 10 and 11 for the cube.*

Exercise 8. *Find a second quadratic cover Y' of the tetrahedron by drawing two copies of a spanning tree of $X = K_4$ and then connecting the rest of the edges of Y so that only two edges in Y' go between sheets of the cover. Compute the Ihara zeta function of Y'.*

One can use the Ihara zeta function to prove the graph prime number theorem. In order to state this result, we need some definitions.

Definition 6. *The* ***prime counting function*** *is*

$$\pi(n) = \#\{\text{primes } [P] \,|\, n = \nu(P) = \text{length of } P\}.$$

Definition 7. *The* ***greatest common divisor of the prime path lengths*** *is*

$$\Delta_X = g.c.d.\{\nu(P) \,|\, [P] \text{ prime of } X\}.$$

Theorem 2. ***The Graph Prime Number Theorem.*** *For a connected graph X if Δ_X divides m, then*

$$\pi(m) \sim \frac{\Delta_X}{m R_X^m}, \quad \text{as } m \to \infty.$$

If Δ_X does not divide m, then $\pi(m) = 0$.

We will see that the proof is much easier than that of the prime number theorem for prime integers. There are also analogs of Dirichlet's theorem on primes in progressions and the Chebotarev density theorem. We will consider this in a later section.

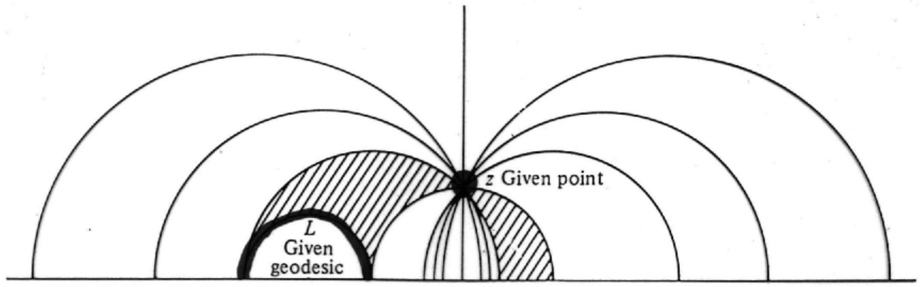

FIGURE 14. The failure of Euclid's 5th postulate is illustrated. All geodesics through z outside the shaded angle fail to meet L (from [93], Vol. I, p. 123).

3. SELBERG ZETA FUNCTION

Some references for this subject are Dennis Hejhal [36], Atle Selberg [77], [78], Audrey Terras [93], and Marie-France Vignéras [99] Another reference is the collection of articles edited by Tim Bedford, Michael Keane, and Caroline Series [10].

The Selberg zeta function is a generating function for "primes" in a compact (or finite volume) Riemannian manifold M. Before we define "prime," we need to think a bit about Riemannian geometry. Assuming M has constant curvature -1 it can be realized as a quotient of the **Poincaré upper half plane**

$$H = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\},$$

with **Poincaré arc length** element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

which can be shown invariant under fractional linear transformation

$$z \longrightarrow \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{R}, \quad ad - bc > 0.$$

The **Laplace operator** corresponding to the Poincaré arc length is

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

It too is invariant under fractional linear transformations.

It is not hard to see that curves minimizing the Poincaré arc length are half lines and semicircles orthogonal to the real axis. Calling these geodesics straight lines creates a model for non-Euclidean geometry since Euclid's 5th postulate fails. There are infinitely many geodesics through a fixed point not meeting a given geodesic. See Figure 14.

The fundamental group Γ of M acts as a discrete group of distance-preserving transformations.

The favorite group of number theorists is the **modular group** $\Gamma = SL(2, \mathbb{Z})$ of 2×2 matrices of determinant one and integer entries or the quotient $\overline{\Gamma} = \Gamma / \{\pm I\}$.

We can identify the quotient $SL(2,\mathbb{Z}) \backslash H$ with the fundamental domain D pictured below in Figure 15, where the sides are identified via $z \longrightarrow z + 1$ and $z \longrightarrow -1/z$. These transformations generate $\overline{\Gamma}$.

The images of D under elements of Γ provide a tessellation of the upper half plane. See Figure 16.

There are 4 types of elements of $\overline{\Gamma}$. They are determined by the Jordan form of the 2x2 matrix. The **corresponding fractional linear map will be one of 4 types:**

identity	$z \longrightarrow z$
elliptic	$z \longrightarrow cz,\quad$ where $\lvert c\rvert = 1, c \neq 1$
hyperbolic	$z \longrightarrow cz,\quad$ where $c > 0, c \neq 1$
parabolic	$z \longrightarrow z + a.$

The Riemann surface $M = \Gamma \backslash H$ is compact and without branch points if Γ has only the identity and hyperbolic elements. The modular group unfortunately has both elliptic and parabolic elements. It is easiest to deal with Selberg zeta functions when Γ has only the identity and hyperbolic elements. Examples of such groups are discussed in Svetlana Katok's book [47].

A hyperbolic element $\gamma \in \Gamma$ will have 2 fixed points on $\mathbb{R} \cup \{\infty\}$. Call these points z and w. Let $C(z, w)$ be a geodesic line or circle in H connecting points z and w in $\mathbb{R} \cup \{\infty\}$. Consider the image $\overline{C(z,w)}$ in the fundamental domain for $\Gamma \backslash H$. We say that $\overline{C(z,w)}$ is a **closed geodesic** if it is a closed curve in the fundamental domain (i.e., the beginning of the curve is the same

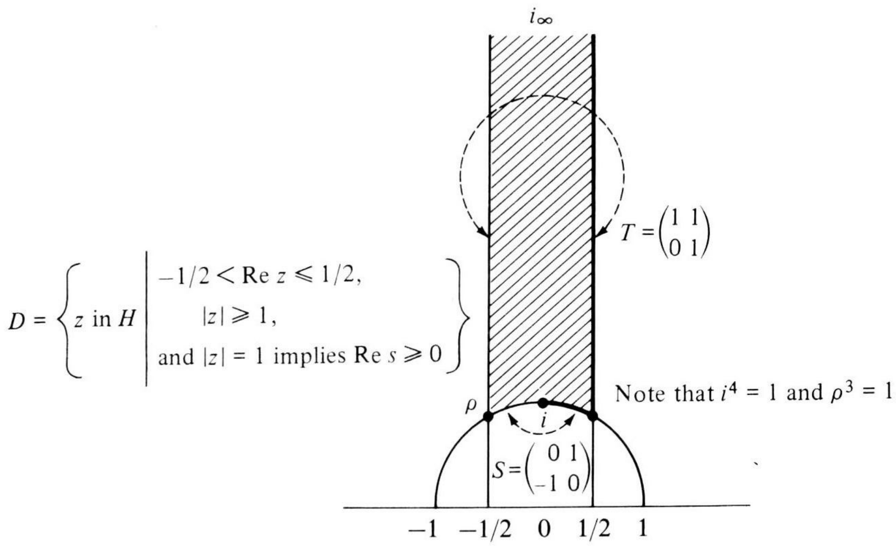

FIGURE 15. A noneuclidean triangle D through the points $\rho, \rho+1, i\infty$, which is a fundamental domain for H mod $SL(2,\mathbb{Z})$. The domain D is shaded. Arrows show boundary identifications by the fractional linear transformation from S and T which generate $SL(2,\mathbb{Z})/\{\pm I\}$, (from [93], p. 164).

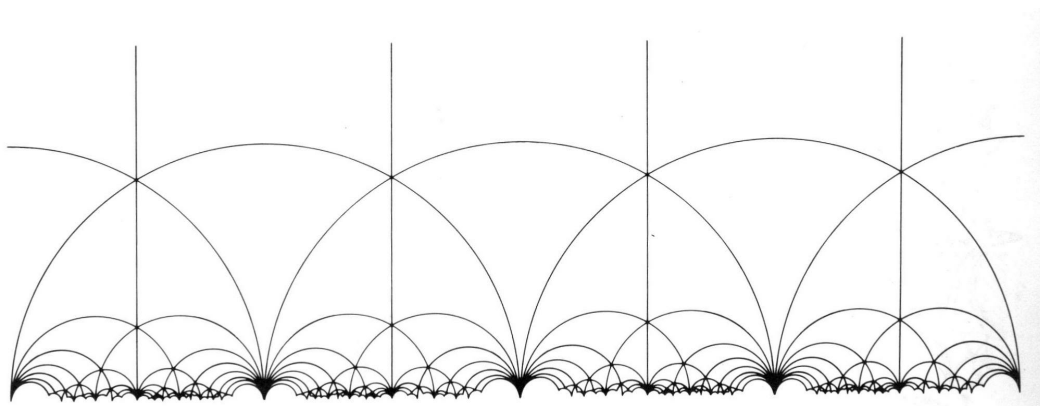

FIGURE 16. The tessellation of the upper half plane arising from applying the elements of the modular group to the fundamental domain D in the preceding figure. The figure is taken from [93], p. 166.

as the end). A **primitive** closed geodesic is traversed only once. One can show that $\overline{C(z,w)}$ is a closed geodesic in $\Gamma\backslash H$ iff there is an element $\gamma \in \Gamma$ such that $\gamma\, C(z,w) \subset C(z,w)$. This means that z and w are the fixed points of a hyperbolic element of Γ. One can show that if a point q lies on $C(z,w)$ then so does γq and the Poincaré distance between q and γq is $\log N\gamma$, where $N\gamma = a^2$, if γ has Jordan form $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$, with a real, $a \neq \pm 1$.

Exercise 9. *Using a computer, graph $\overline{C(z,w)}$ for various choices of z, w. We did this in Figure 17 below and then mapped everything into the unit disc using the* ***Cayley transform***

$$z \longrightarrow \frac{i(z-i)}{z+i}.$$

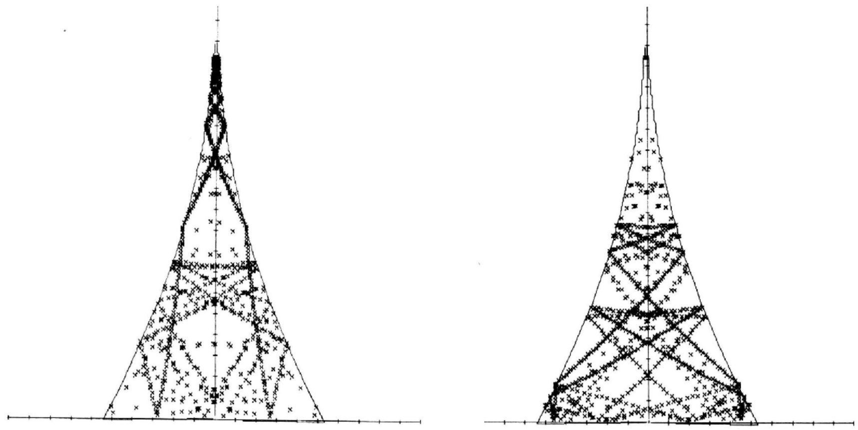

FIGURE 17. Images of points on two geodesic circles after mapping them into the fundamental domain of $SL(2,\mathbb{Z})$ and then by Cayley transform into the unit disc. Here the geodesic circles have center 0 and radii $\sqrt{163}$ on the left and e on the right (from [93], Vol.I, pp. 280-281).

Primes in M.

The lengths of these primitive closed geodesics coming from hyperbolic elements of Γ form the **length spectrum** of $M = \Gamma\backslash H$. These are our "primes" in M. E. Artin was one of the first people to consider the question of whether these geodesics tend to fill up the fundamental domain as the length approaches infinity. This is related to ergodic theory and dynamical systems. It is also related to continued fractions. See Bedford et al [10].

Now we are ready to define the **Selberg zeta function** as

$$Z(s) = \prod_{[C]} \prod_{j \geq 1} \left(1 - e^{-(s+j)\nu(C)}\right).$$

The product is over all primitive closed geodesics C in $M = \Gamma\backslash H$ of length $\nu(C)$. Just as with the Ihara zeta function, this product can also be viewed as a product over conjugacy classes of primitive hyperbolic elements $\{\gamma\} \subset \Gamma$. Here "primitive hyperbolic element" means it generates its centralizer in Γ. The Selberg trace formula gives an exact formula relating the spectrum of the Laplacian on M and the length spectrum of M.

The Selberg zeta function has many properties similar to the Riemann zeta function. Via the Selberg trace formula one shows that the logarithmic derivative of the Selberg zeta function has an analytic continuation as a meromorphic function. See Elstrodt [28], Patterson [68], and Bunke and Olbrich [16]. The non-trivial zeros of $Z(s)$ correspond to the discrete spectrum of the Poincaré Laplacian on $L^2(\Gamma\backslash H)$. This means that the Selberg zeta function satisfies the Riemann hypothesis (assuming that $\Gamma\backslash H$ is compact). Sarnak has said that for nonarithmetic Γ with noncompact fundamental domain he doubts one should think of $Z(s)$ as a zeta function as he conjectures that the discrete spectrum of Δ is finite. Of course this will be the case for the Ihara zeta function of a finite graph so perhaps some might not think it is a zeta at all. Here "arithmetic" Γ means that $\mathbb{Z}$ is lurking somewhere in the definition of Γ. Vignéras [99] proves many properties of the Selberg zeta function for the non-compact quotient $SL(2,\mathbb{Z})\backslash H$.

Exercise 10. *Show that $Z(s+1)/Z(s)$ has a product formula which looks more like that for the Ihara zeta function.*

The theory of Ihara zeta functions for $(p^e+1)-$regular qraphs, p=prime, can be understood via trace formulas for groups of 2x2 matrices over function fields. See Nagoshi [64]. One can also work out the theory of Ihara zetas on d-regular graphs based on trace formulas for groups acting on the d-regular tree. See Horton, Newland, and Terras [41], Terras [92], and Terras and Wallace [96].

4. RUELLE'S ZETA FUNCTION

Some references for this subject are Ruelle [74] as well as Bedford, Keane and Series [10].

Let M be a compact manifold. suppose $f : M \to M$. Assume the set

$$Fix(f^m) = \left\{ x \in M \mid f^m(x) = x \right\}$$

is finite. Suppose $\varphi : M \to \mathbb{C}^{d\times d}$ is a matrix valued function. The first type of **Ruelle zeta function** is defined by

$$\zeta(z) = \exp\left\{ \sum_{m\geq 1} \frac{z^m}{m} \sum_{x\in Fix(f^m)} Tr\left(\prod_{k=0}^{m-1} \varphi\left(f^k(x)\right)\right)\right\}.$$

Ruelle also defines a 2nd type of zeta function associated to a 1-parameter semigroup of maps $f^t : M \to M$. See the reference above for the details.

Now we consider a special case to see that the Ihara zeta function of a graph is a Ruelle zeta function. For this we consider subshifts of finite type. Let I be a finite non-empty set (our alphabet). For a graph X, let I be the set of directed edges of X. Define the transition matrix $t = (t_{ij})_{ij\in I}$ to be a matrix of 0's and 1's.

For the case of a graph let t denote the **edge matrix** W_1 defined below.

Definition 8. *For a graph X, define the* ***0,1 edge matrix*** *W_1 by orienting the m edges of X and labeling them as in formula (2.1). Then W_1 is the $2m \times 2m$ matrix with ij entry 1 if edge e_i feeds into e_j provided that $e_j \neq e_i^{-1}$, and ij entry 0 otherwise. By* **"***a* ***feeds into*** *b,***"** *we mean that the terminal vertex of edge a is the same as the initial vertex of edge b.*

We will show later that

$$\zeta_X(u)^{-1} = \det(I - W_1 u). \tag{4.1}$$

From this we shall derive Theorem 1.

Note that the product $I^{\mathbb{Z}}$ is compact and thus so is the closed subset Λ defined by

$$\Lambda = \left\{ (\xi_k)_{k\in\mathbb{Z}} \,\middle|\, t_{\xi_k\xi_{k+1}} = 1, \text{ for all } k\right\}.$$

In the graph case $\xi \in \Lambda$ corresponds to a path without backtracking.

A continuous function $\tau : \Lambda \to \Lambda$ such that $\tau(\xi)_k = \xi_{k+1}$ is called a **subshift of finite type**.

In the graph case this shifts the path left, assuming the paths go from left to right.

Proposition 1. ***(Bowen and Lanford)***

One has the following formula for the Ruelle zeta function

$$\zeta(z) = \exp\left(\sum_{m\geq 1} \frac{z^m}{m} |Fix(\tau^m)|\right) = (\det(1 - zt))^{-1}.$$

Later, we will consider an edge zeta function attached to X which involves more than one complex variable.

Proof. By the first exercise below, we have the first equality in the theorem. Then $|Fix(\tau^m)| = Tr(t^m)$ which implies

$$\begin{aligned} \zeta(z) &= \exp\left(\sum_{m\geq 1} \frac{z^m}{m} Tr(t^m)\right) \\ &= \exp(Tr(-\log(1 - zt))) = \det(1 - zt)^{-1}. \end{aligned}$$

To see the second equality, use the Taylor series for the matrix log. Here $|z|$ must be small. The last equality comes from the second exercise below. □

Exercise 11. *Show that in the graph case, $|Fix(\tau^m)|$ is the number of length m closed paths without backtracking or tails in the graph X with $t = W_1$, the edge matrix from Definition 8.*

Exercise 12. *Show that* $\exp Tr(A) = \det(\exp A)$, *for any matrix A. To prove this, you need to know that there is a non-singular matrix B such that $BAB^{-1} = T$ is upper triangular.*

Ruelle's motivation for his definition came partially from a paper by Artin and Mazur [4]. They were in turn motivated by the definition of the **zeta function of a projective nonsingular algebraic variety** of dimension n defined over a finite field k with q elements. If N_m is the number of points of V with coordinates in the degree m extension field of k, the zeta function of V is

$$Z(z, V) = exp\left(\sum_{m=1}^{\infty} N_m \frac{z^m}{m}\right).$$

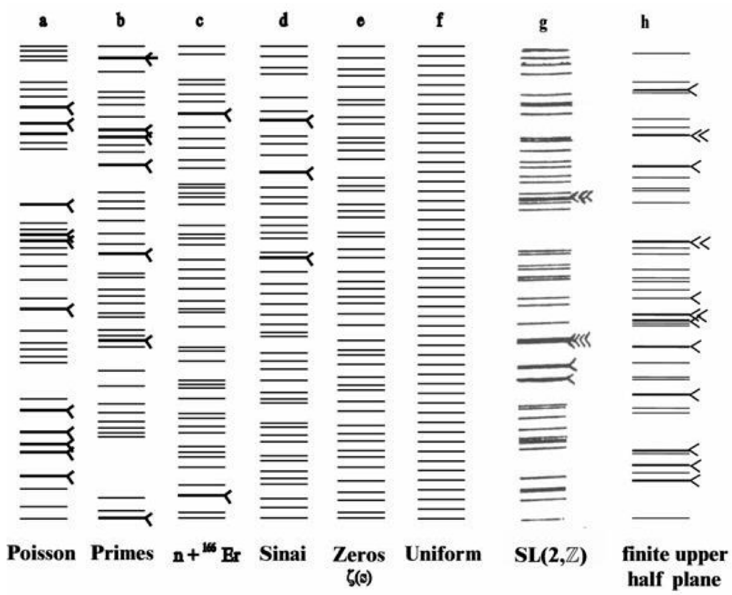

FIGURE 18. Columns a-f are from Bohigas and Giannoni [12] and column g is from Sarnak [76]. Segments of "spectra," each containing 50 levels. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than 1/4. The labels are explained in the text. Column h contains finite upper half plane graph eigenvalues (without multiplicity) for the prime 53, with $\delta = a = 2$.

Note that $N_m = |fix\,(F^m)|$, where F denotes the Frobenius morphism which takes a point of coordinates (x_i) to the point (x_i^q). The Weil conjectures, ultimately proved in the general case by Deligne, say

$$Z(z, V) = \prod_{j=0}^{2n} P_j(z)^{(-1)^{j+1}},$$

where the P_j are polynomials whose zeros have absolute value $q^{-j/2}$. Moreover the P_j have a cohomological meaning (roughly $P_j(z) = \det\left(1 - zF^* | H^j(V)\right)$. Then Artin and Mazur replace the Frobenius of the algebraic variety with a diffeomorphism f of a smooth compact manifold M and look at the zeta function

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} \left|Fix\left(f^m\right)\right|\right).$$

5. CHAOS

References for this subject include Cipra [21], Miller and Takloo-Bighash [61] (see in particular the downloadable papers from the book's website at Princeton University Press), Sarnak [76], Terras [94] and [95].

Quantum chaos is in part the study of the statistics of energy levels of quantum mechanical systems; i.e. the eigenvalues of the Schrödinger operator $\mathcal{L}\phi_n = \lambda_n \phi_n$. A good website for quantum chaos is that of Matthew W. Watkins:

www.maths.ex.ac.uk/~mwatkins.

We quote Oriol Bohigas and Marie-Joya Gionnoni [12], p. 14: "The question now is to discover the stochastic laws governing sequences having very different origins, as illustrated in ... [Figure 18]. There are displayed six spectra, each containing 50 levels ...." Note that the spectra have been rescaled to the same vertical axis from 0 to 49 and we have added 2 more columns to the original figure.

In Figure 18, column a represents a Poisson spectrum, meaning that of a random variable with spacings of probability density e^{-x}. Column b represents primes between 7791097 and 7791877. Column c represents the resonance energies of the compound nucleus observed in the reaction $n +^{166} Er$. Column d comes from eigenvalues corresponding to transverse vibrations of a membrane whose boundary is the Sinai billiard which is a square with a circular hole cut out centered at the center of the square. Then column e is from the positive imaginary part of zeros of the Riemann zeta function from the 1551th to the 1600th zero. Column f is equally spaced - the picket fence or uniform distribution. Column g comes from Sarnak [76] and corresponds to eigenvalues of the Poincaré Laplacian on the fundamental domain of the modular group $SL(2, \mathbb{Z})$ consisting

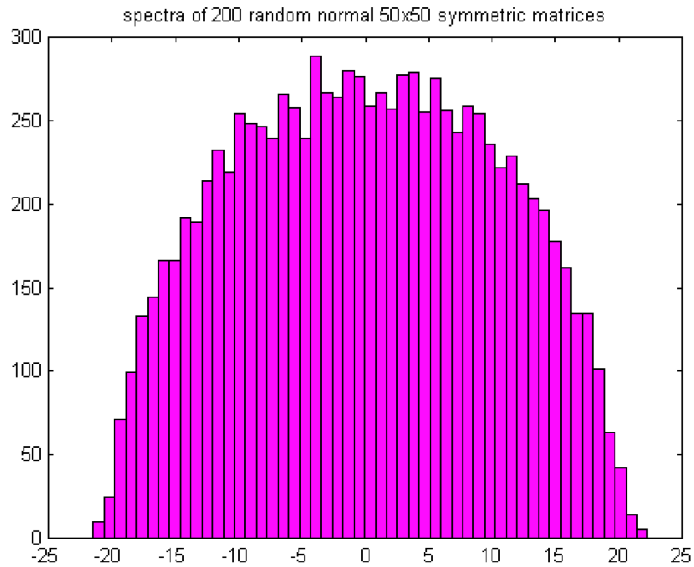

FIGURE 19. Histogram of the spectra of 200 random real 50x50 symmetric matrices created using Matlab.

of 2×2 integer matrices of determinant 1. From the point of view of randomness, columns g and h should be moved to lie next to column b. Column h is the spectrum of a finite upper half plane graph for p=53 ($a = \delta = 2$), without multiplicity. See Terras [92] for the definition of finite upper half plane graphs.

Exercise 13. *Produce your own versions of as many columns of Figure 18 as possible. Also make a column for level spacings of lengths of primitive closed geodesics in* $SL(2,\mathbb{Z})\backslash H$. *See pages 277-280 of Terras* [93], *Vol. I.*

Quantum mechanics says the energy levels E of a physical system are the eigenvalues of a Schrödinger equation $\mathcal{H}\phi = E\phi$, where $\mathcal{H}$ is the Hamiltonian (a differential operator), ϕ is the state function (eigenfunction of $\mathcal{H}$), and E is the energy level (eigenvalue of $\mathcal{H}$). For complicated systems, physicists decided that it would usually be impossible to know all the energy levels. So they investigate the statistical theory of these energy levels. This sort of thing happens in ordinary statistical mechanics as well. Of course symmetry groups (i.e., groups of motions commuting with $\mathcal{H}$) have a big effect on the energy levels.

In the 1950's Wigner (see [101]) considered modelling $\mathcal{H}$ with a large real symmetric $n \times n$ matrices whose entries are independent Gaussian random variables. He found that the histogram of the eigenvalues looks like a semi-circle (or, more precisely, a semi-ellipse). This has been named the **Wigner semi-circle distribution**. For example, he considered the eigenvalues of 197 "random" real symmetric 20x20 matrices. Figure 19 below shows the results of an analogue of Wigner's experiment using Matlab. We take 200 random (normally distributed) real symmetric 50x50 matrices with entries that are chosen according to the normal distribution. Wigner notes on p. 5: "What is distressing about this distribution is that it shows no similarity to the observed distribution in spectra." This may be the case in physics, but the semi-circle distribution is well known to number theorists as the Sato-Tate distribution.

Exercise 14. *Repeat the experiment that produced Figure 19 using uniformly distributed matrices rather than normally distributed ones. This is a problem best done with Matlab which has commands rand(50), giving a random* 50×50 *matrix with uniformly distributed entries, and randn(50) giving a normal random* 50×50 *matrix.*

So physicists have devoted more attention to histograms of level spacings rather than levels. This means that you arrange the energy levels (eigenvalues) E_i in decreasing order:

$$E_1 \geq E_2 \geq \cdots \geq E_n.$$

Assume that the eigenvalues are normalized so that the mean of the level spacings $|E_i - E_{i+1}|$ is 1. Then one can ask for the shape of the histogram of the normalized level spacings. There are (see Sarnak [76]) two main sorts of answers to this question: **Poisson**, meaning e^{-x}, and GOE (see Mehta [60]) which is more complicated to describe exactly but looks like $\frac{\pi}{2}xe^{-\frac{\pi x^2}{4}}$ (the **Wigner surmise**). Wigner (see [101]) conjectured in 1957 that the level spacing histogram for levels having the same values of all quantum numbers is given by $\frac{\pi}{2}xe^{-\frac{\pi x^2}{4}}$ if the mean spacing is 1. In 1960 Gaudin and Mehta found the correct distribution

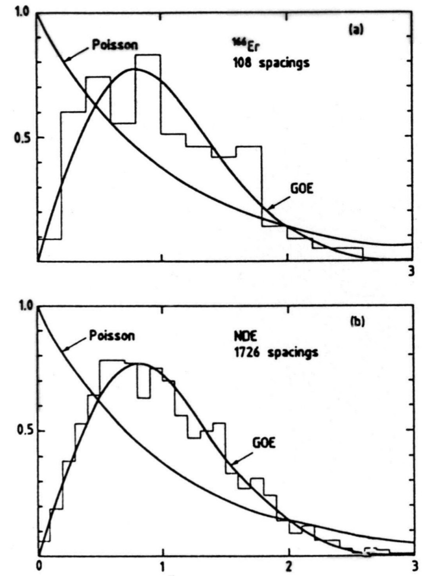

FIGURE 20. (from Bohigas, Haq, and Pandey [13]) Level spacing histogram for (a) ^{166}Er and (b) a nuclear data ensemble.

function which is surprisingly close to Wigner's conjecture but different. The correct graph is labeled GOE in Figure 20. Note the level repulsion indicated by the vanishing of the function at the origin. Also in Figure 20, we see the Poisson density which is e^{-x}.

You can find a Mathematica program to compute the GUE function at the website in [29]. Sarnak [76], p. 160 says: "It is now believed that for integrable systems the eigenvalues follow the Poisson behavior while for chaotic systems they follow the GOE distribution." Here GOE stands for **Gaussian Orthogonal Ensemble** - the eigenvalues of a random $n \times n$ symmetric real matrix as n goes to infinity. And GUE stands for the **Gaussian Unitary Ensemble** (the eigenvalues of a random $n \times n$ Hermitian matrix).

There are many experimental studies comparing GOE prediction and nuclear data. Work on atomic spectra and spectra of molecules also exists. In Figure 20, we reprint a figure of Bohigas, Haq, and Pandey [13] giving a comparison of histograms of level spacings for (a) ^{166}Er and (b) a nuclear data ensemble (or NDE) consisting of about 1700 energy levels corresponding to 36 sequences of 32 different nuclei. Bohigas et al say: "The criterion for inclusion in the NDE is that the individual sequences be in general agreement with GOE."

More references for quantum chaos are F. Haake [33] and Z. Rudnick [73].

Andrew Odlyzko (see www.dtc.umn.edu/~odlyzko/doc/zeta.html) has investigated the level spacing distribution for the non-trivial zeros of the Riemann zeta function. Assume the Riemann hypothesis and look at the zeros ordered by imaginary part

$$\left\{ \gamma_n \middle| \zeta(\frac{1}{2} + i\gamma_n) = 0,\ \gamma_n > 0 \right\}.$$

For the normalized level spacings, replace γ_n by $\widetilde{\gamma_n} = \frac{1}{2\pi}\gamma_n log\, \gamma_n$, since we want the mean spacing to be one. Here one needs to know that the number of γ_n such that $\gamma_n \leq T$ is asymptotic to $\frac{1}{2\pi} T \log T$ as $T \longrightarrow \infty$.

Historically the connections between the statistics of the Riemann zeta zeros γ_n and the statistics of the energy levels of quantum systems were made in a dialogue of Freeman Dyson and Hugh Montgomery over tea at the Institute for Advanced Study, Princeton. Odlyzko's experimental results show that the level spacings $|\gamma_n - \gamma_{n+1}|$, for large n, look like that of the Gaussian unitary ensemble (GUE); i.e., the eigenvalue distribution of a random complex Hermitian matrix. See Figure 21.

The level spacing distribution for the eigenvalues of Gaussian unitary matrices is not a standard function in Matlab, Maple or Mathematica. Sarnak [76] and Katz and Sarnak [49] proceed as follows. Let $K_s : L^2[0,1] \longrightarrow L^2[0,1]$ be the integral operator with kernel defined by

$$h_s(x,y) = \frac{sin\left(\frac{\pi s(x-y)}{2}\right)}{\frac{\pi(x-y)}{2}}, \text{ for } \ s \geq 0.$$

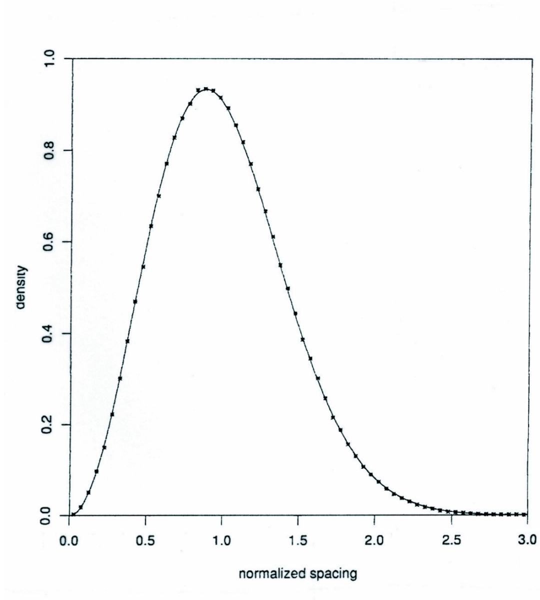

FIGURE 21. From Cipra [21]. Odlyzko's comparison of level spacing of zeros of the Riemann zeta function and that for GUE (Gaussian unitary ensemble). See Odlyzko and Forrester [29]. The fit is good for the 1,041,000 zeros near the 2×10^{20} zero.

Approximations to this kernel have been investigated in connection with the uncertainty principle (see Terras [93], Vol. I, p. 51). The eigenfunctions are spheroidal wave functions. Let $E(s)$ be the Fredholm determinant $det(I - K_s)$ and let $p(s) = E''(s)$. Then $p(s) \geq 0$ and $\int_0^\infty p(s)ds = 1$. The Gaudin-Mehta distribution ν is defined in the GUE case by $\nu(I) = \int_I p(x)ds$. For the GOE case the kernel h_s is replaced by $\{h_s(x,y) + h_s(-x,y)\}/2$. See also Mehta [60].

Katz and Sarnak [48], [49] have investigated many zeta and L-functions of number theory and have found that "the distribution of the high zeroes of any L-function follow the universal GUE Laws, while the distribution of the low-lying zeros of certain families follow the laws dictated by symmetries associated with the family. The function field analogues of these phenomena can be established...." More precisely, they show that "the zeta functions of almost all curves C [over a finite field $\mathbb{F}_q$] satisfy the Montgomery-Odlyzko law [GUE] as q and g [the genus] go to infinity." However, not a single example of a curve with GUE zeta zeros has ever been found.

These statistical phenomena are as yet unproved for most of the zeta functions of number theory; e.g., Riemann's. But the experimental evidence of Rubinstein [72] and Strömbergsson [88] and others is strong. Figures 3 and 4 of Katz and Sarnak show the level spacings for the zeros of the L-function corresponding to the modular form Δ and the L-function corresponding to a certain elliptic curve. Strömbergsson's web site has similar pictures for L-functions corresponding to Maass wave forms (http://www.math.uu.se/~andreas/zeros.html). All these pictures look GUE.

Derek Newland [65] has investigated the spacings of the zeros of the Ihara zeta function for various kinds of graphs. He finds that the level spacing for random k-regular graphs appears to be GOE while that of the Euclidean graphs in Terras [92] appear to be Poisson. This mimics the experimental work on level spacings of eigenvalues of the non-Euclidean Laplacian for discrete groups Γ acting on the upper half plane. For arithmetic Γ the distribution looks Poisson and for non-arithmetic Γ the distribution looks GOE. This led Sarnak to coin the phrase "arithmetical quantum chaos." See Terras [95] for more information. Figure 22 shows the result of one such experiment for a random regular graph as given by Mathematica. See Skiena [80] for more information on the way Mathematica deals with graphs. See also the article of Jakobson, Miller, Rivin, and Rudnick in [37], pp. 317-327.

Note: One expects the zeta function not to be a product of other zetas for the level spacing to look GOE or GUE. See a paper of D. W. Farmer on the ArXiv, Nov., 2005.

Exercise 15. *Compute the change of variables between an element of the spectrum of the adjacency matrix and the imaginary part of s when* q^{-s} *is a pole of* ζ_X *for a* $(q+1)$*-regular graph X.*

Exercise 16. *Do a similar experiment to that of Figure 22 for a Cayley graph of your choice or for an n-cover of* $X = K_4 - edge$.

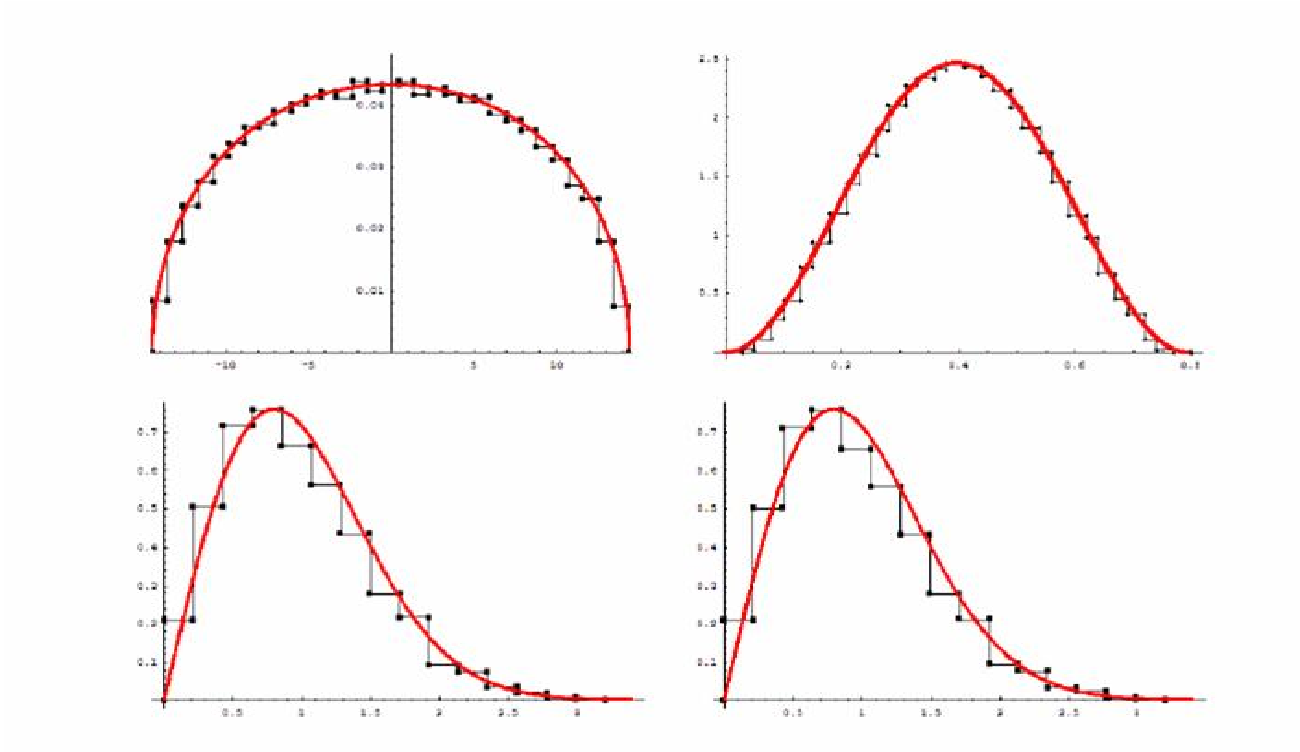

FIGURE 22. Taken from Newland [65]. For a pseudo-random regular graph with degree 53 and 2000 vertices, generated by Mathematica, the top row shows the distributions for eigenvalues of adjacency matrix on left and imaginary parts of the Ihara zeta poles on right. The bottom row contains their respective level spacings. The red line on bottom left is the Wigner surmise for the GOE $y = (\frac{\pi x}{2})e^{-\frac{\pi x^2}{4}}$.

As a final project for Part I, you might try to list all the zeta functions that you can find and figure out what they are good for. There are lots of them. We have left out zeta functions of codes for example.

Part 2. The Vertex Ihara Zeta Function and the Graph Theory Prime Number Theorem

The graph theory zetas first appeared in work of Ihara on p-adic groups in the 1960s (see [43]). Serre (see [79]) made the connection with graph theory. The main authors on the subject in the 1980s and 90s were Sunada [90], [91], Hashimoto [34], [35], and Bass [9]. Other references are Venkov and Nikitin [98] and Northshield's paper in the volume of Hejhal et al [37]. The main properties of the Riemann zeta function have graph theory analogs, at least for regular graphs. For irregular graphs there is no known functional equation and it is difficult to formulate the Riemann hypothesis, but we will try. In later sections, we will consider the multivariable zeta functions known as edge and path zeta functions of graphs. We will show how to specialize the path zeta to the edge zeta and the edge zeta to the original Ihara (vertex) zeta.

Much of our discussion can be found in the papers of the author and Harold Stark [83], [84], [85]. Topics for this section will include the graph theory prime number theorem and our version of Bass's proof of the determinant formula (which was Theorem 1 in the previous section). We will see how to modify the definitions to obtain zeta functions of weighted or metric graphs. We will find out what happens to the edge zeta if you delete an edge (fission) and what happens to the path zeta if you collapse an edge to a point (fusion or contraction). We will cover the Artin L-functions of Galois graph coverings from [84] in later sections.

We do not consider zeta functions of infinite graphs here. Such zeta functions are discussed, for example, by Clair and Mokhtari-Sharghi [22] as well as Grigorchuk and Zuk [31].

Throughout this section we will assume Theorem 1 of Ihara. It will be proved in the next section.

6. The Ihara Zeta Function of a Weighted Graph

Many applications involve weighted or metric graphs; that is, graphs with positive real numbers attached to the edges to provide lengths or resistance or some other physical attribute. In particular, quantum graphs are weighted (see [52]). Other references for weighted graphs are Fan Chung and S.T. Yau [20] or Osborne and Severini [66]. For the most part we will not consider weighted graphs here but let us at least give a natural extension of the definition of the Ihara zeta function to weighted graphs.

Definition 9. *For a graph X with oriented edge set $\overrightarrow{E}$, consisting of $2m$ oriented edges, suppose we have a weighting function $L:\overrightarrow{E}\to\mathbb{R}^{+}$. Then define the* ***weighted length*** *of a closed path $C=a_1a_2\cdots a_s$, where $a_j\in\overrightarrow{E}$, by*

$$\nu(C,L)=\nu_X(C,L)=\sum_{i=1}^{s}L(a_i).$$

Here we usually assume $L(e)=L(e^{-1})$.

Definition 10. *The* ***Ihara zeta function of a weighted (undirected) graph*** *for $|u|$ small and $u\notin(-\infty,0)$ is*

$$\zeta_X(u,L)=\prod_{[P]}\left(1-u^{\nu(P,L)}\right)^{-1}.$$

Clearly when $L=1$, meaning the function such that $L(e)=1$ for all edges e in X, we have $\zeta_X(u,1)=\zeta_X(u)$, our original Ihara zeta function. Then one has the natural question.

Is there an Ihara theorem for the weighted zeta function?

Standardly the adjacency matrix of a weighted graph is defined as follows.

Definition 11. *The* ***adjacency matrix*** *A_L, for the weighted graph with n vertices and length function L, is given by an $n\times n$ matrix with entry $(A_L)_{x,y}=L(e)$, if edge e joins vertex x to vertex y.*

We can find an Ihara determinant formula for positive integer weights by inflating the graph, but this has the drawback of changing the adjacency matrix by changing the number of vertices, among other things. However, it is a natural thing to do from the point of view of zeta functions.

Definition 12. *Given a graph X with positive integer-valued weight function L, define the* ***inflated graph*** *X_L in which each edge e is replaced by an edge with $L(e)-1$ new degree 2 vertices.*

Then clearly $\nu_X(C,L)=\nu_{X_L}(C,1)$,where the 1 means again that $1(e)=1$, for all edges e. It follows that **for positive integer-valued weights L, we have the identity relating the weighted zeta and the ordinary Ihara zeta**:

$$\zeta_X(u,L)=\zeta_{X_L}(u).$$

An Ihara determinant formula therefore holds for this integer weighted graph zeta but involves a different and larger adjacency matrix than A_L. It follows that $\zeta_X(u, L)^{-1}$ is a polynomial for integer valued weights L. For non-integer weights, we get a determinant formula like that of formula (4.1) for the general weighted Ihara zeta function in the next section.

Example. Suppose $Y = K_5$, the complete graph on 5 vertices. Let $L(e) = 5$ for each of the 10 edges of X. Then $X = Y_L$ is the graph on the left in Figure 24. The new graph X has 45 vertices (4 new vertices on the 10 edges of K_5). One sees easily that

$$\zeta_X(u)^{-1} = \zeta_{K_5}(u^5)^{-1} = (1-u^{10})^5(1-3u^5)(1-u^5)(1+u^5+3u^{10}).$$

Exercise 17. *What happens to* $\zeta_Y(u)$ *if* $Y = X_{L_n}$, *for* $L_n = n$, *as* $n \to \infty$?

What if you add edges instead of vertices? Thus, for the preceding example, instead of adding vertices, we would put 4 new edges between each pair of vertices. This would seem to give the correct adjacency matrix A_L, using Definition 1, but would change the zeta function by adding many new paths. The fundamental group would become much larger. Thus we have quite a challenge in attempting to produce an Ihara theorem for the weighted graphs. We can do such things more easily with the edge zeta function to be discussed later.

For the most part, therefore we shall restrict our discussion to non-weighted graphs from now on.

7. Regular Graphs, Location of Poles, Functional Equations

Next we want to consider the Ihara zeta function for regular graphs. We need some facts from graph theory first. References for the subject include Biggs [11], Bollobas [14], [15], Cvetković, Doob and Sachs [23].

Definition 13. *A graph is a* ***bipartite graph*** *iff the set of vertices can be partitioned into 2 disjoint sets* S, T *such that no vertex in* S *is adjacent to any other vertex in* S *and no vertex in* T *is adjacent to any other vertex in* T.

Exercise 18. *Show that an example of a bipartite graph is the cube of Figure 13.*

Proposition 2. ***Facts about Spectrum(A), when A is the adjacency operator of a connected (q+1)-regular graph X.*** *Assume that X is a connected (q+1)-regular graph and that A is its adjacency matrix.*

1) $\lambda \in \mathrm{Spectrum}(A)$ *implies* $|\lambda| \leq q+1$.

2) $q+1 \in \mathrm{Spectrum}(A)$ *and it has multiplicity 1.*

3) $-(q+1) \in \mathrm{Spectrum}(A)$ *iff the graph X is bipartite.*

To prove fact 1), note that $(q+1)$ is clearly an eigenvalue of A corresponding to the constant vector. Suppose $Av = \lambda v$, for some vector $v = {}^t(v_1, ..., v_n) \in \mathbb{R}^n$. And suppose that the maximum of the $|v_i|$ occurs at $i = a$. Then, using the notation $b \sim a$, to mean the bth vertex is adjacent to the ath, we have

$$|\lambda|\,|v_a| = \left|(Av)_a\right| = \left|\sum_{b\sim a} v_b\right| \leq (q+1)\,|v_a|\,.$$

Fact 1) follows.

To prove fact 2), suppose $Av = (q+1)v$, for some non-0 vector $v = {}^t(v_1, ..., v_n) \in \mathbb{R}^n$. Again suppose that the maximum of the $|v_i|$ occurs at $i = a$. We can assume $v_a > 0$, by multiplication of the vector v by -1. As in the proof of fact 1,

$$(q+1)v_a = (Av)_a = \sum_{b\sim a} v_b \leq (q+1)v_a.$$

To have equality, there can be no cancellation in this sum and $v_b = v_a$, for each b adjacent to a. If X is connected, by iterating this argument, v must be the constant vector.

Exercise 19. *a) Prove fact 3) above.*

b) Show that, if $(q+1)$ *has multiplicity 1 as an eigenvalue of the adjacency matrix of a* $(q+1)$*-regular graph, then this graph must be connected.*

Definition 14. *Suppose that X is a connected* $(q+1)$*-regular graph (without degree 1 vertices). We say that the Ihara zeta function* $\zeta_X(q^{-s})$ *satisfies the* ***Riemann hypothesis*** *iff when* $0 < \mathrm{Re}\, s < 1$,

$$\zeta_X(q^{-s})^{-1} = 0 \implies \mathrm{Re}\, s = \frac{1}{2}.$$

Note that if $u = q^{-s}$, $\mathrm{Re}\, s = \frac{1}{2}$ corresponds to $|u| = \frac{1}{\sqrt{q}}$.

Theorem 3. *For a connected* $(q+1)$*-regular graph X,* $\zeta_X(u)$ *satisfies the Riemann hypothesis iff the graph X is Ramanujan in the sense of Definition 4.*

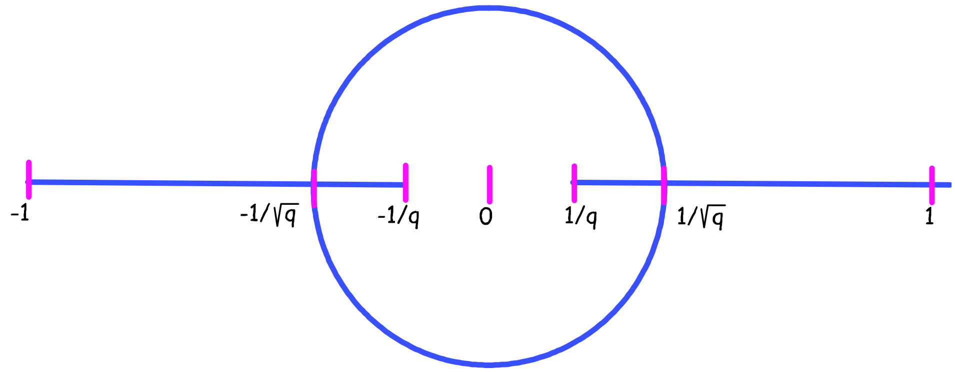

FIGURE 23. Poles of $\zeta_X(u)$ for a regular graph. The circle corresponds to the part of the spectrum of the adjacency matrix satisfying the Ramanujan inequality. The real poles correspond to the non-Ramanujan eigenvalues of A, except the two poles on the circle itself.

Proof. Use theorem 1 to see that

$$\zeta_X(q^{-s})^{-1} = (1-u^2)^{r-1} \prod_{\lambda \in spectrum(A)} \left(1 - \lambda u + qu^2\right).$$

Write $1 - \lambda u + qu^2 = (1-\alpha u)(1-\beta u)$, where $\alpha\beta = q$ and $\alpha + \beta = \lambda$. Note that α, β are the reciprocals of poles of $\zeta_X(u)$. Using the facts above, we have 3 cases.

Case 1. $\lambda = \pm(q+1)$ implies $\alpha = \pm q$ and $\beta = \pm 1$.

Case 2. $|\lambda| \leq 2\sqrt{q}$ implies $|\alpha| = |\beta| = \sqrt{q}$ and $\alpha, \beta \notin \mathbb{R} - \{\pm\sqrt{q}\}$.

Case 3. $2\sqrt{q} < |\lambda| < q+1$ implies $\alpha, \beta \in \mathbb{R}$ and $1 < |\alpha| = |\beta| < \sqrt{q}$.

To see these things, let u be either α^{-1} or β^{-1}. Then by the quadratic formula, we have α or $\beta = u^{-1}$ where

$$u = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}.$$

Cases 1 and 2 are easily seen. To understand case 3, first assume $\lambda > 0$ and note that $u = \frac{\lambda + \sqrt{\lambda^2-4q}}{2q}$ is a monotone increasing function of λ. This implies that u is in the interval $\left(\frac{1}{\sqrt{q}}, 1\right)$. A similar argument works for negative λ (**Exercise**). Where is the smaller root $u' = \frac{\lambda - \sqrt{\lambda^2-4q}}{2q}$? Answer: $|u'| \in \left(\frac{1}{q}, \frac{1}{\sqrt{q}}\right)$.

Exercise. Prove this using the fact that $uu' = \frac{1}{q}$.

The theorem is proved by noting that when $u = q^{-s}$, case 2 is $\mathrm{Re}\, s = \frac{1}{2}$. □

Exercise 20. *Show that the radius of convergence of the product defining the Ihara zeta function of a (q+1)-regular graph is* $R_X = \frac{1}{q}$.

Figure 23 shows the possible locations of poles of the Ihara zeta function of a (q+1)-regular graph. The poles satisfying the Riemann hypothesis are those on the circle. The circle basically corresponds to case 2 in the preceding proof. The real axis corresponds to Cases 1 and 3.

The following proposition gives some functional equations of the Ihara zeta function for a regular graph. If we set $u = q^{-s}$, the functional equations relate the value at s with that at $1-s$, just as is the case for the Riemann zeta function.

Proposition 3. *Suppose that X is a $(q+1)$-regular connected graph without degree 1 vertices with $n = |V|$. Then we have the following* ***functional equations*** *among others.*

1) $\Lambda_X(u) = (1-u^2)^{r-1+\frac{n}{2}} \left(1 - q^2u^2\right)^{\frac{n}{2}} \zeta_X(u) = (-1)^n \Lambda_X(\frac{1}{qu})$.

2) $\xi_X(u) = (1+u)^{r-1}(1-u)^{r-1+n}(1-qu)^n \zeta_X(u) = \xi_X\left(\frac{1}{qu}\right)$.

3) $\Xi_X(u) = (1-u^2)^{r-1}(1+qu)^n \zeta_X(u) = \Xi_X\left(\frac{1}{qu}\right)$.

Proof. We will prove part 1) and leave the rest as an **Exercise**. To see part 1), write

$$\begin{aligned}\Lambda_X(u) &= (1-u^2)^{\frac{n}{2}}\left(1-q^2u^2\right)^{\frac{n}{2}}\det\left(I-Au+qu^2I\right)^{-1}\\ &= \left(\frac{q^2}{q^2u^2}-1\right)^{\frac{n}{2}}\left(\frac{1}{q^2u^2}-1\right)^{\frac{n}{2}}\det\left(I-A\frac{1}{qu}+\frac{q}{(qu)^2}I\right)^{-1}\\ &= (-1)^n\Lambda_X(\frac{1}{qu}).\end{aligned}$$

□

Exercise 21. *Prove parts 2 and 3 of Proposition 3.*

Look at figure 23, and ask what sort of symmetry is indicated by the functional equations which imply that if $\zeta_X(u)$ has a pole at u, then it must also have a pole at $\frac{1}{qu}$. If u is on the circle then $\frac{1}{qu}$ is the complex conjugate of u. If u is in the interval $\left(\frac{1}{\sqrt{q}},1\right)$, then $\frac{1}{qu}$ is in the interval $\left(\frac{1}{q},\frac{1}{\sqrt{q}}\right)$.

For examples of regular graphs the easiest method is to start with a generating set S of your favorite finite group G. Assume that S is symmetric, meaning that $s\in S$ implies $s^{-1}\in S$. Create a graph called a **Cayley graph** denoted $X(G,S)$ whose vertices are the elements of G and whose edges are beween vertex g and gs for all $g\in G$ and $s\in S$. The degree of $X(G,S)$ is $|S|$. The cube is $X(\mathbb{F}_2^3,S)$, where $\mathbb{F}_2$ denotes the field with 2 elements, $\mathbb{F}_2^3$ is the additive group of 3-vectors with entries in this field, and $S=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Another example is the Paley graph considered in subsection 9.2.

We considered a large number of such graphs in Terras [92]. One example is $X(\mathbb{F}_q^n,S)$, where S consists of solutions $x\in\mathbb{F}_q^n$ of the equation $x_1^2+\cdots+x_n^2=a$ for some $a\in\mathbb{F}_q$. We called such graphs "Euclidean." There are also "non-Euclidean" graphs associated to finite fields where the distance is replaced by a finite analog of the Poincaré distance in the upper half plane. The question of whether these Euclidean and non-Euclidean graphs are Ramanujan can be translated into a question about bounds on exponential sums.

More examples of regular graphs come from Lubotzky, Phillips and Sarnak. See [25].

Mathematica will create "random" regular graphs with the command X=RegularGraph[d,n]; where d=degree and n=number of vertices.

Exercise 22. *Consider examples of regular graphs such as those mentioned above and find out whether they are Ramanujan graphs. Then plot the poles of the Ihara zeta function. You might also look at the level spacings of the poles as in Figure 22.*

Exercise 23. *Show if X is a non-bipartite k-regular graph with* $k\geq 3$*,then the g.c.d.* Δ_X *of the prime lengths from Definition 7 is 1.*

8. Irregular Graphs: What is the RH?

Next let us speak about irregular graphs. Kotani and Sunada [51] show the following theorem.

Theorem 4. *1) Every pole u of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$, with R_X from Definition 3, and*

$$q^{-1} \leq R_X \leq p^{-1}. \tag{8.1}$$

2) For a graph X, if $q+1$ is the maximum degree of X and $p+1$ is the minimum degree of X, then every non-real pole u of $\zeta_X(u)$ satisfies the inequality

$$q^{-1/2} \leq |u| \leq p^{-1/2}. \tag{8.2}$$

3) The poles of ζ_X on the circle $|u| = R_X$ have the form $R_X e^{2\pi i a/\Delta_X}$, where $a = 1, ..., \Delta_X$. Here Δ_X is from Definition 7.

Proof. We postpone this proof until the section on the edge zeta function. You can find it in Kotani and Sunada [51] if you are desperate. □

Horton gives examples of graphs such that R_X is as close as you want to a given positive real number such as π or e.

Now let us define two constants associated to the graph X.

Definition 15.

$$\begin{aligned} \rho_X &= \max \{|\lambda| \mid \lambda \in \text{spectrum}(A_X)\}, \\ \rho'_X &= \max \left\{|\lambda| \;\middle|\; \lambda \in \text{spectrum}(A_X),\ |\lambda| \neq \rho_X \right\}. \end{aligned}$$

We will say that the **naive Ramanujan inequality** is

$$\rho'_X \leq 2\sqrt{\rho_X - 1}. \tag{8.3}$$

Lubotzky [55] has defined X to be **Ramanujan** if

$$\rho'_X \leq \sigma_X. \tag{8.4}$$

where σ_X is the **spectral radius of the adjacency operator on the universal covering tree** of X. Both inequalities (8.3) and (8.4) reduce to the usual definition of Ramanujan for connected regular graphs.

Definition 16. *$\overline{d_X}$ denotes the **average degree of the vertices** of X.*

Hoory [38] has proved the following theorem.

Theorem 5.

$$2\sqrt{\overline{d_X} - 1} \leq \sigma_X.$$

Proof. For the special case that the graph is regular, the proof will be given below when we prove the result of Alon and Boppana which is Theorem 8. For the irregular case, the reader is referred to Hoory [38]. □

Thus one has a criterion for a graph X to be Ramanujan in Lubotzky's sense. It need only satisfy the **Hoory inequality**

$$\rho'_X \leq 2\sqrt{\overline{d_X} - 1}. \tag{8.5}$$

To develop the RH for irregular graphs, the natural change of variable is $u = R_X^s$ with R_X from Definition 3. All poles of $\zeta_X(u)$ are then located in the "critical strip", $0 \leq \text{Re}(s) \leq 1$ with poles at $s = 0$ $(u = 1)$ and $s = 1$ $(u = R_X)$. From this point of view, it is natural to say that the Riemann hypothesis for X should require that $\zeta_X(u)$ has no poles in the open strip $1/2 < \text{Re}(s) < 1$. This is the graph theory RH below. After looking at examples, it seems that one rarely sees an Ihara zeta satisfying this RH so we weaken it to the weak graph theory RH below. The examples below show that in general one cannot expect a functional equation relating $f(s) = \zeta(R_X^s, X)$ and $f(1-s)$.

Graph theory RH $\zeta_X(u)$ is pole free for

$$R_X < |u| < \sqrt{R_X}, \tag{8.6}$$

Weak graph theory RH $\zeta_X(u)$ is pole free for

$$R_X < |u| < 1/\sqrt{q}. \tag{8.7}$$

Note that (8.6) and (8.7) are the same if the graph is regular. We have examples (such as Example 1 below) for which $R_X > q^{-\frac{1}{2}}$ and in such cases the weak graph theory RH is true but vacuous. In [85] we give a longer discussion of the preceding 2 versions of the RH for graphs showing the connections with the versions for the Dedekind zeta function and the existence of Siegel zeros. We will define Siegel zeros for the Ihara zeta function later.

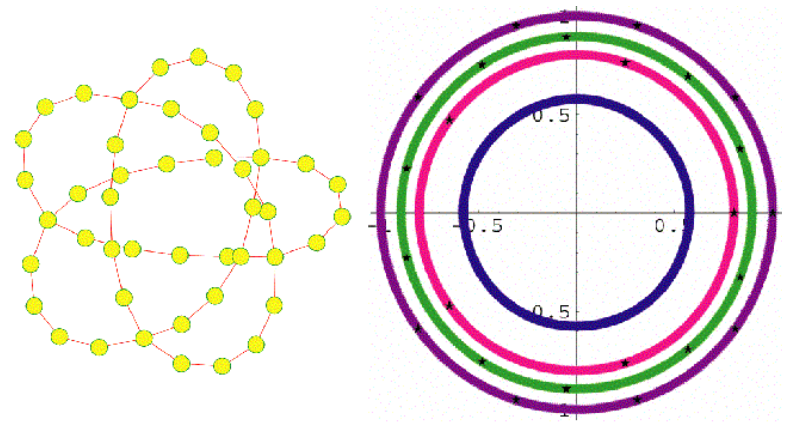

FIGURE 24. On the left is the graph $X = Y_5$ obtained by adding 4 vertices to each edge of $Y = K_5$, the complete graph on 5 vertices. On the right the poles ($\neq -1$) of the vertex Ihara zeta function of X are shown by stars. The circles have centers at the origin and radii $\left\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\right\}$.

Sometimes number theorists state a modified GRH = Generalized Riemann Hypothesis for the Dedekind zeta function and this just ignores all possible real zeros while only requiring the non-real zeros to be on the line $\mathrm{Re}(s) = \frac{1}{2}$. The graph theory analog of the modified weak GRH would just ignore the real poles and require that there are no non-real poles of $\zeta_X(u)$ in $R_X < |u| < q^{-1/2}$. But this is true for all graphs by Theorem 4: if μ is a pole of $\zeta_X(u)$ and $|\mu| < q^{-1/2}$ then μ is real!

One may ask about the relations between the constants $\rho_X, \overline{d_X}, R_X$. One can show (**Exercise**) that

$$\rho_X \geq \overline{d_X}. \tag{8.8}$$

This is easily seen using the fact that ρ_X is the maximum value of the Rayleigh quotient $\langle Af, f\rangle / \langle f, f\rangle$, for any non-0 vector f in $\mathbb{R}^n$, while $\overline{d_X}$ is the value when f is the vector all of whose entries are 1. In all the examples to date we see that $\rho_X \geq 1 + \frac{1}{R_X} \geq \overline{d_X}$ but can only show that $\rho_X \geq \frac{p}{q} + \frac{1}{R_X}$. As a **research problem**, the reader might want to investigate this.

Next we give some examples including answers to the questions: Do the spectra of the adjacency matrices satisfy the naive Ramanujan inequality (8.3) or the Hoory inequality (8.5)? Do the Ihara zeta functions for the graphs have the pole-free region (8.7) of the weak graph theory RH or the pole-free region (8.6) of the full graph theory RH?

Example 5. *Let X be* ***the graph obtained from the complete graph on 5 vertices by adding 4 vertices to each edge*** *as shown on the left in Figure 24.*

We thank Adam O'Neill for his help with this example. For the graph X, we find that $\rho' \approx 2.32771$ and

$$\{\rho, 1 + 1/R, \overline{d_X}\} \approx \{2.39138, 2.24573, 2.22222\}.$$

It follows that this graph satisfies the naive Ramanujan inequality (8.3) but not the Hoory inequality (8.5). The picture on the right in Figure 24 shows stars for the poles not equal to -1 of $\zeta(u, X)$. Here

$$\zeta(u, X)^{-1} = \zeta(u^5, K_5)^{-1} = \left(1 - u^{10}\right)^5 \left(1 - 3u^5\right) \left(1 - u^5\right) \left(1 + u^5 + 3u^{10}\right).$$

The circles in the picture on the right in Figure 24 are centered at the origin with radii

$$\left\{q^{-\frac{1}{2}}, R, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\right\} \approx \{0.57735, 0.802742, 0.895958, 1\}.$$

The zeta function satisfies the RH and thus the weak RH. However the weak RH is vacuous.

Example 6. ***Random graph with probability 1/2 of an edge.***

See Figure 25 for the location of the poles not equal to ± 1 of the vertex Ihara zeta function of a random graph produced by Mathematica with the command RandomGraph[100,1/2]. This means there are 100 vertices and the probability of an edge between any 2 vertices is 1/2. The graph satisfies the Hoory inequality (8.5) and it is thus Ramanujan in Lubotzky's sense and satisfies the naive Ramanujan inequality (8.3). We find that $\rho' \approx 10.3357$ and $\{\rho, 1 + 1/R, \overline{d_X}\}$ $\approx \{48.5972, 48.585, 48.\}$. The circles in Figure 25 are centered at the origin and have radii given by $\{R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\}$ $\approx \{0.021015, 0.129099, 0.144966, 0.174078\}$. The poles of the zeta function satisfy the weak RH but not the RH. See Skiena [80] *for more information on the model that Mathematica uses to produce random graphs.*

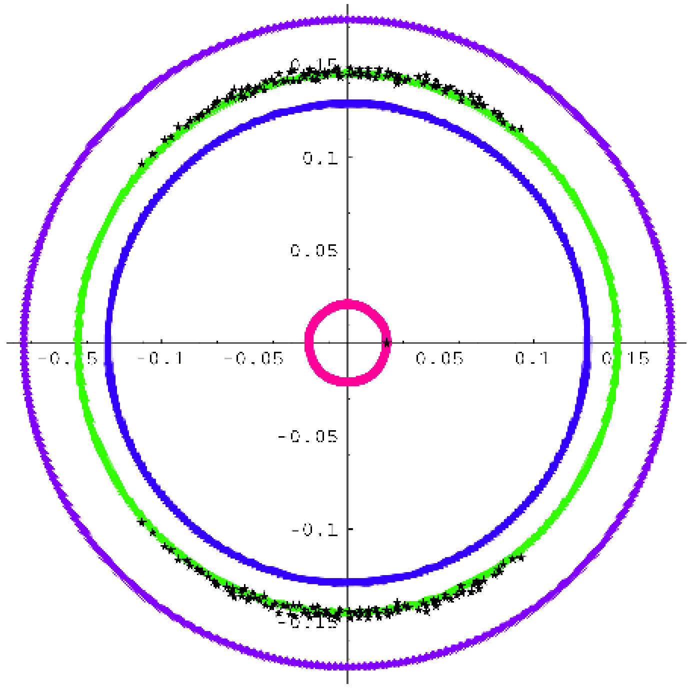

FIGURE 25. The poles ($\neq \pm 1$) of the vertex Ihara zeta function for a random graph produced by Mathematica with the command RandomGraph[100,$\frac{1}{2}$]. The circles have centers at the origin and radii $\left\{R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\right\}$.

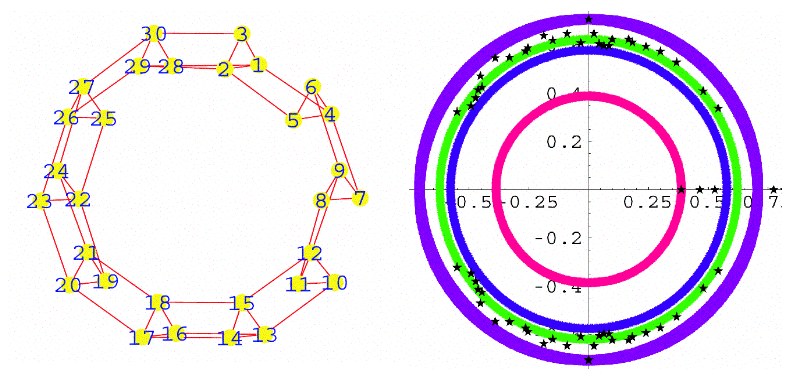

FIGURE 26. The graph N on the left results from deleting 7 edges from the product of a 3-cycle and a 10-cycle. In the picture on the right, stars indicate the poles ($\neq \pm 1$) of the vertex Ihara zeta function of N. The circles are centered at the origin with radii $\{R, q^{-1/2}, R^{1/2}, p^{-1/2}\}$.

Example 7. ***Torus minus some edges.***

From the torus graph T which is the product of a 3-cycle and a 10-cycle, we delete 7 edges to obtain a graph we will call N which is on the left in Figure 26. The spectrum of the adjacency matrix of N satisfies neither the Hoory inequality (8.5) nor the naive Ramanujan inequality (8.3). We find that $\{\rho, 1 + 1/R, d\} \approx \{3.60791, 3.58188, 3.53333\}$, *and* $\rho' \approx 3.39492$. *The right hand side of Figure 26 shows the poles of the vertex zeta for N as stars. The circles have center at the origin and radii* $\{R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}\} \approx \{0.387315, 0.57735, 0.622347, 0.707107\}$. *The zeta poles satisfy neither the graph theory weak RH nor the RH.*

Exercise 24. *Do more examples in the spirit of the preceding figures. One possibility comes from work of Friedman and Hoory who take graphs which are n-covers of K_4 − edge, for large n. Another is the zig-zag product graphs of Wigderson et al* [39].

See the websites of Joel Friedman (www.math.ubc.ca/~jf) and Avi Wigderson(www.math.ias.edu/~avi). There are also examples in later sections of this book.

9. Discussion of Regular Ramanujan Graphs

In this section we restrict ourselves to regular graphs. Our goals are

(1) to explain why a random walker gets lost fast on a Ramanujan graph,
(2) give an example of regular Ramanujan graphs,
(3) show why the Ramanujan bound is best possible, and
(4) explain why Ramanujan graphs are good expanders.
(5) diameters

9.1. **Random Walks on Regular Graphs.** Suppose that A is the adjacency matrix of a k-regular graph X with n vertices. We get a **Markov chain** from A as follows. The states are the vertices of X. At time t, the process (walker) goes from the ith state to the jth state with probablility p_{ij} given by $\frac{1}{k}$ if vertex i is adjacent to vertex j and with 0 probability otherwise. A **probability vector** $p \in \mathbb{R}^n$ has non-negative entries p_i such that $\sum_{i=1}^{n} p_i = 1$. Here p_i represents the probability that the random walker is at vertex i of the graph.

Notation 1. *All our vectors in $\mathbb{R}^n$ are column vectors and we write $^t p$ to denote the* ***transpose*** *of a column vector p in $\mathbb{R}^n$. The same notation will also be used for matrices.*

The **Markov transition matrix** is $T = (p_{ij})_{1 \leq i.i \leq n} = \frac{1}{k} A$. Let $p_i^{(m)}$ denote the probability that the walker is a vertex i at time m. The **probability vector** is $p^{(m)} = {}^t\left(p_1^{(m)}, ..., p_n^{(m)}\right)$. Then

$$p^{(m+1)} = Tp^{(m)} \text{ and } p^{(m)} = T^m p^{(0)}.$$

Theorem 6. ***(A Random Walker Gets Lost).*** *Suppose that X is a connected non-bipartite k-regular graph with n vertices and adjacency matrix A. If $T = \frac{1}{k}A$, for every initial probability vector $p^{(0)}$, we have*

$$\lim_{m\to\infty} p^{(m)} = \lim_{m\to\infty} T^m p^{(0)} = u = {}^t\left(\frac{1}{n}, ..., \frac{1}{n}\right);$$

i.e., the limit is the uniform probability vector.

Proof. Since T is a real symmetric matrix, the spectral theorem from linear algebra says that there is a real orthogonal matrix U; i.e., $^tUU = I$, the identity matrix, such that $^tUTU = D$, where D is a diagonal matrix with the eigenvalues λ_i of T down the diagonal. Let $U = (u_1, ..., u_n)$, with column vectors u_i. Then these columns are orthonormal, meaning the inner products of the columns satisfy

$$< u_i, u_j >= {}^tu_i u_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Now any vector v can be written as a linear combination of the column vectors of U

$$v = \sum_{i=1}^{n} < v, u_i > u_i, \text{ and then } Tv = \sum_{i=1}^{n} < v, u_i > \lambda_i u_i.$$

Then

$$T^m v = \sum_{i=1}^{n} < v, u_i > \lambda_i^m u_i.$$

Assume that u_1 is a constant vector of norm 1 so that $\lambda_1 = 1$ and the entries of u_1 are $\frac{1}{\sqrt{n}}$. By the hypothesis on X, we know (via Proposition 2) that $|\lambda_i| < 1$, for $i > 1$. It follows that

$$\lim_{m\to\infty} \lambda_i^m = \begin{cases} 1, & i = 1; \\ 0, & i \neq 1. \end{cases}$$

Thus

$$\lim_{m\to\infty} T^m v =< v, u_1 > u_1 = \frac{1}{n}.$$

This proves the theorem. □

But we want to know how long it takes the random walker to get lost. This depends on the second largest eigenvalue of the adjacency matrix, assuming the graph is non-bipartite. The next theorem answers the question. If the graph is bipartite, one can modify the random walk to make the walker get lost, by allowing the walker to stay in place with equal probability. We will use the $1-$norm $\|\|_1$ to measure distances between vectors in $\mathbb{R}^n$ Statisticians seem to prefer this to the 2-norm. See Diaconis [26]. Define

$$\|v\|_1 = \sum_{i=1}^{n} |v_i| . \tag{9.1}$$

Theorem 7. ***(How long to get lost?)*** *Suppose that X is a connected non-bipartite k-regular graph with n vertices and adjacency matrix A. If $T = \frac{1}{k}A$, for every initial probability vector $p^{(0)}$, we have*

$$\left\| T^m p^{(0)} - u \right\|_1 \leq \sqrt{n} \left(\frac{\mu}{k} \right)^n ,$$

where $u = {}^t\left(\frac{1}{n}, ..., \frac{1}{n}\right)$ *and*

$$\mu = \max \{ |\lambda| \, | \lambda \in Spectrum(A), |\lambda| \neq q+1 \} .$$

Proof. See my book [92], pp. 104-106. The proof is in the same spirit as that of the preceding theorem. □

Corollary 1. *If the graph in Theorem 7 is Ramanujan as in Definition 4, then $\mu \leq 2\sqrt{k-1}$, and*

$$\left\| T^m p^{(0)} - u \right\|_1 \leq \sqrt{n} \left(\frac{2\sqrt{k-1}}{k} \right)^n .$$

The moral of this story is that for large values of the degree k, it does not take a very long time before the walker is lost.

Exercise 25. *Redo the preceding results for irregular graphs.*

9.2. Examples, The Paley Graph, 2D Euclidean Graphs, and Graphs of Lubotzky, Phillips and Sarnak.

Example 8. ***The Paley Graph.*** $P = X(\mathbb{Z}/p\mathbb{Z}, \square)$ *this is a Cayley graph for the group $\mathbb{Z}/p\mathbb{Z}$, where p is an odd prime of the form $p = 1 + 4n, n \in \mathbb{Z}$. The vertices of the graph are elements of $\mathbb{Z}/p\mathbb{Z}$ and two vertices a, b are connected iff $a - b$ is a non-zero square in $\mathbb{Z}/p\mathbb{Z}$. If $p \equiv 1 (\text{mod } 4)$, then -1 is a square and conversely (**Exercise**). It follows that when $p \equiv 1 (\text{mod } 4)$ the Paley graph is undirected.*

*The **additive characters** of the group $\mathbb{Z}/p\mathbb{Z}$ are of the form $\chi_a(y) = e^{\frac{2\pi i a y}{p}}$, for $a, y \in \mathbb{Z}/p\mathbb{Z}$. They form a complete orthogonal set of eigenfunctions of the adjacency operator of the Paley graph.*

$$A\chi_a(y) = \sum_{x \sim y} e^{\frac{2\pi i a x}{p}} = \frac{1}{2} \sum_{\substack{x=y+u^2 \\ 0 \neq u \in \mathbb{Z}/p\mathbb{Z}}} e^{\frac{2\pi i a x}{p}} = \lambda_a \chi_a(y).$$

The eigenvalues λ_a have the form

$$\lambda_a = \frac{1}{2} \sum_{u=1}^{p-1} e^{\frac{2\pi i a u^2}{p}} .$$

*Recall that the **Gauss sum** is*

$$G_a = \sum_{u=0}^{p-1} e^{\frac{2\pi i a u^2}{p}} . \tag{9.2}$$

Thus $\lambda_a = \frac{1}{2}(G_a - 1)$. Use the Exercise below then to see that if a is not congruent to $0 (\text{mod } p)$,

$$|\lambda_a| \leq \frac{1 + \sqrt{p}}{2} .$$

Thus the graph is Ramanujan if $p \geq 5$, since the degree is $\frac{p-1}{2}$.

Exercise 26. *Show that when a is not congruent to $0 (\text{mod } p)$, the Gauss sum defined in the preceding example satisfies $|G_a| = \sqrt{p}$.*

Hint. This can be found in most elementary number theory books and in my book [92].

Exercise 27. *Fill in the details showing that the Paley graph above is Ramanujan when $p \geq 5$. Then find out how large p must be in order that $\|T^m v - u\|_1 \leq \frac{1}{100}$, where $v = {}^t(1, 0, 0, ..., 0)$.*

Example 9. ***2 Dimensional Euclidean Graphs.***

Suppose that p is an odd prime. Define the Cayley graph $X(G, S)$ for the group $G = \mathbb{F}_p^2$ consisting of 2-vectors with entries in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, with the operation of vector addition. The generating set S is the set of vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_p^2$ satisfying $x^2 + y^2 = 1$. This is a special case of the Euclidean graphs considered in Terras [92] where they are connected with finite analogs of symmetric spaces.

The additive characters of $G = \mathbb{F}_p^2$ are $\psi_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \exp\left(\frac{2\pi i(ax+by)}{p}\right)$, . for (a, b) and $(x, y) \in G$. They form a complete orthogonal set of eigenfunctions of the adjacency operator of the 2D Euclidean graph:

$$A\psi_{a,b}\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} u \\ v \end{pmatrix}} e^{\frac{2\pi i ax+by}{p}} = \sum_{r^2+s^2=1} e^{\frac{2\pi i(a(r+u)+b(s+v))}{p}} = \lambda_{a,b}\psi_{a,b}\begin{pmatrix} u \\ v \end{pmatrix}.$$

The corresponding eigenvalues $\lambda_{a,b}$ are

$$\lambda_{a,b} = \sum_{r^2+s^2=1} e^{\frac{2\pi i(ar+bv)}{p}}.$$

These numbers can be identified with a sum which is a favorite of number theorists called a Kloosterman sum.

If κ is a character of the multiplicative group $\mathbb{F}_p^* = \mathbb{F}_p - 0$ and $a, b \in \mathbb{F}_p^*$, define the **generalized Kloosterman sum** as

$$K(\kappa|a, b) = \sum_{t \in \mathbb{F}_p^*} \kappa(t) e^{\frac{-2\pi i(at+b/t)}{p}}.$$

It turns out that the non-trivial eigenvalues of the adjacency matrix for the 2D Euclidean Cayley graph are

$$\lambda_{a,b} = \frac{1}{p} G_1^2 K\left(\varepsilon^2 \middle| 1, a^2 + b^2\right),$$

where G_1 is the Gauss sum in formula (9.2) and the quadratic character is

$$\varepsilon(t) = \begin{cases} 1, & t \equiv u^2 (\text{mod } p), \text{ for some } u \in \mathbb{F}_p^* \\ 0, & t \equiv 0 (\text{mod } p) \\ -1, & \text{otherwise.} \end{cases} \tag{9.3}$$

As a consequence of the Riemann hypothesis for zeta functions of curves over finite fields one has a bound on the Kloosterman sums. This was proved by A. Weil. See Rosen [71] for more information. The bound implies that for $(a, b) \neq (0, 0)$, we have

$$\left|\lambda_{a,b}\right| \leq 2\sqrt{q}.$$

The degrees of the 2D Euclidean Cayley graph may be computed exactly (see Rosen [71]) to be $p - \varepsilon(-1)$, where ε is defined by formula (9.3).

It follows that if $p \equiv 3 (\text{mod } 4)$, the graphs are Ramanujan. But when $p = 17, 53$, for example, the graphs are not Ramanujan. Katz has proved that these Kloosterman sums do not vanish. He also proved that the distribution of the Kloosterman sums approaches the semicircle distribution as $p \to \infty$. See Terras [92] for the references. Derek Newland [65] has found that for these Euclidean graphs, the level spacings of $\text{Im}\, s$ corresponding to poles of $\zeta(q^{-s}, X)$ look Poisson for large p. The contour maps of the eigenfunctions are beautiful pictures of the finite circles $x^2 + y^2 \equiv a (\text{mod } p)$. There are movies on my website of these pictures as p runs through an increasing sequence of primes.

In Terras [92] we also define non-Euclidean finite upper half plane graphs where the Euclidean distance is replaced by a finite analog of the Poincare distance. These also give Ramanujan graphs which provide interesting spectra of their adjacency matrices. It takes quite a bit of knowledge of group representations plus Weil's result proving the Riemann hypothesis for curves over finite fields in order to prove that these graphs are Ramanujan.

Exercise 28. *Consider some examples of the finite upper half plane graphs in* [92]. *Experiment with the spectra of the adjacency matrices to see whether the graphs are Ramanujan. Look at the level spacings of the poles of Ihara zeta.*

Such examples are not really the expander graphs sought after by computer scientists since the degree blows up with the number of vertices. It is more difficult to find families of Ramanujan graphs of fixed degree with number of vertices approaching infinity. The first examples were due to Margulis [58] and independently Lubotzky, Phillips and Sarnak [56]. Recently J. Friedman (see his website www.math.ubc.ca/~jf) has shown that for fixed degree k and and $\epsilon > 0$, the probability that $\lambda_1(X_{m,k}) \leq 2\sqrt{k-1}+\epsilon$ approaches 1 as $n \to \infty$.

Let us finish by presenting the example of Lubotzky, Phillips and Sarnak [56].

Example 10. ***The Lubotzky, Phillips and Sarnak Graphs*** $\mathbf{X}_{p,q}$.

Let p and q be distinct primes congruent to 1 modulo 4. These are Cayley graphs for the group $G = PGL(2, \mathbb{F}_q) = GL(2, \mathbb{F}_q)/Center$. Here $GL(2, \mathbb{F}_q)$ is the group of non-singular 2×2 matrices with elements in the field with q elements and we identify matrices which are scalar multiples. Fix some integer i so that $i^2 \equiv -1 (\mathrm{mod}\, q)$. Define S to be

$$S = \left\{ \left(\begin{array}{cc} a_o + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{array} \right) \middle| \; a_0^2 + a_1^2 + a_2^2 + a_3^2 = p, \;\; \textit{for odd} \;\; a_0 > 0 \;\; \textit{and even} \;\; a_1, a_2, a_3 \right\}.$$

A theorem of Jacobi says there are exactly $p+1$ integer solutions to $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$ so that $|S| = p+1$. One can show that S is closed under matrix inverse. The graph $X_{p,q}$ is then the connected component of the identity in the Cayley graph $X(G,S)$. It can be proved that either $X(G,S)$ is connected or it has 2 connected components of equal size. Using Weil's proof of the Riemann hypothesis for zeta functions of curves over finite fields, Lubotzky, Phillips and Sarnak show that these graphs are Ramanujan. For fixed p we then have a family of Ramanujan graphs of degree $p+1$ having $O(q^3)$ vertices as $q \to \infty$.

9.3. Why the Ramanujan Bound is Best Possible (Alon and Boppana).

We want to prove the following theorem.

Theorem 8. ***(Alon and Boppana)*** *Suppose that X_n is a sequence of k-regular connected graphs with the number of vertices of X_n approaching infinity with n. Let $\lambda_1(X_n)$ denote the second largest eigenvalue of the adjacency matrix of X_n. Then*

$$\lim_{n\to\infty} \left(\inf_{m \geq n} \lambda_1(X_m) \right) \geq 2\sqrt{k-1}.$$

Proof. **(Lubotzky, Phillips, and Sarnak).** Let the set of eigenvalues of the adjacency matrix A_n of X_n be

$$Spec(A_n) = \left\{ \lambda_0 = k > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V(X_n)|-1} \right\}.$$

Let $N_v(m, X_n)$ be the number of paths of length m going from vertex v to v in graph X_n. Note that these paths can have backtracking and tails. Then

$$\sum_{j=0}^{|V(X_n)|-1} \lambda_j^m = Tr(A_n^m) = \sum_{v \in X} N_v(m, X_n).$$

The universal covering space of X_n is the **k-regular tree T_k** (meaning it is an infinite graph which is k-regular, connected and having no cycles). Part of T_4 is pictured in Figure 9. The lower bound we seek is actually the spectral radius of the adjacency operator on T_k.

Let τ_m be the number of paths of length m on T_k going from any vertex $\widetilde{v}$ back to $\widetilde{v}$. Since T_k is the k-regular tree, τ_m is 0 unless m is even and τ_m is independent of $\widetilde{v}$. Then $N_v(m, X_n) \geq \tau_m$, since any path on T_k projects down 1-1 to a path on X_n. Therefore

$$\sum_{j=0}^{|V(X_n)|-1} \lambda_j^m = \sum_{v \in X} N_v(m, X_n) \geq |V(X_n)|\, \tau_m.$$

It follows that

$$k^m + (|V(X_n)| - 1)\, \lambda_1^m \geq |V(X_n)|\, \tau_m.$$

We will be done if we can show that

$$\tau_{2m}^{1/2m} \to 2\sqrt{k-1}, \;\text{ as }\; m \to \infty. \tag{9.4}$$

For then we would have

$$\lambda_1 \geq \left(\frac{|V(X_n)|\, \tau_{2m} - k^{2m}}{|V(X_n)| - 1} \right)^{1/2m} = \left(\frac{|V(X_n)|}{|V(X_n)| - 1} \right)^{1/2m} \tau_{2m}^{1/2m} \left(1 - \frac{k^{2m}}{\tau_{2m}\, |V(X_n)|} \right)^{1/2m}.$$

The first factor approaches 1 as $n \to \infty$. The second factor approaches $2\sqrt{k-1}$. The third factor approaches 1.

Now we must prove formula (9.4). For this part of the proof we follow the reasoning of H. Stark. Let x and y be any two points of T_k such that the distance between them $d(x, y) = j$; i.e., the number of edges in the unique path in T_k joining x and y is j.

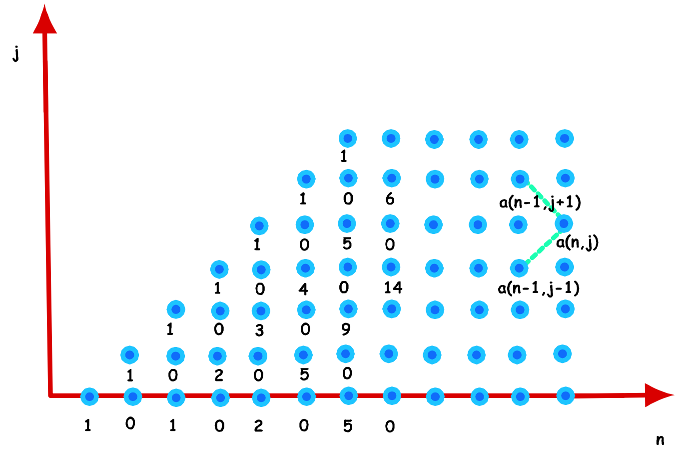

FIGURE 27. Values of $a(n,j)$ defined by the recursion $a(n,j) = a(n-1,j-1) + a(n-1,j+1)$, with $a(0,0) = 1$, $a(n,j) = 0$ unless $0 \le j \le n$.

We define $\tau(n,j)$ to be the number of ways starting at x to get to a point y at distance j from x by a path of length n in T_k. It is $\tau(m,0) = \tau_m$ that we want to study. It is an **Exercise** to see that $\tau(n,j) \neq 0$ implies that $j \equiv n(\bmod 2)$ and $n \ge j$. For $j > 0$ and $n > 1$, we have the recursion

$$\tau(n,j) = (k-1)\tau(n-1,j+1) + \tau(n-1,j-1). \tag{9.5}$$

For one must be at one of the k neighbors of y at the $(n-1)$st step, and $(k-1)$ of these neighbors are a distance $(j+1)$ from x, while the last neighbor is at a distance $(j-1)$ from x.

The recursion (9.5) is reminiscent of Pascal's triangle. It is an **Exercise** to show that

$$\tau(2m,0) \ge a(2m,0)(k-1)^m,$$

where $a(n,j)$ is defined by the following recursive definition

$$\begin{aligned} a(n,j) &= a(n-1,j-1) + a(n-1,j+1); \\ a(0,0) &= 1; \\ a(n,0) &= 0, \text{ unless } 0 \le j \le n, \ a(0,0) = 1. \end{aligned}$$

Set $a_{2m} = a(2m,0)$. Note that a_{2m} satisfies the recursion

$$a_{2m} = \sum_{k=1}^{m} a_{2k-2} a_{2m-2k}.$$

This recursion arises in many ways in combinatorics. See Vilenkin [100]. For example a_{2m} is the number of permutations of $2m$ letters, m of which are b's and m of which are f's, such that for every r with $1 \le r \le 2m$, the number of b's in the first r terms of the permutations is $\ge$ the number of f's. The solution is the **Catalan number**

$$a_{2m} = \frac{1}{m+1}\binom{2m}{m}.$$

Stirling's formula implies that

$$\binom{2m}{m}^{1/2m} \sim 2, \quad \text{as } m \to \infty.$$

Formula (9.4) follows and thus the theorem. □

Exercise 29. *Fill in the details in the preceding proof. It may help to look at Figure 27 showing some of the values of $a(n,j)$.*

9.4. **Why are Ramanujan graphs good expanders?** First, what is an expander graph? Roughly it means that the graph is highly connected but sparse (meaning that there are relatively few edges). Such graphs are quite useful in computer science - for building efficient communication networks, for creating error-correcting codes with efficient encoding and decoding. See Davidoff et al [25], Hoory et al [39], Lubotzky [54], Sarnak [75] for more information.

Suppose that X is an undirected k-regular graph satisfying our usual assumptions.

Definition 17. *For sets of vertices S,T of X, define*

$$E(S,T)=\{e\,|e \text{ is edge of } X \text{ with one vertex in } S \text{ and the other vertex in } T\}.$$

Definition 18. *If S is a set of vertices of X, we say the **boundary** is $\partial S=E(S,X-S)$.*

Definition 19. *A graph X with vertex set V and $n=|V|$ has **expansion ratio***

$$h(X)=\min_{\{S\subset V\,\big|\,|S|\le\frac{n}{2}\}}\frac{|\partial S|}{|S|}.$$

Note that there are many variations on this definition. We follow Sarnak [75] and Hoory et al [39] here. The expansion constant is the discrete analog of the Cheeger constant in differential geometry. See Lubotzky [54].

Definition 20. *A sequence of $(q+1)$-regular graphs $\{X_j\}$ such that $\left|V(X_j)\right|\to\infty$, as $j\to\infty$, is called a **family of expanders**if there is an $\varepsilon>0$ such that the expansion ratio $h(X_j)\ge\varepsilon$, for all j.*

For connected $k-$regular graphs X whose adjacency matrix has spectrum $k=\lambda_1>\lambda_2\ge\cdots\ge\lambda_n$, one can prove that

$$\frac{k-\lambda_2}{2k}\le h(X). \tag{9.6}$$

See my book [92], p. 80. There is also a discussion in Hoory, Lineal and [39], pages 474-476, who prove an upper bound as well. Such results were originally proved by Dodziuk, Alon, and Milman. It follows that for large expansion constant, one needs small λ_2.

Next we prove the expander-mixing lemma (from Alon and Fan Chung [1]) which says that $E(S,T)$ will be closer to the expected number of edges between S and T in a random $k-$regular graph X of edge density $\frac{k}{n}$, (where $n=|V|$) provided that μ, the 2nd largest eigenvalue (in absolute value) of the adjacency matrix of X is small as possible.

Lemma 1. ***The Expander Mixing Lemma.***

Suppose X is a connected k-regular non bipartite graph with n vertices and

$$\mu=\max\{|\lambda|\,|\lambda\in Spectrum(A),|\lambda|\ne k\}.$$

Then for all sets S,T of vertices of X, we have

$$\left|E(S,T)-\frac{k\,|S|\,|T|}{n}\right|\le\mu\sqrt{|S|\,|T|}.$$

Proof. By our hypotheses $\mu<k$.

Let δ_S denote the vector whose entries are 1 for vertices of S and 0 otherwise. Recall the spectral theorem for the symmetric matrix $A=$the adjacency matrix of X. This says there is a complete orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors ϕ_j of A, with $A\phi_j=\lambda_j\phi_j$ and giving

$$(A)_{a,b}=\sum_{j=1}^n\lambda_j\phi_j(a)\phi_j(b).$$

Here we write $\phi_j(a)$ to denote the entry of ϕ_j corresponding to vertex a of X. We may assume that we have numbered things so that $\phi_1(a)=\frac{1}{\sqrt{n}}$ and $\lambda_1=k$. Then, pulling out the 1st term of the sum gives

$$|E(S,T)|={}^T\delta_S A\delta_T=\sum_{j=1}^n\lambda_j\sum_{\substack{a\in S\\ b\in T}}\phi_j(a)\phi_j(b)=\frac{k}{n}\,|S|\,|T|+\sum_{j=2}^n\lambda_j\sum_{\substack{a\in S\\ b\in T}}\phi_j(a)\phi_j(b).$$

Now, by the definition of μ, since our graph is not bipartite, there is only one eigenvalue with absolute value equal to k, and

$$\left|\sum_{j=2}^n\lambda_j\sum_{\substack{a\in S\\ b\in T}}\phi_j(a)\phi_j(b)\right|\le\mu\sum_{j=2}^n\sum_{\substack{a\in S\\ b\in T}}\phi_j(a)\phi_j(b).$$

To finish the proof, use the Cauchy-Schwarz inequality. Note that the Fourier coefficients of δ_S with respect to the basis ϕ_j are

$$< \delta_S, \phi_j >= \sum_{a \in S} \phi_j(a).$$

This implies by Bessel's equality that

$$|S| = \|\delta_S\|_2^2 = \sum_{j=1}^{n} < \delta_S, \phi_j >^2 .$$

So Cauchy-Schwarz says

$$\left| \sum_{j=2}^{n} \sum_{\substack{a \in S \\ b \in T}} \phi_j(a)\phi_j(b) \right| \leq \left| \sum_{j=1}^{n} \sum_{\substack{a \in S \\ b \in T}} \phi_j(a)\phi_j(b) \right| \leq \sqrt{|S|\,|T|}.$$

This completes the proof of the lemma. □

9.5. **Why do Ramanujan graphs have small diameters?** In this section, we present a theorem of Fan Chung [19] which bounds the diameter of a connected k-regular graph in terms of the second largest eigenvalue in absolute value. We assume the graph is not bipartite to avoid the problem that $-k$ could also be an eigenvalue. From the theorem, we see that Ramanujan graphs will have as small diameter as possible for sequences of k-regular graphs with number of vertices approaching infinity. Thus, the Ramanujan graphs found by Lubotzky, Phillips and Sarnak [56] were shown to have small diameters.

Definition 21. *Define the distance $d(x,y)$ between 2 vertices x,y of a graph X to the the length of a smallest path connecting the vertices. Then the **diameter**of X is*

$$\max_{x,y \in V(X)} d(x,y).$$

Theorem 9. *Suppose that X is a connected, non bipartite $k-$regular graph with n vertices and*

$$\mu = \max \{|\lambda| \mid \lambda \in Spectrum(A),\ |\lambda| \neq k\}.$$

Then

$$\text{diameter}(X) \leq 1 + \frac{\log(n-1)}{\log \frac{k}{\mu}}.$$

Proof. As in the proof in the last subsection, we will use the spectral theorem for the adjacency matrix A of X. This says there is a complete orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors ϕ_j of A, with $A\phi_j = \lambda_j \phi_j$ and giving

$$(A)_{a,b} = \sum_{j=1}^{n} \lambda_j \phi_j(a)\phi_j(b).$$

Write $\phi_j(a)$ to denote the entry of ϕ_j corresponding to vertex a of X. Assume that we have numbered things so that $\phi_1(a) = \frac{1}{\sqrt{n}}$ and $\lambda_1 = k$.

Note that for vertices a,b of X, we have $\left(A^t\right)_{a,b} = \#\{\text{paths of length } t \text{ connecting } a \text{ to } b\}$. If d is the diameter of X, then $\left(A^d\right)_{a,b} \neq 0$, for some a,b with $d(a,b) = d$. Then $\left(A^{d-1}\right)_{a,b} = 0$, as there is no shorter path connecting a and b. Therefore, if $t = d-1$, and

$$0 = \left(A^t\right)_{a,b} = \sum_{j=1}^{t} \lambda_j^t \phi_j(a)\phi_j(b).$$

Use the Cauchy-Schwarz inequality to see that

$$\begin{aligned} 0 &\geq \frac{k^t}{n} - \mu^t \sum_{j=2}^{n} |\phi_j(a)|\,|\phi_j(b)| \geq \frac{k^t}{n} - \mu^t \left(\sum_{j=2}^{n} |\phi_j(a)|^2\right)^{1/2} \left(\sum_{j=2}^{n} |\phi_j(b)|^2\right)^{1/2} \\ &= \frac{k^t}{n} - \mu^t \sqrt{1-\phi_1(a)^2}\sqrt{1-\phi_1(b)^2} = \frac{k^t}{n} - \mu^t \left(1 - \frac{1}{n}\right). \end{aligned}$$

This implies that

$$\frac{k^t}{n} \leq \mu^t \left(1 - \frac{1}{n}\right).$$

Thus

$$\left(\frac{k}{\mu}\right)^t \leq n-1.$$

Taking logs,

$$t \log \frac{k}{\mu} \leq \log(n-1).$$

So recalling that $t = d-1$, we have

$$d - 1 \leq \frac{\log(n-1)}{\log \frac{k}{\mu}}.$$

The theorem follows. □

10. The Graph Theory Prime Number Theorem

The main application of the Ihara zeta function is to give an asymptotic estimate of the number of primes of length m. This is the content of the next the theorem. We will use results proved in the next part. Before we do this we need to consider the generating function obtained from the logarithmic derivative of the Ihara zeta function. We computed this function for K_4 in Example 2.

Definition 22. *$N_m = N_m(X)$ is the* ***number of closed paths of length m in the graph X without backtracking and tails.***

Lemma 2. *With N_m as in the preceding definition, we have the generating function:*

$$u\frac{d}{du}\log\zeta_X(u) = \sum_{m\geq 1} N_m u^m. \tag{10.1}$$

Proof. Recall Definition 2 of the Ihara zeta function. Then

$$\begin{aligned} u\frac{d}{du}\log\zeta_X(u) &= -u\frac{d}{du}\sum_{[P]}\log\left(1-u^{\nu(P)}\right) \\ &= \sum_{[P]}\sum_{j\geq 1}\frac{1}{j}u\frac{d}{du}u^{\nu(P)j} = \sum_{[P]}\sum_{j\geq 1}\frac{\nu(P)j}{j}u^{\nu(P)j}. \end{aligned}$$

The outer sum is over equivalence classes $[P]$ of prime closed paths with no backtracking or tails. Sum over all primes $[P]$ with $\nu(P) = d$. Now there are d elements in $[P]$. So we can drop the [ ] and obtain a sum over all closed paths $C = P^j$ without backtracking or tails. This gives the lemma. □

Exercise 30. *Consider the graph $X = K_4 - e$. Find N_m, for $m = 3, 4, 5, ..., 11$. Then go on to find $\pi(m)$, $m = 3, 4, 5, ..., 11$. We did a similar computation for K_4 in Example 2.*

Theorem 10. ***Graph Prime Number Theorem.*** *Suppose that R_X is as in Definition 3. If $\pi(m)$ and Δ_X are as in Definitions 6 and 7, then $\pi(m) = 0$ unless Δ_X divides m. If Δ_X divides m, we have*

$$\pi(m) \sim \Delta_X \frac{R_X^{-m}}{m}, \quad \text{as} \quad m \to \infty.$$

Proof. We imitate the proof of the analogous result for zeta functions of function fields in Rosen [71]. Observe that the defining formula for the Ihara zeta function can be written as

$$\zeta_X(u) = \prod_{n\geq 1}\left(1-u^n\right)^{-\pi(n)}.$$

Then

$$u\frac{d}{du}\log\zeta_X(u) = \sum_{n\geq 1}\frac{n\pi(n)u^n}{1-u^n} = \sum_{m\geq 1}\sum_{d|m} d\pi(d)u^m.$$

Here the inner sum is over all positive divisors of m. Thus we obtain the **relation between N_m and $\pi(n)$.**

$$N_m = \sum_{d|m} d\pi(d).$$

This sort of relation occurs frequently in number theory and combinatorics. It is inverted using the **Möbius function** $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^r & n = p_1 \cdots p_r, \text{ for distinct primes } p_i \\ 0 & \text{otherwise.} \end{cases}$$

Then by the **Möbius inversion formula**

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d. \tag{10.2}$$

Next we look at formula (4.1) where W_1 is from Definition 8. This formula was proved in Proposition 1 and will be reproved in the next part. Apply the differential operator after using the Schur upper triangularization of W_1 (see Horn and Johnson [40]). This gives

$$u\frac{d}{du}\log\zeta_X(u) = u\frac{d}{du}\sum_{\lambda\in Spec(W_1)}\log(1-\lambda u) = \sum_{\lambda\in Spec(W_1)}\sum_{n\geq 1}(\lambda u)^n.$$

It follows from Lemma 2 that we have the **formula relating N_m and the spectrum of the edge matrix W_1 :**

$$N_m = \sum_{\lambda\in Spec(W_1)} \lambda^m. \tag{10.3}$$

The dominant terms in this last sum are those coming from $\lambda \in Spec(W_1)$ such that $|\lambda| = R^{-1}$, with $R = R_X$ from Definition 3.

By Theorem 4, the largest absolute value of an eigenvalue λ occurs Δ_X times with these eigenvalues having the form $e^{2\pi ia/\Delta_X}R^{-1}$, where $a = 1, ..., \Delta_X$. Using the orthogonality relations for exponential sums (see [92]) which are basic to the theory of the finite Fourier transform, we see that

$$\pi(n) \sim \frac{1}{n}\sum_{|\lambda| \text{ maximal}} \lambda^n = \frac{R^{-n}}{n}\sum_{a=1}^{\Delta_X} e^{\frac{2\pi ian}{\Delta_X}} = \frac{R^{-n}}{n}\begin{cases} 0, & \Delta_X \text{ does not divide } n \\ \Delta_X, & \Delta_X \text{ divides } n. \end{cases} \tag{10.4}$$

The graph prime number theorem follows from formulas (10.2), (10.3), and (10.4). □

Exercise 31. *Prove* $\sum_{a=1}^{\Delta_X} e^{\frac{2\pi ian}{\Delta_X}} = \begin{cases} 0, & \Delta_X \text{ does not divide } n \\ \Delta_X, & \Delta_X \text{ divides } n. \end{cases}$

Exercise 32. *If the graph $X = K_4$, the tetrahedron, find $\pi(m)$ for $m = 3, 4, 5, , , , 11$.*

If the Riemann hypothesis (either version for irregular graphs) holds for $\zeta_X(u)$, then one has a good bound on the error term in the prime number theorem by formula (4.1). By Theorem 8 of Alon and Boppana, the bound on the error term will be best possible for a family of connected degree $q+1$ graphs with number of vertices approaching infinity.

The vertex Ihara zeta function can be used to determine the rank of the fundamental group, for it is the order of the pole of the Ihara zeta function at $u = 1$. The **complexity** κ_X of a graph is defined to be the number of spanning trees in X. One can use the matrix-tree theorem (see Biggs [11]) to prove that

$$\left[\frac{d^r}{du^r}\zeta_X^{-1}(u)\right]\bigg|_{u=1} = r!(-1)^{r+1}2^r(r-1)\kappa_X.$$

This result is an exercise on the last page of Terras [92], where some hints are given. It is an analog of the formula for the Dedekind zeta function of a number field at 0 (a formula involving the class number and the regulator of the number field). See Lang [53].

Exercise 33. *Prove the preceding formula.*

Exercise 34. *Prove the prime number theorem for a (q+1)-regular graph using Theorem 1 of Ihara with the 3-term determinant rather thatn the* $\det(I - W_1)^{-1}$ *formula.*

Research Question: What other basic invariants of the graph X can be determined by the Ihara zeta function? See Horton [42].

We have found that the Ihara zeta function possesses many analogous properties to the Dedekind zeta function of an algebraic number field. There are other analogs as well. For example there is an an analog of the ideal class group called the Jacobian of a graph. It has order equal to κ_X, the complexity. It has been considered by Bacher, de la Harpe and Tatiana Nagnibeda [5] as well as Baker and Norine [6] (on Baker's web page at www.math.gatech.edu/~mbaker).

Part 3.

Functions

In this part, we consider 2 multivariable zeta functions associated to a finite graph, the edge zeta and the path zeta. We will give a matrix analysis version of the Bass proof of Ihara's determinant formula. This implies that there is a determinant formula for the vertex zeta function of weighted graphs even if the weights are not integers. In an example we will discuss what deleting an edge of a graph (fission) does to the edge zeta function. We will also discuss what happens if a graph edge is fused ; i.e,. shrunk to a point. There is an application of the edge zeta to error correcting codes. See Koetter et al [50].

11. The Edge Zeta Function

Notation 2. *From now on, we change our notation for the Ihara zeta function of the last section, replacing $\zeta_X(u)$ by $\zeta_V(u, X)$, where the "V" is for **vertex**. We call the Ihara zeta a **"vertex zeta."***

Definition 23. *The **edge matrix** W for graph X is a $2m \times 2m$ matrix with a, b entry corresponding to the oriented edges a and b. This a, b entry is the complex variable w_{ab} if edge a feeds into edge b and $b \neq a^{-1}$ and the a, b entry is 0 otherwise.*

Definition 24. *Given a path C in X, which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the **edge norm** of C is*

$$N_E(C) = w_{a_1a_2} w_{a_2a_3} \cdots w_{a_{s-1}a_s} w_{a_sa_1}.$$

*The **edge Ihara zeta function** is*

$$\zeta_E(W, X) = \prod_{[P]} (1 - N_E(P))^{-1},$$

where the product is over primes in X. Here assume that all $|w_{ab}|$ are sufficiently small for convergence.

Specializing Variables to Obtain other Zetas

1) Clearly if you set all non-zero variables in W equal to u, the **edge zeta function specializes to the vertex zeta function**; i.e,

$$\zeta_E(W, X)\big|_{0 \neq w_{ab} = u} = \zeta_V(u, X). \tag{11.1}$$

2) If X is a **weighted graph** with weight function L, and you specialize the non-zero variables

$$w_{ab} = u^{(L(a)+L(b))/2}, \tag{11.2}$$

you get the weighted Ihara zeta function. Or you could specialize

$$w_{ab} = u^{L(a)}. \tag{11.3}$$

3) To obtain the Hashimoto edge zeta function discussed in Stark and Terras [83], specialize $w_{ab} = u_a$.

4) If you cut or delete an edge of a graph (something we think of as "fission"), you can compute the edge zeta for the new graph with one less edge by setting all variables equal to 0 if the cut or deleted edge or its inverse appear in a subscript. Note that graph theorists usually call an edge a "cut edge" only if its removal disconnects the graph.

5) You can also use the variables w_{ab} in the edge matrix W corresponding to $b = a^{-1}$ to produce a zeta function that keeps track of paths with backtracking. See Bartholdi [8].

Before proving the determinant formula for the edge zeta, we need a Lemma.

Lemma 3. *Suppose we have 2 power series*

$$f(x_1, ..., x_t) = \sum_{i_1,...,i_t} a(i_1, ..., i_t) x_1^{i_1} \cdots x_t^{i_t}$$

and

$$g(x_1, ..., x_t) = \sum_{i_1,...,i_t} b(i_1, ..., i_t) x_1^{i_1} \cdots x_t^{i_t},$$

where the sums are over t-tuples of non-negative integers. Suppose that $a(0, ..., 0) = b(0, ..., 0)$. Then $f(x_1, ..., x_t) = g(x_1, ..., x_t)$ if and only if

$$\sum_{j=1}^{t} x_j \frac{\partial}{\partial x_j} f(x_1, ..., x_t) = \sum_{j=1}^{t} x_j \frac{\partial}{\partial x_j} g(x_1, ..., x_t).$$

Proof. Note that when we apply the Euler differential operator $L = \sum_{j=1}^{t} x_j \frac{\partial}{\partial x_j}$ to the monomial $x_1^{i_1} \cdots x_t^{i_t}$, we get

$$Lx_1^{i_1} \cdots x_t^{i_t} = (i_1 + \cdots + i_t)x_1^{i_1} \cdots x_t^{i_t}.$$

The result follows. □

The edge zeta again has a determinant formula and is the reciprocal of a polynomial in the w_{ab} variables. This is the following theorem whose proof should be compared with that of Proposition 1.

Theorem 11. ***(Determinant Formula for the Edge Zeta).***

$$\zeta_E(W, X) = \det(I - W)^{-1}.$$

Proof. First note that, from the Euler product for the edge zeta function, we have

$$-\log \zeta_E(W, X) = \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} N_E(P)^j.$$

Let $L = \sum_{i,j} w_{ij} \frac{\partial}{\partial w_{ij}}$ be the Euler operator. Then, since there are $\nu(P)$ elements in $[P]$, we have

$$-L \log \zeta_E(W, X) = \sum_{\substack{m \geq 1 \\ j \geq 1}} \frac{1}{jm} \sum_{\substack{P \\ \nu(P)=m}} L\left(N_E(P)^j\right).$$

Now $\nu(P) = m$ implies $L\left(N_E(P)^j\right) = jmN_E(P)^j$. It follows that

$$-L \log \zeta_E(W, X) = \sum_C N_E(C).$$

Here we sum over paths C which need not be prime paths, but are still closed without backtracking or tails. Then by the Exercise below, we see that

$$-L \log \zeta_E(W, X) = \sum_{m \geq 1} Tr(W^m).$$

Finally, again using the Exercise below, we see that the right hand side of the preceding formula is $L \log \det(I - W)^{-1}$ This proves L(log (determinant formula)). To finish the proof, observe that both sides are 1 when all the w_{ab} are 0 and then make use of the preceding Lemma. □

Exercise 35. *Prove that*

$$\sum_C N_E(C) = \sum_{m \geq 1} Tr(W^m) = L \log \det(I - W)^{-1}.$$

Hints. 1) For the first equality, you need to think about $Tr(W^m)$ as an $(m+1)$-fold sum of products of w_{ij} in terms of closed paths C of length m.

2) For the second equality, recall that Gram-Schmidt tells us that you can write a complex square matrix $B = U^{-1}TU$, where T is upper triangular and U is unitary. Apply this to show that

$$\det(\exp(B)) = e^{Tr(B)}.$$

Then write $I - W = \exp(B)$, using the matrix exponential, and see that $\log \det(I - W) = Tr(\log(I - W))$. *Now apply the Euler operator to both sides, using the power series for the matrix logarithm.*

By formula (11.1), we have the following Corollary, since specializing all the non-zero variables in W to be u, yields the matrix uW_1, where W_1 is from Definition 8.

Corollary 2. $\zeta_X(u) = \zeta_V(u, X) = \det(I - uW_1)^{-1}$.

Moral: The poles of $\zeta_X(u)$ are the reciprocals of the eigenvalues of W_1.

Exercise 36. *Write W_1 in block form with $|E| \times |E|$–blocks:*

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Show that $D = {}^tA,\ B = {}^tB,\ C = {}^tC$.

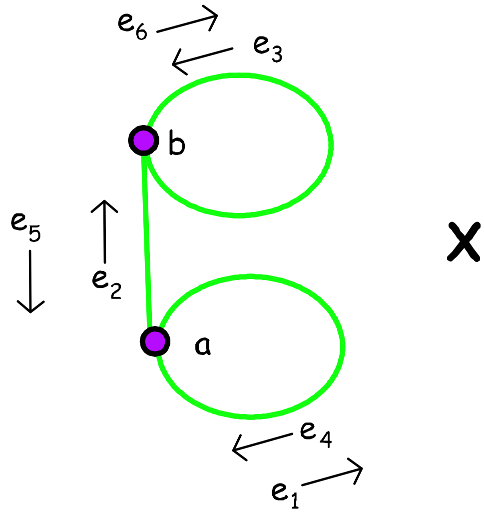

FIGURE 28. The dumbell graph

Suppose we have a weighted graph X with weight function L. Consider the formula obtained from Theorem 11 by specializing the non-zero variables in W as in formula (11.2) or formula (11.3). This gives a determinant formula for the weighted zeta function, but it does not necessarily lead to an Ihara type formula.

Exercise 37. *Consider some weighted graphs and their zeta functions. What are the locations of the poles? What happens to the pole spacings for regular graphs if one introduces weights?*

Example. Dumbbell Graph.

Figure 28 shows the labeled picture of the dumbbell graph X. For this graph we find that $\zeta_E(W,X)^{-1} =$

$$\det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix}.$$

Note that if we cut or delete the vertical edges which are edges e_2 and e_5, we should specialize all the variables with 2 or 5 in them to be 0. This yields the edge zeta function of the subgraph with the vertical edge removed, and incidentally diagonalizes the matrix W. We call this **"fission"**. The edge zeta is particularly suited to keeping track of such fission.

Exercise 38. *Do another example computing the edge zeta function of your favorite graph. Then see what happens if you delete an edge.*

Next we give a version of **Bass's proof of the Ihara determinant formula** (Theorem 1) using the preceding theorem. In what follows, n is the number of vertices of X and m is the number of unoriented edges of X.

First define some matrices. Set $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. Then define the $n \times 2m$ **start matrix** S and the $n \times 2m$ **terminal matrix** T by setting

$$s_{ve} = \begin{cases} 1, & \text{if } v \text{ is starting vertex of edge } e, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$t_{ve} = \begin{cases} 1, & \text{if } v \text{ is terminal vertex of edge } e, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4. ***Some Matrix Identities*** *Using the preceding definitions, the following formulas hold. We write* ${}^tM$ *for the transpose of the matrix* M.

1) $SJ = T, \quad TJ = S$.

2) If A *is the adjacency matrix of* X *and* $Q + I_n$ *is the diagonal matrix whose jth diagonal entry is the degree of the jth vertex of* X, *then* $A = S\,{}^tT$, *and* $Q + I_n = S\,{}^tS = T\,{}^tT$.

3) The 0,1 edge matrix W_1 *from Definition 8 satisfies* $W_1 + J = {}^tTS$.

Proof. 1) This comes from the fact that the starting (terminal) vertex of edge e_j is the terminal (starting) vertex of edge $e_{j+|E|}$, according to our edge numbering system from formula (2.1).

2) Consider

$$(S\,{}^tT)_{a,b} = \sum_e s_{ae} t_{be}.$$

The right hand side is the number of oriented edges e such that a is the initial vertex and b is the terminal vertex of e, which is the a, b entry of A. Note that $A_{a,a}$ =2*number of loops at vertex a. Similar arguments prove the second formula.

3) We have

$$({}^tTS)_{ef} = \sum_v t_{ve} s_{vf}.$$

The sum is 1 iff edge e feeds into edge f, even if $f = e^{-1}$. □

Finally we come to the proof we have advertised for so long.

Bass's Proof of the Generalized Ihara Determinant Formula Theorem 1.

Proof. In the following identity all matrices are $(n + 2m) \times (n + 2m)$, where the 1st block is $n \times n$, if n is the number of vertices of X and m is the number of unoriented edges of X. Use the preceding proposition to see that

$$\begin{pmatrix} I_n & 0 \\ {}^tT & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix}$$
$$= \begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2m} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ {}^tT - {}^tSu & I_{2m} \end{pmatrix}.$$

Exercise 39. *Check this equality. Relate it to the Schur complement of a block in a matrix.*

Take determinants to obtain

$$(1-u^2)^n \det(I - W_1 u) = \det\left(I_n - Au + Qu^2\right)\det\left(I_{2m} + Ju\right).$$

To finish the proof of Theorem 1, observe that

$$I + Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix}$$

implies

$$\begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix}(I + Ju) = \begin{pmatrix} I & Iu \\ 0 & I\left(1-u^2\right) \end{pmatrix}.$$

Thus $\det(I + Ju) = \left(1-u^2\right)^m$. Since $r - 1 = m - n$, for a connected graph, Theorem 1 follows. □

Next we want to prove Theorem 4 of Kotani and Sunada. First we will need some facts about the W_1 matrix.

Definition 25. *An* $s \times s$ *matrix* A, *with* $s > 1$, *whose entries are nonnegative is* ***irreducible*** *iff there does* ***not*** *exist a permutation matrix* P *such that* $A = {}^tP \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} P$, *where* B *is a* $t \times t$ *matrix with* $1 \leq t < s$.

Theorem 12. ***Facts About*** W_1

Let $n = |V|$ *the number of vertices of graph* X *and* $m = |E|$ *the number of edges of* X.

1) The row sums of the entries of W_1 *are* $q_j = -1+$*degree jth vertex.*

2) (Horton) The singular values of W_1 *(i.e., the square roots of the eigenvalues of* $W_1\,{}^tW_1$*) are* $\left\{q_1, ..., q_n, \underbrace{1, ..., 1}_{2m-n}\right\}$.

3) If the rank r *of the fundamental group of the graph* X *satisfies* $r \geq 2$, *then* $(I + W_1)^{2m-1}$ *has all positive entries. This says that the matrix* W_1 *is irreducible.*

Proof. 1) We leave this as an Exercise.

2) (Horton [42]). Modify W_1 to list all edges ending at the same vertex together. Note that then

$$W_1\,{}^tW_1 = \begin{pmatrix} A_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & A_n \end{pmatrix}, \text{ and } A_j = (q_j - 1)J + I,$$

where J is a $(q_j+1)\times(q_j+1)$ matrix of ones. Since the spectrum of J is $\{q_j+1, 0, ..., 0\}$, the spectrum of A_j is $\{q_j^2, 1, ..., 1\}$. The result follows.

3) This follows from Lemma 4 below. □

Exercise 40. *Consider the graph which consists of one vertex with 2 loops and another vertex on one of the loops. Modify W_1 to list all edges ending at the same vertex together and compute $W_1\,{}^tW_1$.*

Lemma 4. *Suppose X has rank at least 2. Given a directed edge e_1 starting at a vertex v_1 and a directed edge e_2 ending at a vertex v_2 in X ($v_1 = v_2$, $e_1 = e_2$, $e_1 = e_2^{-1}$ are allowed), there exists a backtrackless path $P = P(e_1, e_2)$ from v_1 to v_2 with initial edge e_1, terminal edge e_2, and length $\leq 2|E|$.*

Proof. See Figure 29 which shows our construction of $P(e_1, e_2)$ in two cases. First we construct a path P without worrying about its length. This construction is not minimal, but it has relatively few cases to consider.

Choose a spanning tree T of X. Recall that by "cut" edge of X we mean an edge left out of T. Begin by creating 2 backtrackless paths $P_1 f_1$ and $P_2 f_2$ with initial edges e_1 and e_2^{-1} and terminal edges f_1 and f_2 such that f_1 and f_2 are cut edges (i.e., non-tree edges of X). If e_1 is a cut edge, we let P_1 have length 0 and $f_1 = e_1$ (i.e., $P_1 f_1 = e_1$). If e_1 is not a cut edge, take P_1 to be a backtrackless path in the tree with initial edge e_1 which proceeds along T until it is impossible to go any further along the tree. Symbolically we write $P_1 = e_1 T_1$, where T_1 is a path along the tree, possibly of length zero. Let v_1' be the terminal vertex of P_1. With respect to the tree T, v_1' is a dangler or leaf (vertex of degree 1), but X has no danglers. Thus there must be a directed cut edge in X, which we take to be f_1, with initial vertex v_1'. By construction, $P_1 f_1$ is backtrackless also since P_1 is in the tree and f_1 isn't.

Similarly, if e_2 is a cut edge, let P_2 have length 0 and $f_2 = e_2^{-1}$ (i.e., $P_2 f_2 = e_2^{-1}$). If e_2 is not a cut edge, then as above form a backtrackless path $P_2 f_2 = e_2^{-1} T_2 f_2$ where T_2 is in the tree, possibly of length 0 and f_2 is a cut edge. In all cases, we let v_1' and v_2' be the initial vertices of f_1 and f_2.

Now, if we can find a path P_3 beginning at the terminal edge of f_1 and ending at the terminal vertex of f_2 such that the path $f_1 P_3 f_2^{-1}$ has no backtracking, then $P = P_1 f_1 P_3 f_2^{-1} P_2^{-1}$ will have no backtracking, with e_1 and e_2 as its initial and terminal edges, respectively. Of course, creating the path $f_1 P_3 f_2^{-1}$ was the original problem in proving Lemma. However, we now have the additional information that f_1 and f_2 are cut edges of the graph X.

We now have two cases. Case 1 is the case that $f_1 \neq f_2$, which is pictured at the top of Figure 29. In this case we can take $P_3 = T_3$ = the path within the tree T running from the terminal vertex of f_1 to the terminal vertex of f_2. Then, even if the length of T_3 is 0, the path $f_1 T_3 f_2^{-1}$ has no backtracks and we have created P.

Case 2 is $f_1 = f_2$. Thus f_1 and f_2 are the same cut edge of X. The worst case scenario in this case would have $e_2 = e_1^{-1}$, $T_2 = T_1$, $f_2 = f_1$. See the lower part of Figure 29. Since X has rank at least 2, there is another cut edge f_3 of X with $f_3 \neq f_1$ or f_1^{-1}. Let T_3 be the path along the tree T from the terminal vertex of f_1 to the initial vertex of f_3 and let T_4 be the path along the tree T from the terminal vertex of $f_2 = f_1$ to the terminal vertex of f_3. Then $P_3 = T_3 f_3 T_4^{-1}$ has the desired property that $f_1 P_3 f_2^{-1}$ has no backtracking, even if T_3 and/or T_4 have length 0. Thus we have created in all cases a backtrackless path P with initial edge e_1 and terminal edge e_2.

One can create a path P of length $\leq 2|E|$ as follows. If an edge is repeated, it is possible to delete all the edges in between the 1st and 2nd versions of that edge as well as the 2nd version of the edge without harming the properties of P.

Look at the e, f entry of $(I + W_X)^{2|E|-1}$. Take a backtrackless path P starting at e and ending at f. We have just shown that we can assume that the length of P is $\nu = \nu(P) \leq 2|E|$. Look at the e, f entry of the matrix $W^{\nu-1}$ which is a sum of terms of the form $w_{e_1 e_2} \cdots w_{e_{\nu-1} e_\nu}$, where each e_{i_j} denotes an oriented edge and $e_1 = e, e_\nu = f$. The term corresponding to the path P will be positive and the rest of the terms are non-negative. □

It follows from the preceding Lemma that all entries of $(I + W_1)^{2m-1}$ are positive.

We have the following Corollary to the preceding Theorem.

Corollary 3. *The poles of the Ihara zeta function of X are contained in the region $\frac{1}{q} \leq R_X \leq |u| \leq 1$.*

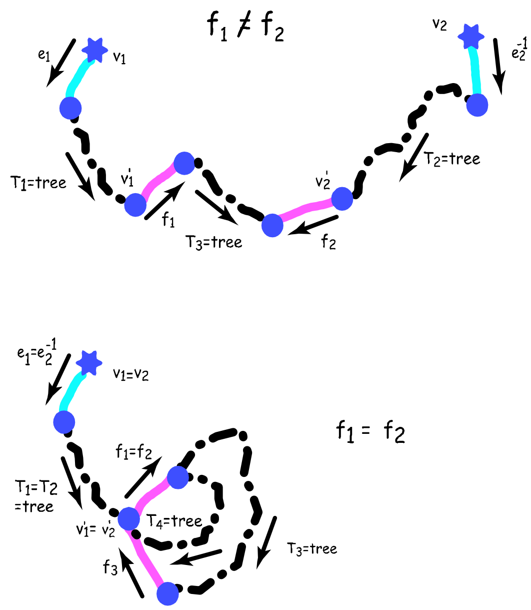

FIGURE 29. The paths in Lemma 4. Here dashed paths are along the spanning tree of X. The edges e_1 and e_2 may not be edges of X which are cut to get the spanning tree T. But f_1, f_2 and (in the second case) f_3 are cut or non-tree edges. The lower figure does not show the most general situation as f_3 need not touch $f_1 = f_2$.

Proof. The poles are reciprocals of the eigenvalues of the W_1 matrix. The singular values of W_1 are $\left\{ q_1, ..., q_n, \underbrace{1, ..., 1}_{2m-n} \right\}$. Assume that $q_j \geq q_{j+1}$. This means that q_j^2 and 1 are successive maxima of the Rayleigh quotient $\frac{{}^t\overline{(W_1 v)} W_1 v}{{}^t\overline{v} v}$ over $v \in \mathbb{C}^n$ orthogonal to the vectors at which the preceding maxima are taken. If $W_1 v = \lambda v$, for non-0 v, then the Rayleigh quotient is λ^2. □

Next recall the Perron-Frobenius Theorem. A proof can be found in Horn and Johnson [40].

Theorem 13. ***(Perron and Frobenius)*** ***Let*** *A be an $s \times s$ matrix all of whose entries are nonnegative. Assume that A is irreducible. Then we have the following facts.*

1) The spectral radius $\rho(A)$ is positive and is an eigenvalue of A which is simple (both in the algebraic and geometric senses). There is a corresponding eigenvector of A all of whose entries are positive.

2) Let $S = \{\lambda_1, ..., \lambda_k\}$ be the eigenvalues of A having maximum modulus. Then

$$S = \left\{ \rho(A) e^{2\pi i a/k} \middle| a = 1, ..., k \right\}.$$

3) The spectrum of A is invariant under rotation by $\frac{2\pi}{k}$.

If $A = W_1$, the spectral radius $\rho(W_1) = R^{-1}$ and the number $k = \Delta = g.c.d.$ of the lengths of the primes of X.

Exercise 41. *Prove the last statement.*

Hint. Part 3) of the Perron-Frobenius theorem implies that $\zeta_X(u)^{-1} = \det(I - W_1 u) = f(u^k)$. By the definition of Δ, we have $\log \zeta_X(u)^{-1} = F(u^\Delta)$. Thus $\pi(m) = 0$ unless Δ divides m.

Now we proceed to prove the Kotani and Sunada Theorem.

Proof of Theorem 4 of Kotani and Sunada.

Let us restate what we are proving. Suppose $q+1$ **is the maximum degree of** X **and** $p+1$ **is the minimum degree of a graph** X.

1) Every pole u **of** $\zeta_X(u)$ **satisfies** $R_X \leq |u| \leq 1$, **with** R_X **from Definition 3, and** $q^{-1} \leq R_X \leq p^{-1}$.

2) For a graph X, **every non-real pole** u **of** $\zeta_X(u)$ **satisfies the inequality** $q^{-1/2} \leq |u| \leq p^{-1/2}$.

3) The poles of ζ_X **on the circle** $|u| = R_X$ **have the form** $R_X e^{2\pi i a/\Delta_X}$, **where** $a = 1, ..., \Delta_X$. **Here** Δ_X **is from Definition 7.**

Proof. The 2nd inequality in Part 1) comes from a result of Frobenius saying that $\rho(W_1) = R^{-1}$ is bounded above and below by the maximum and minimum row sums of W_1, respectively. See Minc [62], p. 24 or Horn and Johnson [40], p. 492.

We know that $R_X \leq |u| \leq 1$ by Corollary 3. Here we will give the Kotani and Sunada proof that $|u| \leq 1$.

If u is a pole of $\zeta_X(u)$ with $|u| \neq 1$, then there is a non-zero vector f so that $\left(I - Au + u^2 Q\right) f = 0$.

We denote the inner product $< f, g >= {}^t\overline{f} g$, for column vectors f, g in $\mathbb{C}^n$. Then

$$0 =< \left(I - uA + u^2 Q\right) f, f >= \|f\|^2 - u < Af, f > +u^2 < Qf, f > .$$

Set $\lambda = \frac{<Af,f>}{\|f\|^2}$, $\mu = \frac{<Df,f>}{\|f\|^2}$, and $D = Q + I$, . So we have $1 - u\lambda + u^2(\mu - 1) = 0$. The quadratic formula gives $u = \frac{\lambda \pm \sqrt{\lambda^2 - 4(\mu-1)}}{2(\mu-1)}$.

Clearly $p \leq \mu \leq q$. We also have $|\lambda| \leq \mu$. To prove this, note that we can make use of the S and T matrices in Proposition 4. Note that $S = (M\ N)$ and $T = (N\ M)$ where M and N have $m = |E|$ columns. Then it is an Exercise using Proposition 4 to show that

$$D - A = (M - N)\,{}^t(M - N) \text{ and } D + A = (M + N)\,{}^t(M + N).$$

So $|< Af, f >| \leq < Df, f >$. It follows that $|\lambda| \leq \mu$.

There are now two cases.

Case 1. **The pole u is real.**

Then

$$\frac{\lambda + \sqrt{\lambda^2 - 4(\mu - 1)}}{2(\mu - 1)} \leq \frac{\mu + \sqrt{\mu^2 - 4(\mu - 1)}}{2(\mu - 1)} = 1$$

and

$$\frac{\lambda - \sqrt{\lambda^2 - 4(\mu - 1)}}{2(\mu - 1)} \geq \frac{-\mu - \sqrt{\mu^2 - 4(\mu - 1)}}{2(\mu - 1)} = -1.$$

Thus $|u| \leq 1$.

Case 2. **The pole u is not real.**

Then

$$|u|^2 = \frac{\lambda^2 + \left(4(\mu - 1) - \lambda^2\right)}{4(\mu - 1)^2} = \frac{1}{\mu - 1}.$$

The second statement of Theorem 4 follows from this and $p \leq \mu \leq q$.

Finally the third part of Theorem 4 follows from the Perron - Frobenius Theorem 13. □

12. Path Zeta Functions

Here we look at a zeta function invented by Stark. It has several advantages over the edge zeta. It can be used to compute the edge zeta with smaller determinants. It gives the edge zeta for a graph in which an edge has been fused; i.e., shrunk to one vertex.

First recall that the fundamental group of X can be identified with the group generated by the edges left out of a spanning tree T of X. Then T has $|V| - 1 = n - 1$ edges. We call the oriented versions of these **edges left out of the spanning tree** T **(**or **"deleted" edges** of T**)** (and their inverses)

$$e_1, \ldots, e_r, e_1^{-1}, \ldots, e_r^{-1}.$$

Call the remaining (oriented) **edges in the spanning tree** T

$$t_1, \ldots, t_{n-1}, t_1^{-1}, \ldots, t_{n-1}^{-1}.$$

Any backtrackless, tailless cycle on X is uniquely (up to starting point on the tree beteen last and first e_k) determined by the ordered sequence of e_k's it passes through. In particular, if e_i and e_j are 2 consecutive e_k's in this sequence, then the part of the cycle between e_i and e_j is the unique backtrackless path on T joining the last vertex of e_i to the first vertex of e_j. For such e_i and e_j, we know that e_j is not the inverse of e_i, as the cycle is backtrackless. Nor is the last edge the inverse of the first. Conversely, if we are given any ordered sequence of edges from the e_k's with no 2 consecutive edges being inverses of each other and with the last edge not being inverse to the 1st edge, there is a unique (up to starting point on the tree between the last and first e_k) backtrackless tailless cycle on X whose sequence of e_k's is the given sequence.

The free group of rank r generated by the e_k's puts a group structure on backtrackless tailless cycles which is completely equivalent to the fundamental group of X. When dealing with the fundamental group of X, any closed path starting at a fixed vertex v_0 on X is completely determined up to homotopy by the ordered sequence of e_k's that it passes through. If we do away with backtracking, such a path will be composed of a tail on the tree and then a backtrackless, tailless cycle corresponding to the same sequence of e_k's, followed by the original tail in the reverse direction, ending at v_0 again. Thus the free group of rank

r generated by the e_k's is identified with the fundamental group of X. We will therefore refer to the free group generated by the e_k's as the **fundamental group** of X.

There are 2 elementary reduction operations for paths written down in terms of directed edges just as there are elementary reduction operations for words in the fundamental group of X. This means that if $a_1, ..., a_s$ and e are taken from the e_k's and their inverses, the **2 elementary reduction operations** are:

i) $a_1 \cdots a_{i-1} e e^{-1} a_{i+2} \cdots a_s \cong a_1 \cdots a_{i-1} a_{i+2} \cdots a_s$;

ii) $a_1 \cdots a_s \cong a_2 \cdots a_s a_1$.

Using the 1st elementary reduction operation, each equivalence class of words corresponds to a group element and a word of minimum length in an equivalence class is **reduced** word in group theory language. Since the second operation is equivalent to conjugating by a_1, an equivalence class using both elementary reductions corresponds to a conjugacy class in the fundamental group. A word of minimum length using both elementary operations corresponds to finding words of minimum length in a conjugacy class in the fundamental group. If $a_1, ..., a_s$ are taken from $e_1, ..., e_{2r}$, a word $C = a_1 \cdots a_s$ is of minimum length in its conjugacy class iff $a_{i+1} \neq a_i^{-1}$, for $1 \leq i \leq s-1$ and $a_1 \neq a_s^{-1}$. This is equivalent to saying that C corresponds to a **backtrackless, tailless** cycle under the correspondence above. Equivalent cycles correspond to conjugate elements of the fundamental group. A conjugacy class $[C]$ is **primitive**if a word of minimal length in $[C]$ is not a power of another word. We will say that a word of minimal length in its conjugacy class is **reduced in its conjugacy class**. From now on, we assume a representative element of $[C]$ is chosen which is reduced in $[C]$.

Definition 26. *The $2r \times 2r$ **path matrix** Z has ij entry given by the complex variable z_{ij} if $e_i \neq e_j^{-1}$ and by 0 if $e_i = e_j^{-1}$.*

Note that the path matrix Z has only one zero entry in each row unlike the edge matrix W from Definition 23 which is rather sparse. Next we imitate the definition of the edge zeta function.

Definition 27. *Define the **path norm** for a primitive path $C = a_1 \cdots a_s$ reduced in its conjugacy class $[C]$, where $a_i \in \left\{e_1^{\pm 1}, \ldots, e_s^{\pm 1}\right\}$ as*

$$N_P(C) = z_{a_1 a_2} \cdots z_{a_{s-1} a_s} z_{a_s a_1}.$$

*Then the **path zeta** is defined for small $\left|z_{ij}\right|$ to be*

$$\zeta_P(Z, X) = \prod_{[C]} (1 - N_F(C))^{-1},$$

where the product is over primitive reduced conjugacy classes $[C]$ other than the identity class.

We have similar results to those for the edge zeta.

Theorem 14.

$$\zeta_P(Z, X)^{-1} = \det (I - Z).$$

Proof. Imitate the proof of Theorem 11 for the edge zeta. □

Next we want to find a way to get the edge zeta out of the path zeta. To do this requires a procedure called **specializing the path matrix to the edge matrix.** Use the notation above for the edges e_i left out of the spanning tree T and the edges t_j of T. A prime cycle C is first written as a product of generators of the fundamental group and then as a product of actual edges e_i and t_k. Do this by inserting $t_{k_1} \cdots t_{k_s}$ which is the unique non backtracking path on T joining the terminal vertex of e_i and the starting vertex of e_j if e_i and e_j are successive deleted or cut edges in C. Now **specialize the path matrix Z to $Z(W)$ with entries**

$$z_{ij} = w_{e_i t_{k_1}} w_{t_{k_1} t_{k_2}} \cdots w_{t_{k_{n-1}} t_{k_s}} w_{t_{k_s} e_j}. \tag{12.1}$$

Then the path zeta function at $Z(W)$ specializes to the edge zeta function

Theorem 15. *Using the specialization procedure defined above, we have*

$$\zeta_P(Z(W), X) = \zeta_E(W, X).$$

Proof. The result should be clear since the two defining infinite products coincide. □

M. Horton [42] has a Mathematica program to do the specialization in formula (12.1).

Note that $\zeta_P(Z, X) = \zeta_E(Z, X^\#)$, where $X^\#$ is the graph obtained from X by fusing all the edges of the spanning tree T to a point. Thus $X^\#$ consists of one vertex and r loops.

Example 11. ***The Dumbbell Again***

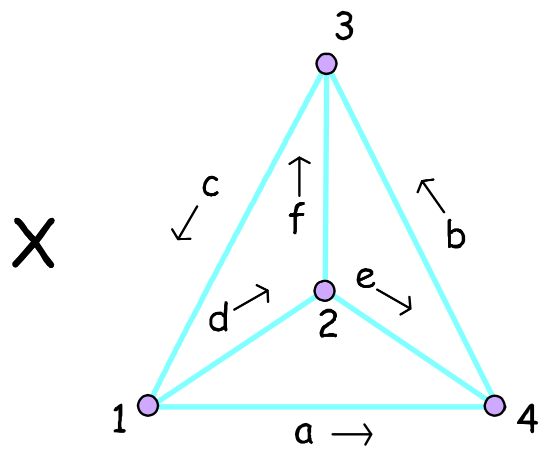

FIGURE 30. Labelling the edges of the tetrahedron.

Recall that the edge zeta of the dumbbell graph of Figure 28 was evaluated by a 6×6 determinant. The path zeta requires a 4×4 determinant. Take the spanning tree to be the vertical edge. That is really the only choice here. One finds using the determinant formula for the path zeta and the specialization of the path to edge zeta:

$$\zeta_E(W,X)^{-1} = \det \begin{pmatrix} w_{11}-1 & w_{12}w_{23} & 0 & w_{12}w_{26} \\ w_{35}w_{51} & w_{33}-1 & w_{35}w_{54} & 0 \\ 0 & w_{42}w_{23} & w_{44}-1 & w_{42}w_{26} \\ w_{65}w_{51} & 0 & w_{65}w_{54} & w_{66}-1 \end{pmatrix}. \quad (12.2)$$

If we shrink the vertical edge to a point (which we call **"fusion"**or contraction), the edge zeta of the new graph is obtained by replacing any $w_{x2}w_{2y}$ (for $x, y = 1, 3, 4, 6$) which appear in formula (12.2) by w_{xy} and any $w_{x5}w_{5y}$ (for $x, y = 1, 3, 4, 6$) by w_{xy}. This gives the zeta function of the new graph obtained from the dumbbell, by fusing the vertical edge.

Example 12. ***The Path Zeta Function of the Tetrahedron Specializes to the Edge Zeta Function of the Tetrahedron.***

Refer to Figure 30 and label the inverse edges with the corresponding capital letters. List the edges that index the entries of the matrix Z as a, b, c, A, B, C. You will then find that the matrix $Z(W)$ for the tetrahedron is

$$\begin{pmatrix} w_{aE}w_{ED}w_{Da} & w_{ab} & w_{aE}w_{Ef}w_{fc} & 0 & w_{aE}w_{Ef}w_{fB} & w_{aE}w_{ED}w_{DC} \\ w_{bF}w_{FD}w_{Fa} & w_{bF}w_{Fe}w_{eb} & w_{bc} & w_{bF}w_{Fe}w_{eA} & 0 & w_{bF}w_{FD}w_{DC} \\ w_{ca} & w_{cd}w_{de}w_{eb} & w_{cd}w_{df}w_{fc} & w_{cd}w_{de}w_{eA} & w_{cd}w_{df}w_{fB} & 0 \\ 0 & w_{Ad}w_{de}w_{eb} & w_{Ad}w_{df}w_{fc} & w_{Ad}w_{de}w_{eA} & w_{Ad}w_{df}w_{fB} & w_{AC} \\ w_{BE}w_{ED}w_{Da} & 0 & w_{BE}w_{Ef}w_{fc} & w_{BA} & w_{BE}w_{Ef}w_{fB} & w_{BE}w_{ED}w_{DC} \\ w_{CF}w_{FD}w_{Da} & w_{CF}w_{Fe}w_{eb} & 0 & w_{CF}w_{Fe}w_{eA} & w_{CB} & w_{CF}w_{FD}w_{DC} \end{pmatrix}.$$

Exercise 42. *As a check on the preceding example, specialize all the variables in the $Z(W)$ matrix to $u \in \mathbb{C}$ and call the new matrix $Z(u)$. Check that $\det(I - Z(u))$ is the reciprocal of the Ihara zeta function $\zeta_X(u)$.*

Exercise 43. *Compute the path zeta function for your favorite graph.*

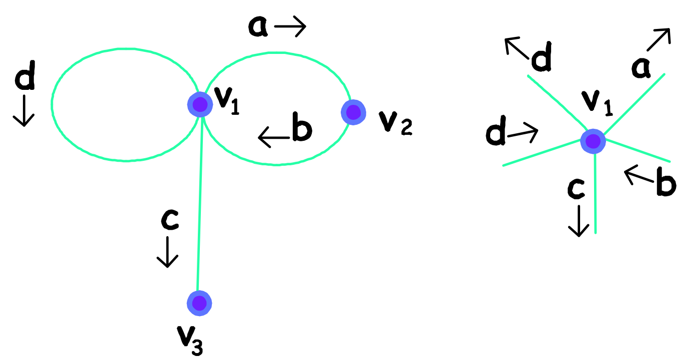

FIGURE 31. a directed graph and a neighborhood of vertex. v_1

Part 4. **Finite Unramified Galois Coverings of Connected Graphs**

As usual we assume that the connected graph X has vertex set V and (undirected) edge set E, possibly iregular and possibly having loops and multiple edges. All the coverings considered here will be unramified.

We will usually assume our graphs are not weighted. Coverings of weighted graphs have been considered by Chung and Yau [20] as well as Osborne and Severini [66]. The latter paper applies graph coverings to quantum computing.

13. Unramified Coverings and Galois Groups

In this section we begin the study of Galois theory for covering graphs. It leads to a generalization of Cayley and Schreier graphs and it provides factorizations of zeta functions of normal coverings into products of Artin L-functions associated to representations of the Galois group of the covering. Coverings can also be used in constructons of Ramanujan graphs and in constructions of pairs of graphs that are isospectral but not isomorphic. Most of this things we cover are taken from from Stark and Terras [84]. Other references are Sunada [90] and Hashimoto [35]. Another theory of graph covering which is essentially equivalent can be found in Gross and Tucker [32]. Our coverings differ in that we require all our graphs to be connected.

First we need to think about directed coverings since we want to prove the fundamental theorem of Galois theory. The one third in the next definition could be replaced by any $\varepsilon > 0$.

Definition 28. *A **neighborhood** N of a vertex v in a directed graph X is obtained by taking one–third of each edge at v. The labels and directions are to be included. See Figure 31.*

Definition 29. *An undirected finite graph Y is a **covering** of an undirected graph X if, after arbitrarily directing the edges of X, there is an assignment of directions to the edges of Y and an onto **covering map** $\pi : Y \longrightarrow X$ sending neighborhoods of Y 1-1, onto neighborhoods of X preserving directions.*

The fact that Y is a covering of X is independent of the choice of directions on X. See Figure 32 for an example of an invalid assignment of directions in Y over X. Note also that if you lift a loop you may get a graph with multiple edges. Thus once you allow loops, you cannot discuss the general covering without allowing multiple edges.

Definition 30. *If Y/X is a d-sheeted covering with projection map $\pi : Y \longrightarrow X$, we say that it is a **normal covering** when there are d graph automorphisms $\sigma : Y \longrightarrow Y$ such that $\pi \circ \sigma = \pi$. The Galois group $G(Y/X)$ is the set of these maps σ.*

To explain roughly how to construct a covering Y of a connected graph X with d vertices, first find a spanning tree T in X. For a d-fold covering, make d copies of T. This gives the nd vertices of our graph Y. That is, Y can be viewed as the set of points (x, i), $i = 1, ..., d$. Then lift to Y the edges of X left out of T to get edges of Y. Later we will see that if we want to make Y a normal cover of X, with Galois group G,then if we are given a permutation representation π of G, then π will tell us how to lift edges. See Figure 33.

Let us consider a few basic examples.

Example 13. ***The cube is a normal quadratic covering of the tetrahedron.***

See Figure 34, where the edges in a spanning tree for X are shown as pink dotted lines. The edges of the corresponding two sheets of Y are also shown as dotted lines.

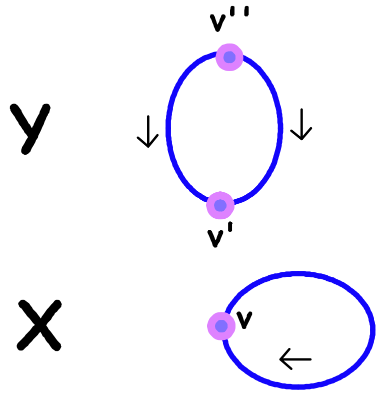

FIGURE 32. example of an illegal covering map

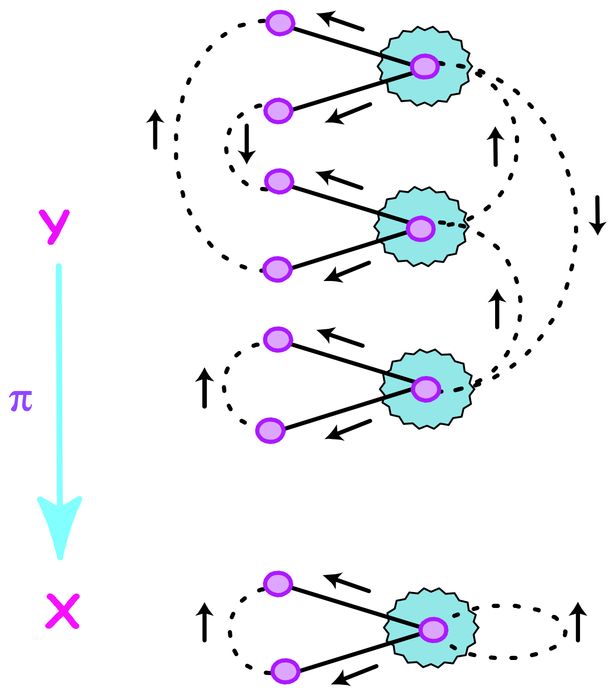

FIGURE 33. A 3-sheeted covering. The blue fuzzy area in X is a neighborhood of a selected vertex. The 3 blue fuzzies in Y are all of its inverse images under π .

Exercise 44. *Create another 2-cover of K_4 using the same spanning tree as we used in Figure 34, except this time when you lift the 3 non-tree edges of K_4, arrange it so that the lift of only 1 non-tree edge goes from sheet 1 to sheet 2 while the other 2 lifts do not change sheet.*

Example 14. ***A Non-Normal Cubic Covering of K_4.***
See Figure 35.

Exercise 45. *Explain why the 3-sheeted covering in Figure 35 is not a normal covering of the tetrahedron. Hint: a' is adjacent to b' but a'' is not adjacent to b''.*

Even non-normal coverings have the nice property that the inverse zeta below divides the inverse zeta above.

Proposition 5. ***Zeta Functions of Covers.***
Suppose Y is a d-sheeted (possibly non-normal) covering of X. Then $\zeta_V(u,X)^{-1}$ divides $\zeta_V(u,Y)^{-1}$.

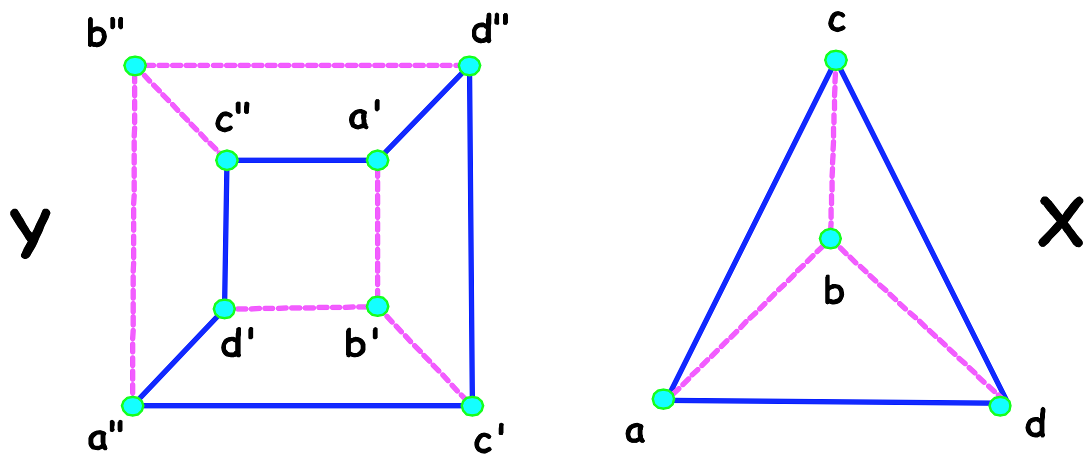

FIGURE 34. The cube is a normal quadratic covering of the tetrahedron. The 2 sheets of Y are copies of the spanning tree in X pictured with pink dotted lines.

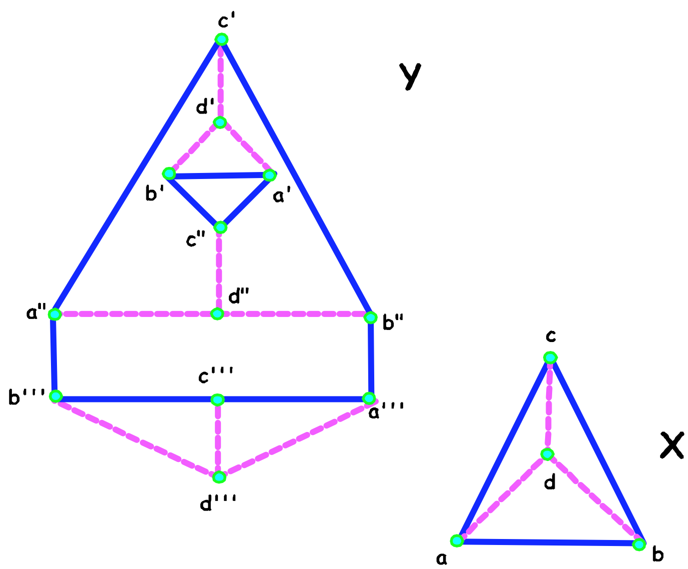

FIGURE 35. A non-normal cubic (3-sheeted) covering of the tetrahedron. The spanning tree in $X = K_4$ is shown with dashed red lines. The sheets of the covering Y are similarly colored.

Proof. Start with the Ihara formula $\zeta_V(u, Y)^{-1} = (1-u^2)^{r_Y - 1} \det \left(I_Y - A_Y u + Q_Y u^2 \right)$. Note that $r_Y - 1 = |E_Y| - |V_Y| = d\left(|E_X| - |V_X|\right)$. Thus $(1-u^2)^{r_X - 1}$ divides $(1-u^2)^{r_Y-1}$.

Now order the vertices of Y in blocks corresponding to the sheets of the cover, so that A_Y consists of blocks $\widetilde{A}_{ij}$, with $1 \leq i, j \leq d$ such that $\sum_j \widetilde{A}_{ij} = A$.

The same ordering puts Q_Y in block diagonal form with n copies of Q_X down the diagonal. Similarly I_Y has block diagonal form consisting of d copies of I_X down the diagonal.

Consider $I_Y - A_Y u + Q_Y u^2$. Without changing the determinant, we can add the right $d-1$ block columns to the first block column. The new first column is

$$\begin{pmatrix} I_X - A_X u + Q_X u^2 \\ \vdots \\ I_X - A_X u + Q_X u^2 \end{pmatrix}.$$

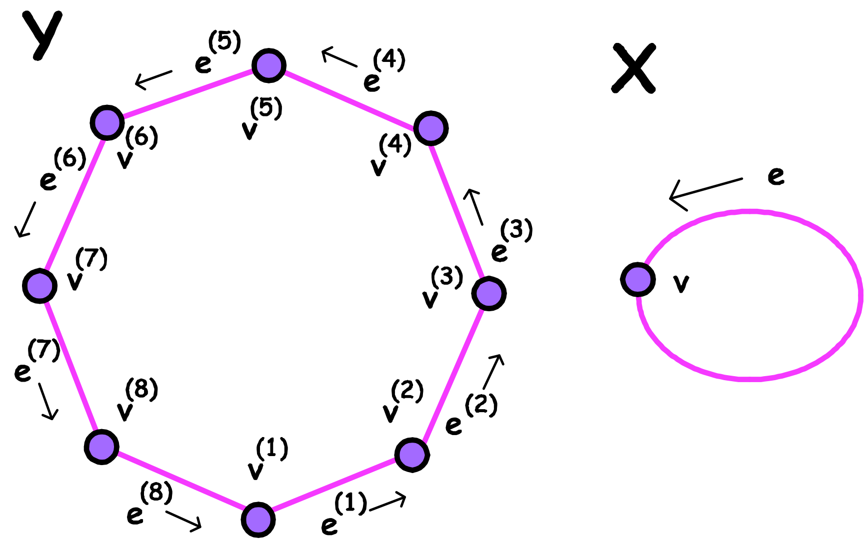

FIGURE 36. An n-cycle is a normal n-fold covering of a loop with cyclic Galois group.

Then subtract the first block row from all the rest of the block rows. Then the first block column becomes:

$$\begin{pmatrix} I_X - A_X u + Q_X u^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The proposition follows. □

Proposition 6. *Suppose Y/X is a normal covering. The Galois group $G = G(Y/X)$ acts transitively on the sheets of the covering.*

Proof. Every path in X has a unique lift once the initial vertex in Y is fixed - even when there are loops and multiedges assuming that you have assigned a direction to all edges. So now each spanning tree has a unique lift starting at any point in $\pi^{-1}(v_0)$, where v_0 is a fixed point in X. These d lifts are the sheets of the covering.

An automorphism $\sigma \in G$ that fixes a point $\widetilde{v_0} \in \pi^{-1}(v_0)$ is the identity. To see this, suppose $\widetilde{v}$ is any point in Y. Then a path $\widetilde{P}$ from $\widetilde{v_0}$ to $\widetilde{v}$ in Y projects under π to a path from v_0 to $v = \pi(\widetilde{v})$ in X and this path has a unique lift starting at $\widetilde{v_0}$ which must be the original path $\widetilde{P}$. So if $\widetilde{v_0} = \sigma(\widetilde{v_0})$ then $\widetilde{v} = \sigma(\widetilde{v})$ by the uniqueness of lifting starting at a given sheet. Thus σ must be the identity.

So each distinct $\sigma \in G$ takes $\widetilde{v_0}$ to a different point and there are only d different points in Y above v_0. It follows that the action of G is transitive. Otherwise two different automorphisms would take v_0 to the same point and we just showed that is impossible. □

Notation 3. *We choose one of the sheets of Y and call it sheet 1. The image of sheet 1 under an element g in G will be called **sheet** g. Any vertex $\widetilde{x}$ on Y can then be uniquely denoted $\widetilde{x} = (x, g)$, where $x = \pi(\widetilde{x})$ and g is the sheet containing $\widetilde{x}$.*

Definition 31. ***Action of the Galois Group*** *The Galois group $G(Y/X)$ moves sheets of Y via $g \circ (sheet\ \ h) = sheet(gh)$:*

$$g \circ (x, h) = (x, gh), \quad for \;\; x \in X, g, h \in G$$

It follows that g moves a path in Y as follows:

(13.1) $$g \circ (\text{path from } (a, h) \text{ to } (b, j)) = \text{path from } (a, gh) \text{ to } (b, gj).$$

Example 15. ***An n-cycle is a normal n-fold covering of a loop with cyclic Galois group.*** *See Figure 36 for this example.*

The Ihara zeta functionn of the loop X in Figure 36 is $\zeta_X(u) = (1-u)^{-2}$, and the zeta function of the n-cycle is $\zeta_Y(u) = (1-u^n)^{-2}$. Thus

$$\zeta_Y(u) = (1-u^n)^{-2} = \prod_{j=0}^{n-1} (1 - w^j u)^{-2}, \quad \text{where} \quad w = e^{2\pi i/n}.$$

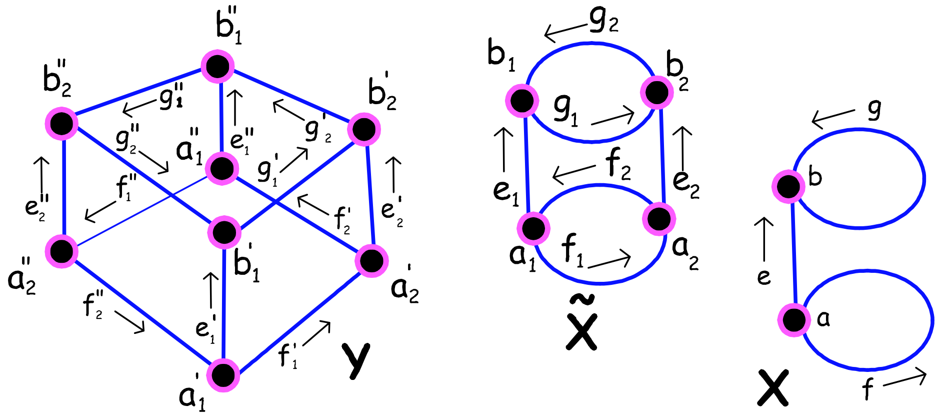

FIGURE 37. An order 4 cyclic cover Y/X, where Y is the cube. Included is the intermediate quadratic cover $\widetilde{X}$. The notation makes clear the covering projections $\pi : Y \longrightarrow X$, $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi_1 : \widetilde{X} \longrightarrow X$,

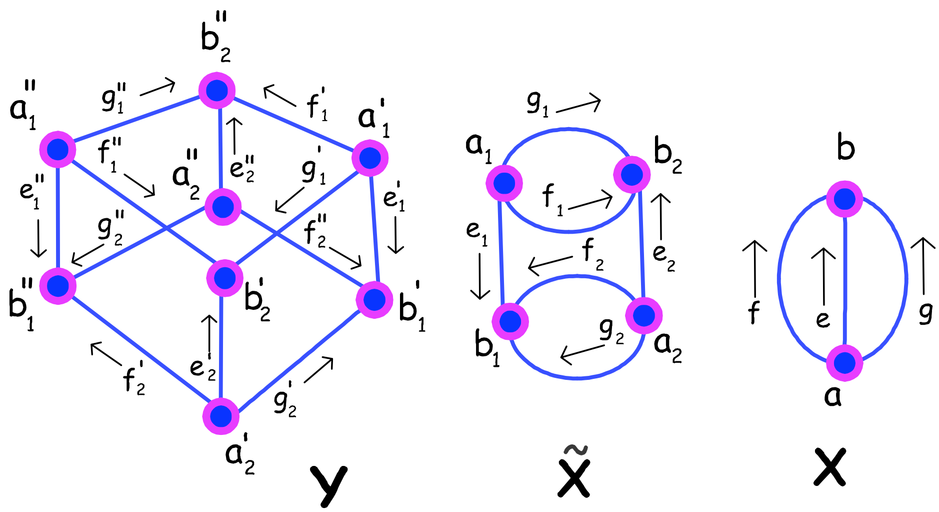

FIGURE 38. A Klein 4-group Cover Y/X, where Y = the cube. Included is one of the 3 intermediate quadratic covers.

This factorization of $\zeta_Y(u)$ will later be seen as a special case of the factorization of the Ihara zeta function of Y into a product of Artin L-functions associated to representations of the Galois group of Y/X.

Question. **Suppose a graph Y has a large symmetry group S and G is a subgroup of S. Is there a graph X such that Y is a normal cover of X with group G?**

Answer. Not always. For example, the cube has S_4 symmetry group - a group of order 6. But S_4 cannot be the Galois group $G(Y/X)$. Why? If Y/X were a Galois cover with 6 sheets, it would follow that 6 divides the number of vertices (and edges) of Y . But the cube has 8 vertices. It would also mean that 6 divides $r_Y - 1 = |E| - |V| = 12 - 8 = 4$.

Example 16. *Two graphs with the cube as a normal cover. See Figures 37 and 38 for these examples.*

Let Y be the cube. Then $|V| = 8$, $|E| = 12$ and $|G|$ divides $g.c.d.(8, 12) = 4$. Figure 37 is a normal covering Y/X where the cube=Y such that $G = G(Y/X)$ is a cyclic group of order 4. Figure 38 is another such covering Y/X in where $G = G(Y/X)$ is the Klein 4-group. We include in the figures an intermediate quadratic cover in each case. The concept of intermediate cover will be discussed in the next section.

Example 17. *The Octahedron as a Cyclic 6-fold Cover of 2 Loops.*

The octahedron has $|V| = 6$, $|E| = 12$ which implies that $|G| = 6$ may be possible. An example where X is a double loop is given in Figure 39.

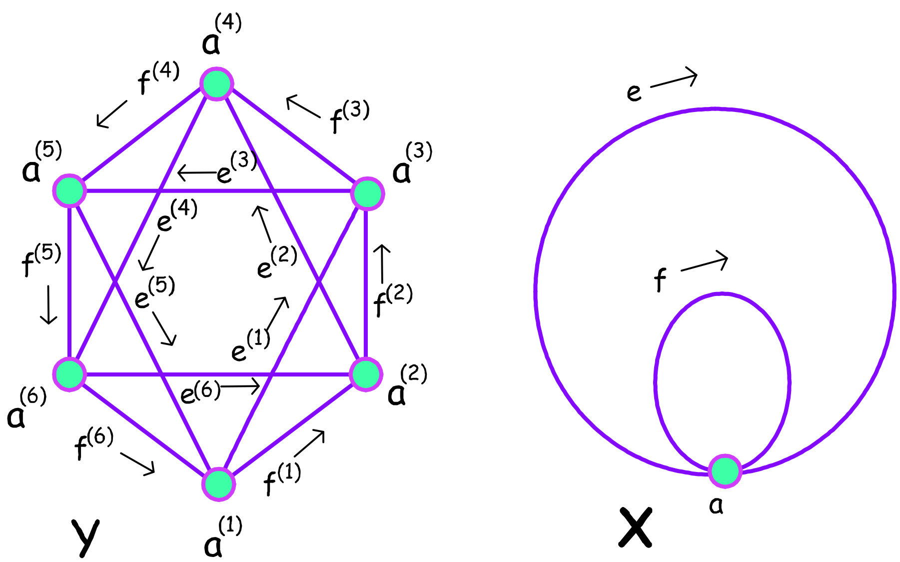

FIGURE 39. A Cyclic 6-fold cover Y/X, where Y is the octahedron.

Exercise 46. *Motivated by the example of a ramified point on a Riemann surface (e.g. the origin in the Riemann surface of the function $w = \sqrt{z}$), give an example of a ramified covering Y of a loop X on one vertex. Does the inverse zeta of X divide the inverse zeta of Y ?*

14. Fundamental Theorems of Galois Theory

Question. What does it mean to say that $\widetilde{X}$ is intermediate to Y/X? In order to be able to prove the fundamental theorem of graph Galois theory giving a 1-1 correspondence between subgroups H of the Galois group G of Y/X and intermediate graphs $\widetilde{X}$ to Y/X, we need a definition which is stronger than just saying $Y/\widetilde{X}$ is a covering and $\widetilde{X}/X$ is a covering. To see this, consider Figure 38 where there could be 3 intermediate graphs like the one drawn corresponding to the same subgroup of the Galois group. That would say that the fundamental theorem of Galois theory is false for graph coverings. To avoid this problem, we make the following definition.

Exercise 47. *Draw the other 2 fake intermediate graphs for Figure 38.*

Definition 32. *Suppose that Y is a covering of X with projection map π. A graph $\widetilde{X}$ is an **intermediate covering** to Y/X if $Y/\widetilde{X}$ is a covering and $\widetilde{X}/X$ is a covering and the projection maps $\pi_1 : \widetilde{X} \longrightarrow X$ and $\pi_2 : Y \longrightarrow \widetilde{X}$ have the property that $\pi = \pi_1 \circ \pi_2$.*

See Figure 40. Technically, it is the triple $(\widetilde{X}, \pi_1, \pi_2)$ that gives the intermediate covering.

A second definition will tell us when two intermediate graphs are to be considered the same or equal

Definition 33. *Let $\widetilde{X}$ and $\widetilde{X}'$ be intermediate to Y/X with projection maps as in Figure 40. Suppose i is an isomorphism between $\widetilde{X}$ and $\widetilde{X}'$. If the notation is as in Figure 40, and we have $\pi_1 = \pi'_1 \circ i$, then we say i is a **covering isomorphism**. We say that $\widetilde{X}$ and $\widetilde{X}'$ are **covering isomorphic**. If, in addition, we have $i \circ \pi_2 = \pi'_2$, we say that $\widetilde{X}$ and $\widetilde{X}'$ are **the same** or **equal**.*

Now we can prove the fundamental theorem. Note that most of these proofs are based on the uniqueness of lifts of paths in X to paths in the covering graph Y starting on a given sheet of Y.

Theorem 16. ***Fundamental Theorem of Galois Theory.*** *Suppose Y/X is an unramified normal covering with Galois group $G = G(Y/X)$.*

1) Given a subgroup H of G, there exists a graph $\widetilde{X}$ intermediate to Y/X such that $H = G(Y/\widetilde{X})$. Write $\widetilde{X} = \widetilde{X}(H)$.

2) Suppose $\widetilde{X}$ is intermediate to Y/X. Then there is a subgroup $H = H(\widetilde{X})$ of G which is $G(Y/\widetilde{X})$.

3) Two intermediate graphs $\widetilde{X}$ and $\widetilde{X}'$ are equal (as in Definition 33) if and only if $H(\widetilde{X}) = H(\widetilde{X}')$.

4) We have $H(\widetilde{X}(H)) = H$ and $\widetilde{X}(H(\widetilde{X})) = \widetilde{X}$. So we write $\widetilde{X} \leftrightarrow H$ for the correspondence between intermediate graphs $\widetilde{X}$ to Y/X and subgroups H of the Galois group $G = G(Y/X)$.

5) If $\widetilde{X_1} \leftrightarrow H_1$ and $\widetilde{X_2} \leftrightarrow H_2$ then $\widetilde{X_1}$ is intermediate to $Y/\widetilde{X_2}$ iff $H_1 \subset H_2$.

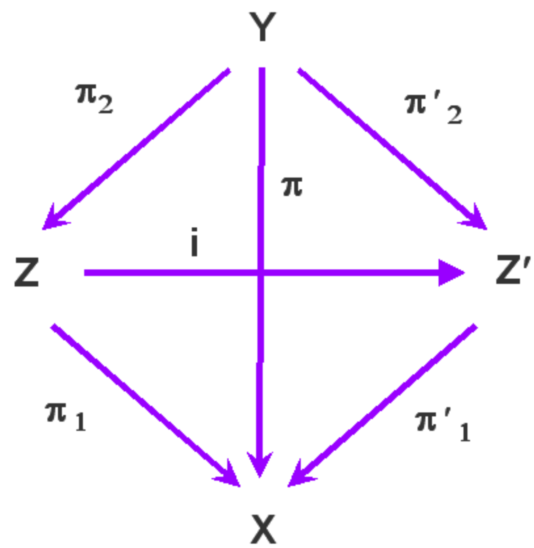

FIGURE 40. A covering isomorphism i of intermediate graphs.

Proof. **Part 1)** Let H be a subgroup of G. Points of Y are of the form (x, g), with $x \in X$ and $g \in G$. Define the vertices of $\widetilde{X}$ to be (x, Hg) for $x \in X$, and coset $Hg \in H\backslash G$. Put an edge between (a, Hr) and (b, Hs), for $a, b \in X$ and $r, s \in G$ iff there are $h, h' \in H$ such that (a, hr) and $(b, h's)$ have an edge joining them above in Y.

The edge between (a, Hr) and (b, Hs) in $\widetilde{X}$ is given the label and direction of the projected edge between a and b in X.

Exercise 48. *Show that $\widetilde{X}$ is well-defined, intermediate to Y/X and connected.*

Part 2) Let $\widetilde{X}$ be intermediate tp Y/X, with projections $\pi : Y \longrightarrow X$, $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi_1 : \widetilde{X} \longrightarrow X$. Fix a vertex $v_0 \in X$ with $\widetilde{\widetilde{v_0}} \in \pi^{-1}(v_0)$ on sheet 1 of Y. That is, $\widetilde{\widetilde{v_0}} = (v_0, 1)$ using our labeling of sheets of Y. Let $\widetilde{v_0} = \pi_2(\widetilde{\widetilde{v_0}}) \in \widetilde{X}$. Define

$$H = \left\{ h \in G \;\middle|\; h(\widetilde{\widetilde{v_0}}) \in \pi_2^{-1}(\widetilde{v_0}) \right\} = \{ h \in G \mid \pi_2(v_0, h) = \pi_2(v_0, 1) \} . \tag{14.1}$$

To see that H is a subgroup of G, we must show that H is closed under multiplication . Let h_1 and h_2 be elements of H. Then, by the definition of H, the vertices $\pi_2(v_0, h_1) = \pi_2(v_0, h_2) = \pi_2(v_0, 1) = \widetilde{v_0}$.

Let $\widetilde{\widetilde{p_1}}$ and $\widetilde{\widetilde{p_2}}$ be paths on Y from $(v_0, 1)$ to the vertices (v_0, h_1) and (v_0, h_2), respectively. Then $\widetilde{\widetilde{p_1}}$ and $\widetilde{\widetilde{p_2}}$ project under π_2 to closed paths $\widetilde{p_1}$ and $\widetilde{p_2}$ in $\widetilde{X}$ beginning and ending at $\widetilde{v_0}$. And $\widetilde{\widetilde{p_1}}$ and $\widetilde{\widetilde{p_2}}$ project under $\pi = \pi_1 \circ \pi_2$ to closed paths p_1 and p_2 in X beginning and ending at v_0.

By formula (13.1), $h_1 \circ \widetilde{\widetilde{p_2}}$ starts at (v_0, h_1) and ends at $(v_0, h_1 h_2)$. Thus the lift of $\widetilde{p_1}\widetilde{p_2}$ from $\widetilde{X}$ to Y beginning at $(v_0, 1)$, which is the same as the lift of $p_1 p_2$ from X to Y beginning at $(v_0, 1)$, ends at $(v_0, h_1 h_2)$. It follows that $h_1 h_2$ is in H and H is a subgroup of G.

Part 4). Let $\widetilde{X}$ be intermediate to Y/X. We want to prove that $\widetilde{X}(H(\widetilde{X})) = \widetilde{X}$, with the definitions from the proofs of Parts 1 and 2.

Before attempting to prove the equality, we ned to prove a chacterization of $\pi_2^{-1}(\widetilde{v})$ for any vertex $\widetilde{v}$ of $\widetilde{X}$. This says that there is an element $g_v \in G$ such that

$$\pi_2^{-1}(\widetilde{v}) = \{ (v, hg_v) \mid h \in H \} . \tag{14.2}$$

Let v_0 be the fixed vertex of X from the definition of H in the proof of Part 1. Let $\widetilde{q}$ be a path in $\widetilde{X}$ from $\widetilde{v_0}$ to $\widetilde{v}$. There is also a lift $\widetilde{\widetilde{q}}$ of $\widetilde{q}$ to Y starting at $(v_0, 1)$ and ending at (v, g_v). Write $\widetilde{\widetilde{v}} = (v, g_v)$ in Y with $\widetilde{v} = \pi_2(\widetilde{\widetilde{v}}) \in \widetilde{X}$ and $\pi(\widetilde{\widetilde{v}}) = v$. Projected down to X, we get the path q from v_0 to v.

Look at Figure 41. For $h \in G$, by equation (13.1), the path $\widetilde{q}$ in $\widetilde{X}$ lifts to a path $h \circ \widetilde{\widetilde{q}}$ from (v_0, h) to (v_0, hg_v) in Y. Thus, by the uniqueness of lifts, starting on a given sheet, we must have $\pi_2 \circ h \circ \widetilde{\widetilde{q}} = \widetilde{q}$ if and only if the initial sheet of the lift of q is that of $\widetilde{v_0}$. That is, $\pi_2 \circ h \circ \widetilde{\widetilde{q}} = \widetilde{q}$ iff $h \in H$. This proves formula (14.2).

Now we seek to show that $\widetilde{X}' = \widetilde{X}(H(\widetilde{X})) = \widetilde{X}$. Recall that $\widetilde{X}' = \widetilde{X}(H(\widetilde{X}))$ has vertices (x, Hg) and projections $\pi'_2(x, g) = (x, Hg)$ and $\pi'_1(x, Hg) = x$. Define $i : \widetilde{X} \to \widetilde{X}'$ by $i(\widetilde{v}) = (v, Hg_v)$ using the element $g_v \in G$ from formula (14.2).

Exercise 49. *Prove that i is a graph isomorphism and $i \circ \pi_2 = \pi'_2$, $\pi'_1 \circ i = \pi_1$.*

To complete the proof of Part 4), we must show that $H(\widetilde{X}(H)) = H$. By our definitions made in the proof of Parts 1 and 2, we have
$H(\widetilde{X}(H)) = \{g \in G \mid \pi_2(v_0, g) = \pi_2(v_0, 1)\} = \{g \in G \mid (v_0, Hg) = (v_0, H)\} = H.$

Part 5). Suppose $\pi_2 : Y \to X_1$ and $\pi_1 : X_1 \to X_2$ with $\pi_3 = \pi_1 \circ \pi_2 : Y \to X_2$. Then by the proof of Part 2),

$$\begin{aligned} H(X_1) &= H_1 = \{h \in G \mid \pi_2(v_0, h) = \pi_2(v_0, 1)\}, \\ H(X_2) &= H_2 = \{h \in G \mid \pi_3(v_0, h) = \pi_3(v_0, 1)\}. \end{aligned}$$

Since $\pi_3 = \pi_1 \circ \pi_2$, it follows that $H_1 \subset H_2$.

For the converse, suppose that $H_1 \subset H_2$. Then we have the intermediate graphs $\widetilde{X_i}$ with vertices $(x, H_i\sigma)$ for $x \in X$, and coset $H_i\sigma \in H_i\backslash G$. There is an edge between $(a, H_i\sigma)$ and $(b, H_i\tau)$, for $a, b \in X$ and $\sigma, \tau \in G$ iff there are $h, h' \in H_i$ such that $(a, h\sigma)$ and $(b, h'\tau)$ have an edge in Y. We need to show that $\pi_2 : Y \to X_1$ and $\pi_1 : X_1 \to X_2$ with $\pi_3 = \pi_1 \circ \pi_2 : Y \to X_2$. Here $\pi_2(v, g) = (v, H_1 g)$ and $\pi_3(v, g) = (v, H_2 g)$, for $v \in X, g \in G$. Then since $H_1 \subset H_2$, we see that $\pi_1(v, H_1 g) = (v, H_2 g)$ makes sense as $H_1 a = H_1 b$ iff $ab^{-1} \in H_1 \subset H_2$ implies $H_2 a = H_2 b$.

Parts 3). Suppose we have 2 intermediate graphs $\widetilde{X}$ and $\widetilde{X}'$ to Y/X with the projections $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi_1 : \widetilde{X} \longrightarrow X$ and $\pi'_2 : Y \longrightarrow \widetilde{X}'$, $\pi'_1 : \widetilde{X}' \longrightarrow X$. Set $H = H(\widetilde{X})$ and $H' = H(\widetilde{X}')$. Suppose $\widetilde{X} = \widetilde{X}'$. Then there is a graph isomorphism $i : \widetilde{X} \to \widetilde{X}'$ as in Definition 33 such that $i \circ \pi_2 = \pi'_2$, $\pi'_1 \circ i = \pi_1$.

Then

$$\begin{aligned} H &= \{h \in G \mid \pi_2(v_0, h) = \pi_2(v_0, 1)\}, \\ H' &= \left\{h \in G \;\middle|\; \pi'_2(v_0, h) = \pi'_2(v_0, 1)\right\}. \end{aligned}$$

Since $i \circ \pi_2 = \pi'_2$ and i is 1-1, we find that $H = H'$.

To go the other way suppose that $H(\widetilde{X}) = H(\widetilde{X}')$. Then we need to show that there is a graph isomorphism $i : \widetilde{X} \to \widetilde{X}'$ as in Definition 33 such that $i \circ \pi_2 = \pi'_2$, $\pi'_1 \circ i = \pi_1$. Note first that $\widetilde{X}$ and $\widetilde{X}'$ have the same number of elements. We know from formula (14.2) that

$$\begin{aligned} \pi_2^{-1}(\widetilde{v}) &= \{(v, hg_v) \mid h \in H\}, \\ \pi_2'^{-1}(\widetilde{v}') &= \{(v, hg_{v'}) \mid h \in H\}. \end{aligned}$$

Define $i(\widetilde{v}) = \pi'_2(v, hg_v)$.

Exercise 50. *Show that i is a graph isomorphism such that $i \circ \pi_2 = \pi'_2$, $\pi'_1 \circ i = \pi_1$.*

□

Next we need to think about conjugate subgroups of the Galois group and their corresponding intermediate graphs.

Definition 34. *Suppose we have the following correspondences between intermediate graphs and subgroups of G:*

$$\begin{aligned} &\widetilde{X} \longleftrightarrow H \subset G \\ &\widetilde{X}' \longleftrightarrow gHg^{-1} \subset G, \text{ for some } g \in G. \end{aligned}$$

We say $\widetilde{X}$ and $\widetilde{X}'$ are ***conjugate****.*

This definition turns out to be equivalent to an earlier part of Definition 33.

Theorem 17. *Intermediate graphs $\widetilde{X}$ and $\widetilde{X}'$ are conjugate in the sense of Definition 34 if and only if they are covering isomorphic in the sense of Definition 33.*

Proof. Suppose that H and $H' = g_0 H g_0^{-1}$ are conjugate subgroups of G, where $g_0 \in G$. We want to show that the corresponding intermediate graphs $\widetilde{X} = \widetilde{X}(H)$ and $\widetilde{X}' = \widetilde{X}(H')$ (using the notation of Theorem 16) are covering isomorphic in the sense of Definition 33. We have the disjoint coset decompositions

$$G = \bigcup_{j=1}^{d} Hg_j \quad \text{and} \quad G = \bigcup_{j=1}^{d} H' g_0 g_j.$$

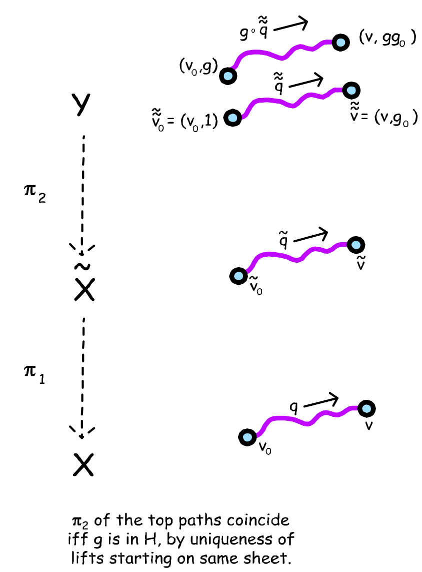

FIGURE 41. Part of the proof of part 4 of Theorem 16 showing that $\{(v, hg_0) \mid h \in H\} = \pi_2^{-1}(\widetilde{v})$. The dashed lines are the projections maps π_1 and π_2.

This means that the graphs $\widetilde{X}$ and $\widetilde{X}'$ have vertices $\left\{(v, Hg_j) \mid v \in X,\ 1 \le j \le d\right\}$ and $\left\{(v, H'g_0g_j) \mid v \in X,\ 1 \le j \le d\right\}$, respectively. The isomorphism $i : \widetilde{X} \longrightarrow \widetilde{X}'$ is defined by $i(v, Hg) = (v, H'g_0g)$.

For the converse, suppose that $\widetilde{X}$ and $\widetilde{X}'$ are covering isomorphic intermediate graphs. We must show that the corresponding subgroups $H = H(\widetilde{X})$ and $H' = H(\widetilde{X}')$ (in the notation of Theorem 16) are conjugate. By Definition 33, there is an isomorphism $i : \widetilde{X} \longrightarrow \widetilde{X}'$ such that $\pi_1 = \pi_1' \circ i$. Fix vertex $v_0 \in X$ and let $\widetilde{\widetilde{v_0}} = (v_0, 1)$ be on sheet 1 of Y, and $\widetilde{v_0} = \pi_2(\widetilde{\widetilde{v_0}})$ in $\widetilde{X}$. For any $\widetilde{v} \in \widetilde{X}$, suppose it projects to $v \in X$ under π_1 and suppose $\widetilde{\widetilde{v}} = (v, g_v) \in Y$ projects to $\widetilde{v}$ under π_2. See Figure 42. The set $\{g \in G \mid \pi_2(v, g) = \widetilde{v}\} = Hg_v$ by formula (14.2). Let $\widetilde{\widetilde{p}}$ be a path on Y from $\widetilde{\widetilde{v_0}}$ to $\widetilde{\widetilde{v}}$. It projects via π_2 to a path $\widetilde{p}$ in $\widetilde{X}$ from $\widetilde{v_0}$ to $\widetilde{v}$ and to a path p in X from v_0 to v.

The path $i(\widetilde{p})$ in $\widetilde{X}'$ from $i(\widetilde{v_0})$ to $i(\widetilde{v})$ also projects under π_1' to p. As $i(\widetilde{v_0})$ projects under π_1' to v_0, there is a $g_0 \in G$ such that $(v_0, g_0) \in Y$ projects via π_2' to $i(\widetilde{v_0})$. Now $\pi\left(g_0 \circ \widetilde{\widetilde{p}}\right) = \pi\left(\widetilde{\widetilde{p}}\right) = p$. Since $\pi = \pi_1' \circ \pi_2'$, it follows that the path $\pi_2'\left(g_0 \circ \widetilde{\widetilde{p}}\right)$ in $\widetilde{X}'$ has initial vertex $i(\widetilde{v_0})$ and projects to p in X. By the uniqueness of lifts, then $i(\widetilde{p}) = \pi_2'\left(g_0 \circ \widetilde{\widetilde{p}}\right)$. However, $g_0 \circ \widetilde{\widetilde{p}}$ ends at (v, g_0g). Therefore π_2' takes (v, g_0g) to $i(\widetilde{v})$. In particular, the set of all such g_0g is $g_0Hg_v = (g_0Hg_0^{-1})g_0g_v$. Therefore $H' = g_0Hg_0^{-1}$. □

Remark 1. *The previous proof showed that the effect of the isomorphism i can be achieved by the element $g_0 \in G$. In fact, g_0 may be replaced by any element of the right coset $(g_0Hg_0^{-1})g_0 = g_0H$, a left coset of H. This gives a* ***1-1 correspondence between left cosets*** g_0H ***of*** H ***and all possible "embeddings" of*** $\widetilde{X}$ ***in*** Y/X.

Theorem 18. *Suppose Y/X is a normal covering with Galois group G and $\widetilde{X}$ is an intermediate covering corresponding to the subgroup H of G. Then $\widetilde{X}$ is itself a normal covering of X if and only if H is a normal subgroup of G and then $G(\widetilde{X}/X) \cong G/H$.*

Proof. Recall the proof of Theorem 17. View $\widetilde{X}$ as $\widetilde{X}(H)$ (using the notation of Theorem 16), with vertex set

$$\left\{(v, Hg_j) \mid v \in X, 1 \le j \le d\right\},$$

where the g_j are right coset representatives for $H\backslash G$.

Suppose H is a normal subgroup of G. A coset Hg acts on $\widetilde{X}(H)$ by sending (v, Hg_j) to (v, Hgg_j). This action preserves edges and is transitive on the cosets Hg.

Exercise 51. *Prove this last statement. You need to use the normality of H to see that the action preserves edges.*

This gives $d = |G/H|$ automorphisms of $\widetilde{X}(H)$ showing that $\widetilde{X}(H)$ is normal over X with Galois group G/H.

For the converse, suppose $\widetilde{X}/X$ is normal and i is an automorphism of $\widetilde{X}$ in $G(\widetilde{X}/X)$. Apply Theorem 17 with $\widetilde{X}' = \widetilde{X}$ and $\pi_1 = \pi_1'$ and $\pi_2 = \pi_2'$. Although i is not the map that makes $\widetilde{X}' = \widetilde{X}$ (that map is the identity map), nevertheless, i is

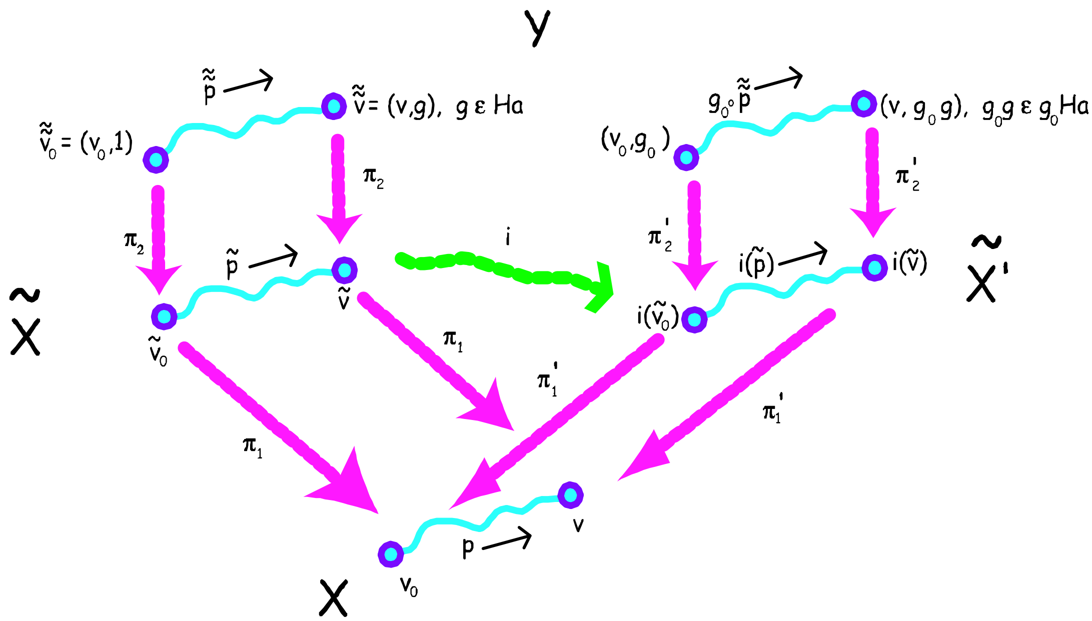

FIGURE 42. Part of the proof of Theorem 17.

an isomorphism between $\widetilde{X}$ and $\widetilde{X}'$ and it is a conjugation map since $\pi'_1 \circ i = \pi_1 \circ i = \pi_1$. Thus Theorem 17 says there is a $g_0 \in G$ such that the intermediate graph $\widetilde{X}'$ corresponds to the subgroup $g_0 H g_0^{-1}$. Since $\widetilde{X}' = \widetilde{X}$, we have $g_0 H g_0^{-1} = H$.

Moreover choosing $\widetilde{v_0} \in \widetilde{X}$ as in the proof of Theorem 17, we have $\pi_2((v_0, g_0)) = \pi'_2((v_0, g_0)) = i(\widetilde{v_0})$. As i runs through the d elements of $G(\widetilde{X}/X)$, the $i(\widetilde{v_0})$ run through the d lifts of v_0 to $\widetilde{X}$. Thus the corresponding d different g_0's run through all d left cosets of H in G, and we have $g_0 H g_0^{-1} = H$ for all of these which says H is normal in G. □

The reader should now go back and reconsider the examples in Figures 37 and 38. As an **exercise**, write down all the intermediate graphs. Next let's consider a new example given in Figure 43.

Example 18. *$G(Y/X) = S_3$, $H = \{(1), (23)\}$ fixes Y_3.*

The top graph Y_6 in Figure 43 is a normal 6-fold cover of the bottom graph X with Galois group S_3, the symmetric group of permutations of 3 objects. We make the identifications

$$a' = (a, (1)), a" = (a, (13)), a^{(3)} = (a, (132), a^{(4)} = (a, (23)), a^{(5)} = (a, (123)), a^{(6)} = (a, (12)).$$

One way to construct this example is obtained by using permutation representations of S_3. See the Exercise below. A spanning tree in the bottom graph X is given in green. The edges in X left out of the spanning tree generate the fundamental group of X. Call the directed edge from vertex 2 to vertex 4 edge c. Call the directed edge from vertex 4 to vertex 3 edge d.

We get the top graph Y_6 using the permutation $\sigma(c) = (14)(23)(56)$. That means we connect vertex $2^{(1)}$ with vertex $4^{(4)}$ in Y_6 and then connect vertex $2^{(4)}$ and $4^{(1)}$. After that connect vertex $2^{(2)}$ with $4^{(3)}$ and vertex $2^{(3)}$ with $4^{(2)}$. Finally connect vertex $2^{(5)}$ with vertex $4^{(6)}$ and vertex $2^{(6)}$ with vertex $4^{(5)}$. Do a similar construction with $\sigma(d) = (12)(36)(45)$. The permutations $\sigma(c)$ and $\sigma(d)$ have order 2 and they generate a subgroup of S_6 isomorphic to S_3. In Section 17 below, we will have more to say about this construction.

We can also identify S_3 with the dihedral group D_3 of rigid motions of an equilateral triangle. Let R be a 120° rotation of an equilateral triangle and F a flip. The we have $D_3 = \{I, R, R^2, F, FR, FR^2\}$, with $R^3 = I$, $FR = R^2 F$. We identify

$$a' = (a, I), a" = (a, FR^2), a^{(3)} = (a, R^2), a^{(4)} = (a, FR), a^{(5)} = (a, R), a^{(6)} = (a, F).$$

We can identify $\sigma(c) = FR$ and $\sigma(d) = FR^2$.

Exercise 52. *Suppose we list the elements of S_3, using cycle notation, as*

$$g_1 = (1), g_2 = (12), g_3 = (123), g_4 = (23), g_5 = (132), g_6 = (13).$$

Then write $g_i g = g_{\mu(g)}$, where $\mu(g) \in S_6$. Show that $\mu(23) = (14)(23)(56)$ and $\mu(12) = (12)(36)(45)$.

Exercise 53. *Create a covering of the cube graph which is normal with Galois group a cyclic group of order 3.*

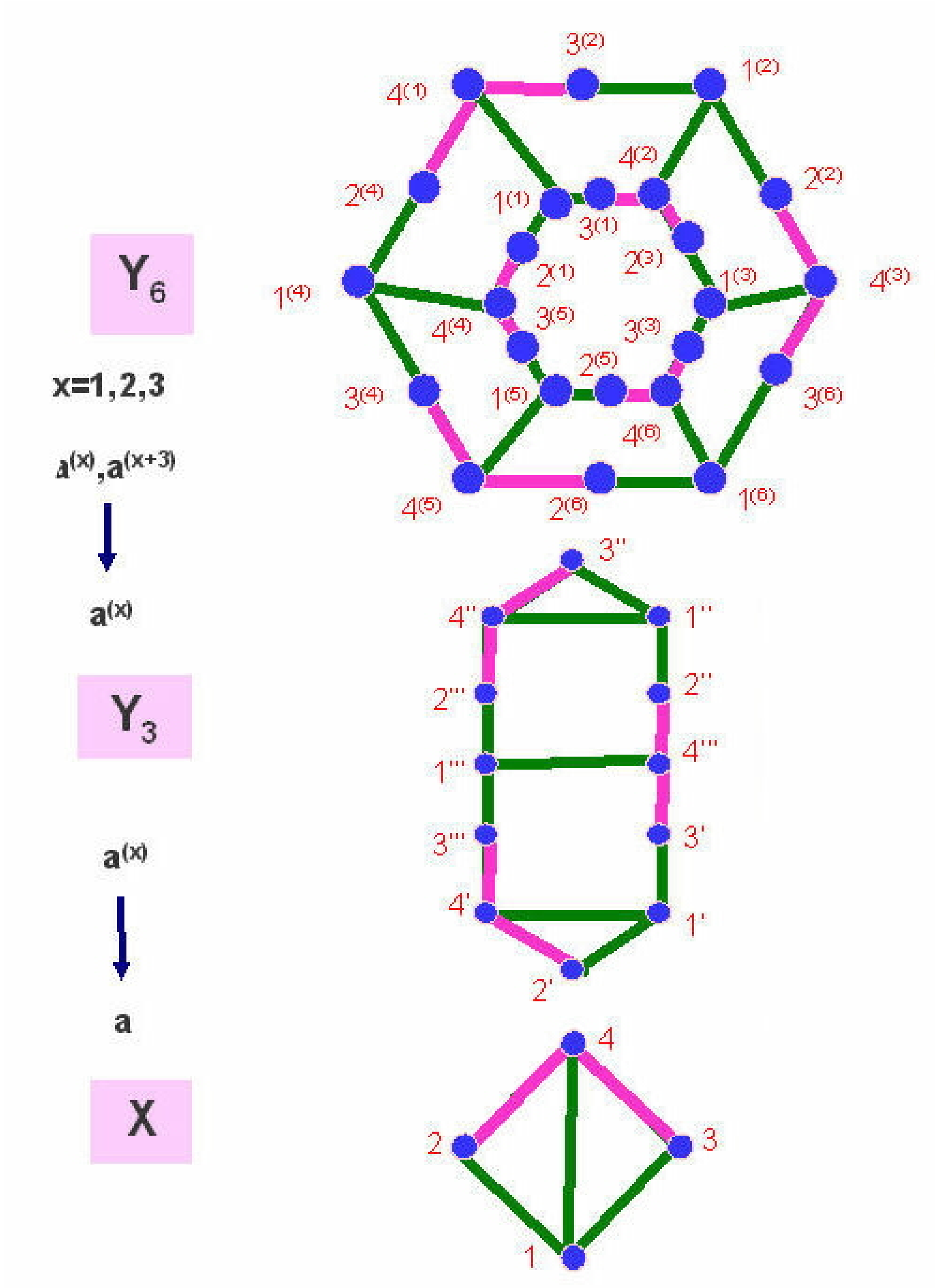

FIGURE 43. A 6-sheeted normal cover Y_6 of X with a non-normal cubic intermediate cover Y_3. Here the Galois group is the symmetric group $G = G(Y/X) = S_6$ and the subgroup $H = \{(1), (23)\}$ fixes Y_3. We write $a^{(1)} = (a, (1)), a^{(2)} = (a, (13)), a^{(3)} = (a, (132)), a^{(4)} = (a, (23)), a^{(5)} = (a, (123)), a^{(6)} = (a, (12))$, using standard cycle notation for elements of the symmetric group.

It is possible to define coverings of weighted graphs. See Chung and Yau [20] or Osborne and Severini [66]. The second paper applied the idea combined with that of quantum walks on graphs.

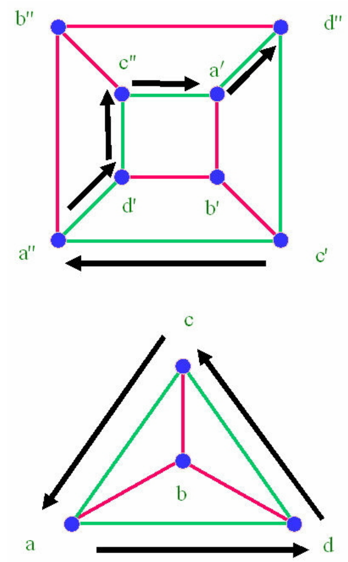

FIGURE 44. Picture of splitting of prime with $f = 2, g = 1, e = 1$. There is 1 prime cycle D above and D is the lift of C^2.

15. BEHAVIOR OF PRIMES IN COVERINGS

We seek analogs of the laws governing the behavior of prime ideals in extensions of algebraic number fields. Figure 6 shows what happens in a quadratic extension of the rationals. Figure 52 shows a cubic extension. See Stark [81] for more information on these examples.

So now let us consider the graph theory analog. The field extension is replaced by a graph covering Y/X, with projection map π. Suppose $[D]$ is a prime in Y. Then $\pi(D)$ is a closed, backtrackless, tailless path in X but it may not be primitive. There will, however, be a prime $[C]$ in X and an integer f such that $\pi(D) = C^f$. The integer f is independent of the choice of D in $[D]$.

Definition 35. *If $[D]$ is a prime in a covering Y/X with projection map π and $\pi(D) = C^f$, where $[C]$ is a prime of X, we will say that $[D]$ is a* ***prime of Y above $[C]$****, or more loosely, that D is a* ***prime above C*** *and write $D|C$ and $f = f(D, Y/X) =$* ***residual degree*** *of D with respect to Y/X.*

If Y/X is normal, for a prime C of X and a given integer j, either every lift of C^j is closed in Y or no lift is closed. Thus the residual degree of $[D]$ above C is the same for all $[D]$ above C.

Definition 36. *Let $g = g(D, Y/X)$ be the* ***number of primes*** *$[D]$ above $[C]$.*

Since our covers are unramified, the analog of the ramification index is $e = e(D, Y/X) = 1$ and we will have the familiar formula from algebraic number theory for normal covers:

$$efg = d = \text{number of sheets of the cover.} \tag{15.1}$$

Example 19. ***Primes in the cube Y over primes in the tetrahedron X.***

In Figure 44 we show a prime $[C]$ of length 3 in the tetrahedron X defined by $C = \langle a, d, c, a \rangle$. Here we list the vertices through which the path passes within <> . The prime $[D]$ in the cube Y, with $D = \langle a', d'', c', a'', d', c'', a' \rangle$, has length 6 and is over $[C]$ in X. Let the Galois group be $G = G(Y/X) = \{1, \sigma\}$. We are using the notation $x' = (x, 1)$ and $x'' = (x, \sigma)$ in Y, for $x \in X$. Then $D = C_1\ (\sigma \circ C_1)$, where $C_1 = \langle a', d'', c', a'' \rangle$. Here $\nu(D) = 2\nu(C) = 6$. In this example $f = 2$ and $g = 1$.

A second example in Y/X is shown in Figure 45. In this case, the prime $[D]$ of Y is represented by $D = \langle a'', c', d'', b'', a'' \rangle$. Then $D|C$ with the prime $[C]$ represented by $C = \langle a, c, d, b, a \rangle$ in X. Here $\nu(D) = \nu(C) = 4$, $f = 1$, and $g = 2$ since there is another prime D' in Y over C, also shown in Figure 45.

Definition 37. *If Y/X is normal and $[D]$ is a prime of Y over $[C]$ in X and σ is in $G(Y/X)$, we refer to $[\sigma \circ D]$ as a* ***conjugate prime*** *of Y over C.*

We then have $f(\sigma \circ D, Y/X) = f(D, Y/X)$. If $f = f(D, Y/X)$, then as g runs through $G(Y/X)$, $g \circ D$ runs through all possible lifts of C^f from X to Y and thus the conjugates of $[D]$ account for all the primes of Y above $[C]$. That is, there are $d = |G(Y/X)|$ lifts of C^f starting on different sheets of Y, but only g of these lifts give rise to different primes of Y.

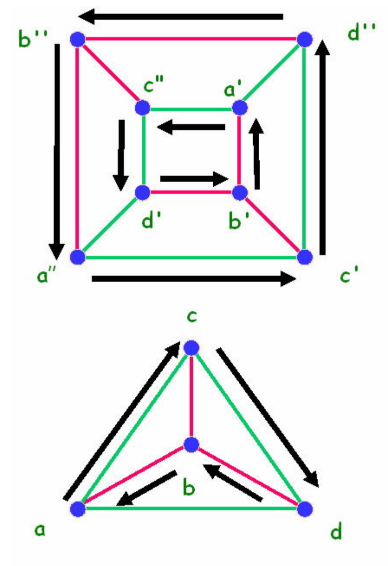

FIGURE 45. Picture of a prime which splits completely; i.e., $f = 1, g = 2, e = 1$. There are 2 prime cycles D, D' in the cube above each with the same length as C below in the tetrahedron.

Thus, when the cover Y/X is not normal, formula (15.1) becomes $\sum_{i=1}^{g} f_i = d$. Here the f_i denote the residual degrees of the primes of Y above some fixed prime of X.

Given Y/X a (finite unramified) graph covering and suppose $\widetilde{X}$ is intermediate to Y/X. Suppose $\pi_1 : \widetilde{X} \longrightarrow X$ and $\pi_2 : Y \longrightarrow \widetilde{X}$, $\pi : Y \longrightarrow X$ are the covering maps, with $\pi = \pi_1 \circ \pi_2$. Let E be a prime of Y over the prime C of X and let $\pi_2(E) = D^{f_2}$, where $f_2 = f(E, Y/\widetilde{X})$. Then we have the **transitivity property**:

$$f(E, Y/X) = f(E, Y/\widetilde{X}) f(D, \widetilde{X}/X). \tag{15.2}$$

This is the graph theoretic analog of a result about the behavior of residual degrees of primes in extensions of algebraic number fields.

Exercise 54. *Prove Formula (15.2).*

Hint. *Note that* $\pi(E) = C^{f(E,Y/X)}$ *and* $\pi_2(E) = D^{f(E,Y/\widetilde{X})}$.

16. FROBENIUS AUTOMORPHISMS

We want to find an analog of the Frobenius automorphism in number theory. See Figure 6 in the Introduction and Figure 49 below. References are Lang [53], and Stark [81].

Assume that Y is a normal cover of the graph X with Galois group G. We want to define the Frobenius automorphism $[Y/X, [D]]$ for a prime $[D]$ in Y over the prime $[C]$ in X. First we can define the normalized Frobenius automorphism $\sigma(p) \in G = G(Y/X)$ associated to a directed path p of X - the existence of which simplifies the graph theory version of things. This normalized Frobenius automorphism should be compared with the voltage assignment map in Gross and Tucker [32]. See Figure 46 for a summary of our definitions.

Definition 38. *Suppose Y/X is normal with Galois group $G = Gal(Y/X)$. For a path p of X, there is a unique lifting to a path $\widetilde{p}$ of Y starting on sheet 1, having the same length as p. If $\widetilde{p}$ has its terminal vertex on the sheet labeled with $g \in G$, define the* ***normalized Frobenius automorphism*** *$\sigma(p) \in G$ by $\sigma(p) = g$.*

Exercise 55. *Compute the normalized Frobenius automorphism $\sigma(C)$ for the curves C in the tetrahedron K_4 pictured in Figures 44 and 45.*

Exercise 56. *Compute the normalized Frobenius automorphism $\sigma(C)$ for the curves C in $K_4 - e$ pictured in Figures 43 and 53.*

Lemma 5. *1) Suppose that p_1 and p_2 are two paths on X such that the terminal vertex of p_1 is the initial vertex of p_2. Then $\sigma(p_1 p_2) = \sigma(p_1)\sigma(p_2)$.*

2)If a path $p = e_1 \cdots e_s$, for directed edges $e_1, ..., e_s$; then $\sigma(p) = \sigma(e_1) \cdots \sigma(e_s)$.

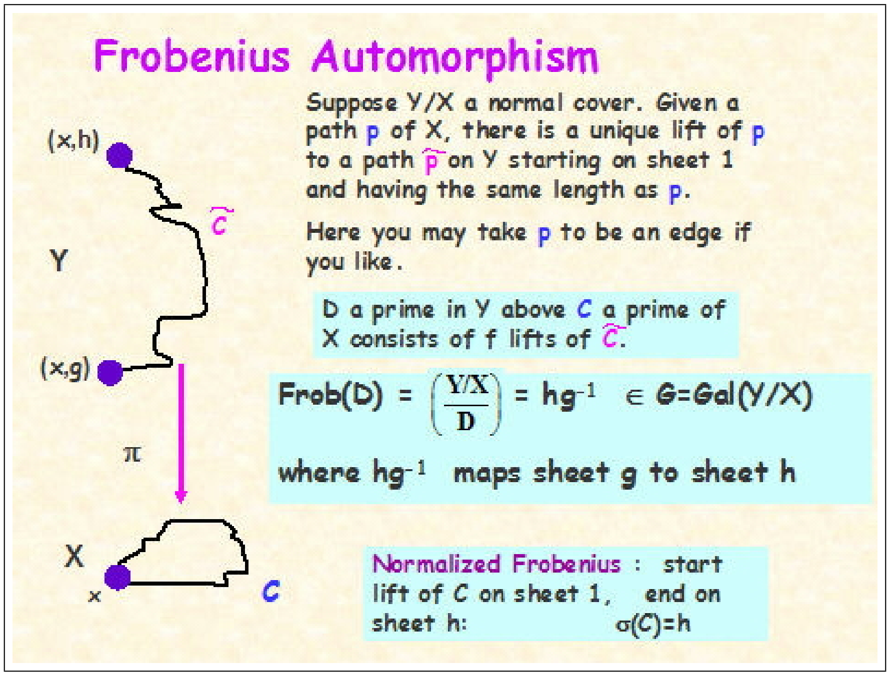

FIGURE 46. The Frobenius automorphism and the normalized version.

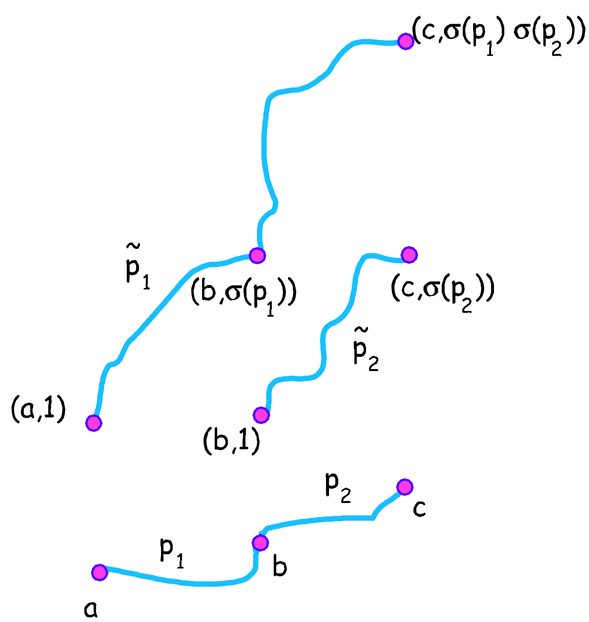

FIGURE 47. The map σ preserves composition of paths.

Proof. 1) If p_1 goes from a to b in X and p_2 goes from b to c in X, then the lift $\widetilde{p_1}$ of p_1 starting on sheet 1 of Y goes from $(a,1)$ to $(b,\sigma(p_1))$ and the lift $\widetilde{p_2}$ of p_2 starting on sheet 1 of Y goes from $(b,1)$ to $(c,\sigma(p_2))$. See Figure 47. Therefore the lift of p_2 starting on sheet $\sigma(p_1)$ goes from $(b,\sigma(p_1))$ to $(c,\sigma(p_1)\sigma(p_2))$. This implies that the lift of p_1p_2 beginning on sheet 1 of Y will end on sheet $\sigma(p_1)\sigma(p_2)$.

2) This follows easily from part 1). □

Now we can define the Frobenius automorphisms and decomposition groups. See Figure 46 again.

Definition 39. *Assume Y/X normal with Galois group G. Let $[C]$ be a prime on X, such that C starts and ends at vertex a. Let $[D]$ be a prime of Y over C such that D starts and ends at vertex (a,g) on sheet $g \in G$ of Y. If the residual degree of D over C is f, then D is the lifting of C^f which begins on sheet g. Suppose C itself lifts to a path $\widetilde{C}$ on Y starting on sheet g at (a,g) and ending on sheet h at (a,h). Define the* ***Frobenius automorphism*** *to be*

$$[Y/X, D] = \left(\frac{Y/X}{D}\right) = hg^{-1} \in G.$$

Note that the Frobenius $[Y/X, D] = \left(\frac{Y/X}{D}\right) = hg^{-1}$ maps sheet g of Y to sheet h of Y. To get the normalized version of the Frobenius you have to take $g = 1$, the identity of G.

Our next definition yields a group analogous to one from algebraic number theory. The letter chosen for it corresponds to the German version of the name.

Definition 40. *The* ***decomposition group*** *of D with respect to Y/X is*

$$Z(D) = Z(D, Y/X) = \{\tau \in G \mid [\tau \circ D] = [D]\}.$$

The next proposition gives analogs of the usual properties of the Frobenius automorphism of a normal extension of number fields (as in Lang [53]).

Proposition 7. ***Properties of the Frobenius Automorphism.***

As usual, Y/X is a normal d-sheeted covering with Galois group G.

1) For a prime cycle D in Y over C in X, the Frobenius automorphism is independent of the choice of D in its equivalence class $[D]$. Thus we can define $[Y/X, [D]] = [Y/X, D]$, without ambiguity.

2) The order of $[Y/X, D]$ in G is the residual degree $f = f(D, Y/X)$.

3) If $\tau \in G$, then $[Y/X, \tau \circ D] = \tau[Y/X, D]\tau^{-1}$.

4) If D begins on sheet 1, then $[Y/X, D] = \sigma(C)$.

5) The decomposition group $Z(D)$ is the cyclic subgroup of G of order f generated by $[Y/X, D]$. In particular, $Z(D)$ does not depend on the choice of D in its equivalence class $[D]$.

6) For a prime cycle D in Y over C in X, if $f = f(D, Y/X)$ is from Definition 35 and $g = g(D, Y/X)$ is as in Definition 36, then $d = fg$. (Here $e =$ the ramification is assumed to be 1.)

Proof. **Part 4)** is proved by noting that the 2 definitions are the same.

Parts 1) and 3). Suppose C has initial (and terminal) point vertex a in X and D is the lifting of C^f beginning at vertex (a, μ_0) on sheet μ_0. In lifting C^f, we lift C a total of f times consecutively, beginning at (a, μ_0) and ending respectively at $(a, \mu_1), (a, \mu_2), \ldots, (a, \mu_{f-1}), (a, \mu_f)$, where $\mu_f = \mu_0$, and $\mu_j \neq \mu_0$, for $j = 1, 2, \ldots, f-1$. See Figure 48.

Suppose that (b, κ) is another vertex on D, where b is on C. Thus (b, κ) lies on one of the f consecutive lifts of C in Figure 48 say the r^{th}. Vertex b splits C into two paths $C = p_1 p_2$, where b is the ending vertex of p_1 and the starting vertex of p_2. The vertex (b, κ) on Y is the ending vertex of the lift of p_1 to D starting at (a, μ_{r-1}). The lift of the version of C in $[C]$ starting at b, namely $p_2 p_1$ to a path on Y which starts at (b, κ) then ends at a vertex (b, λ) on D which lies on the $(r+1)^{st}$ consecutive lift of C.

Let $\widetilde{C'}$ be a path on Y from $(a, 1)$ to (a, μ_0) and let C' be the projection of $\widetilde{C'}$ to X. The vertices $(a, \mu_0), (a, \mu_1), (b, \kappa)$, and (b, λ) of Y are then the end points of the lifts on the paths

$$C', C'C, C'C^{r-1}p_1, C'C^r p_1,$$

respectively, to paths on D starting at $(a, 1)$. Therefore, by Lemma 5, we have

$$\begin{aligned} \mu_0 &= \sigma(C'), \quad \mu_1 = \sigma(C'C) = \sigma(C')\sigma(C); \\ \kappa &= \sigma(C'C^{r-1}p_1) = \sigma(C')\sigma(C)^{r-1}\sigma(p_1); \;\; \lambda = \sigma(C'C^r p_1) = \sigma(C')\sigma(C)^r\sigma(p_1). \end{aligned}$$

It follows that $[Y/X, D]$ is the common value of

$$\lambda\kappa^{-1} = \mu_1\mu_0^{-1} = \sigma(C')\sigma(C)\sigma(C')^{-1}.$$

This proves 1). It also proves 3) in the case $\tau = \mu_0^{-1} = \sigma(C')^{-1}$ and this suffices to prove 3) in general.

Part 2). As above, we see that for each j

$$\mu_j = \sigma(C'C^j) = \sigma(C')\sigma(C)^j$$

and thus

$$\mu_j\mu_0^{-1} = \sigma(C')\sigma(C)^j\sigma(C')^{-1} = [Y/X, D]^j. \tag{16.1}$$

This proves 2).

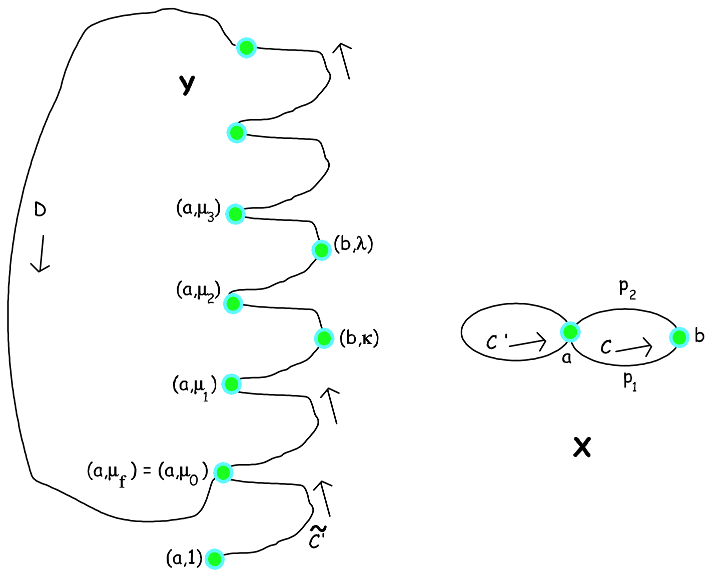

FIGURE 48. Part of the proof of Proposition 7. The vertex (b, κ) lies on the r^{th} consecutive lift of C (shown with $r = 2$). The lift to a path in Y starting at (b, κ) of the version of C in $[C]$ starting at b ends at a vertex (b, λ) which arises on the $(r + 1)^{st}$ consecutive lift of C.

Part 5). Recall that $\tau \in Z(D)$ means that $\tau \circ D$ is equivalent to D.

Suppose $\tau \circ D$ is equivalent to D. If the picture is as in Figure 48, then since $\tau \circ D$ also starts at a vertex projecting under $\pi : Y \to X$ to a, and we must have $\tau\mu_0 = \mu_j$, for one of the μ_j above. Thus, for some j, $\tau = \mu_j\mu_0^{-1} = [Y/X, D]^j$ by (16.1).

Conversely, any such τ has $[\tau \circ D] = [D]$.

To see that the order of the decomposition group $Z(D)$ is $f = f(D, Y/X)$, note that $1 = \mu_j\mu_0^{-1} = [Y/X, D]^j$ iff f divides j.

Part 6) The Galois group $G = G(Y/X)$ of order d acts transitively on primes $[D]$ above $[C]$. The subgroup $Z(D)$ is the subgroup of G fixing $[D]$. Since $Z(D)$ has order f, it follows that the number g of distinct $[D]$ is $g = |G/Z(D)| = d/f$. □

It remains only to discuss the behavior of the Frobenius automorphism with respect to intermediate coverings.

Theorem 19. ***More Properties of the FrobeniusAutomorphism***

1) Suppose $\widetilde{X}$ is an intermediate covering to Y/X and corresponds to the subgroup H of $G = G(Y/X)$. Let $[D]$ be an equivalence class of prime cycles in Y such that D lies above $\widetilde{C}$ in $\widetilde{X}$. Let $f = f(D, Y/X) = f_1 f_2$, where $f_2 = f(D, Y/\widetilde{X})$ and $f_1 = f(\widetilde{C}, \widetilde{X}/X)$. Then f_1 is the minimal power of $[Y/X, D]$ which lies in H and we have

$$[Y/X, D]^{f_1} = [Y/\widetilde{X}, D]. \tag{16.2}$$

2) If further $\widetilde{X}$ is normal over X, then as an element of $H\backslash G$, we have

$$[\widetilde{X}/X, \widetilde{C}] = H[Y/X, D].$$

Proof. **Part 1)** Let C be the prime of X below $\widetilde{C}$. The Frobenius automorphism $[Y/\widetilde{X}, D]$ is found by lifting $\widetilde{C}$ from $\widetilde{X}$ to Y. This is the same as lifting C^{f_1} from X to Y and the analysis in the proof of proposition 7 (equation (16.1) in particular) gives equation (16.2) of 1). The fact that f_1 is the minimal power of $[Y/X, D]$ which lies in H follows from the fact that

$$Z(Y/\widetilde{X}, D) = Z(Y/X, D) \cap H$$

which we know to be cyclic of order f_2. Therefore since $[Y/X, D]$ is of order $f_1 f_2$, we see that $[Y/X, D]^j$ cannot be in H if $j < f_1$.

Part 2) Now let $\widetilde{X}$ be normal over X. View $\widetilde{X}$ as having vertices $(v, H\tau)$, for $v \in X$, $\tau \in G$. Let D start and end at (a, μ_0) in Y and $\widetilde{C}$ start and end at $(a, H\mu_0)$ in $\widetilde{X}$. If C in X lifts to a path in Y starting at (a, μ_0) and ending at (a, μ_1), then C lifts to a path in $\widetilde{X}$ starting at $(a, H\mu_0)$ and ending at $(a, H\mu_1)$. Then 2) follows from Definition 39 of the Frobenius automorphism. □

17. How to Construct Intermediate Coverings using the Frobenius Automorphism

Let us now explain how to construct intermediate coverings.

Lemma 6. *Suppose Y/X is normal with Galois group G. Fix a spanning tree T of X. Let $e_1, ..., e_r$ be the non-tree edges of X (i.e., those corresponding to generators of the fundamental group) with directions assigned. The r normalized Frobenius automorphisms $\sigma(e_j)$, $j = 1, ..., r$, generate G.*

Proof. Since $\sigma(t) = 1$ for all edges t on the tree of X, for any path p on X, $\sigma(p)$ is a product of the $\sigma(e_j)$ and their inverses, by Lemma 5. Since we can get to every sheet of Y by lifting paths of X to paths starting on sheet 1 of Y, we can write any $g \in G$ as a product $\sigma(e_j)$. □

Lemma 7. *Suppose Y/X is normal with Galois group G and $\widetilde{X}$ is an intermediate graph corresponding to the subgroup H of G. Let $H_0 = \bigcap_{g \in G} gHg^{-1}$. Then $H_0 = \{1\}$ if and only if there are no intermediate graphs, other than Y, which are normal over X and intermediate between Y and $\widetilde{X}$.*

Proof. This is a standard fact from Galois theory. A normal intermediate graph covering $\widetilde{X}$ would correspond to a normal subgroup of G (a subgroup which must be contained in H) and conversely. Any normal subgroup of G contained in H is also contained in every conjugate of H and hence is contained in H_0. Since H_0 is a normal subgroup of G, the result is proved. □

Lemma 8. *Suppose $\widetilde{X}$ is a covering of X and that Y/X is a normal covering of X of minimal degree such that $\widetilde{X}$ is intermediate to Y/X. Let $G = G(Y/X)$ and $H = G(Y/\widetilde{X})$. Let $Hg_1, ..., Hg_n$ be the right cosets of H. We have a 1-1 group anti-homomorphism μ from G into the symmetric group S_n defined by setting $\mu(g)(i) = j$ if $Hg_i g = Hg_j$.*

Proof. By the Exercise below, $\mu(g')\mu(g) = \mu(gg')$. The kernel of μ is the set of $g \in G$ such that $Hg'g = Hg'$, $\forall\ g' \in G$. This means $Hg'gg'^{-1} = H$, $\forall\ g' \in G$, which is equivalent to $g \in g'^{-1}Hg'$, $\forall\ g' \in G$. By Lemma 7, $g = 1$ and μ is 1-1. □

Exercise 57. *Check the claim in the preceding proof that $\mu(g')\mu(g) = \mu(gg')$.*

Now put these three Lemmas together.

Theorem 20. *Let the graphs $Y, \widetilde{X}, X$, the groups G, H, and the representation μ be as in Lemma 8. Let T be a fixed spanning tree of X. Suppose that e is one of the non-tree edges of X. Let $\sigma(e)$ be the corresponding normalized Frobenius automorphism of G. Suppose that v is the starting vertex of e and v' is the terminal vertex of e. If $\mu = \mu(\sigma(e))$ is the permutation of $1, ..., n$ such that $\mu(i) = \mu(\sigma(e))(i) = j$, then the directed edge e lifts to an edge in $\widetilde{X}$ starting at (v, Hg_i) and terminating at (v', Hg_j).*

Proof. By the definition of μ, $Hg_i\sigma(e) = Hg_j$. This means that $g_i\sigma(e) = hg_j$ for some element $h \in H$. By definition of $\sigma(e)$, the edge e lifts to an edge on Y from $(v, 1)$ to $(v', \sigma(e))$. If we apply g_i to this edge, we get an edge on Y starting at (v, g_i) and ending at $(v', g_i\sigma(e)) = (v', hg_j)$. Hence e lifts to a directed edge on $\widetilde{X}$ from (v, Hg_i) to (v', Hg_j). □

This theorem shows us how to create intermediate graphs given a normal cover and it also allows us to construct the minimal normal cover Y of X having a given intermediate covering graph $\widetilde{X}$ of X as well as the Galois group $G(Y/X)$. Examples of this theorem can be found in Figures 37, 38, and 43 We will give another series of examples based on the simple group of order 168 later.

Exercise 58. *Construct your own examples of n-fold cyclic covers of the graph $K_4 - edge$.*

18. Vertex Artin L-Functions

Before defining vertex Artin L-functions of normal graph coverings, we should perhaps recall what Artin L-functions do for number fields. First the reader needs to know a bit about representations of finite groups. One reference is the author's [92]. References for Artin L-functions of number fields are Lang [53] and Stark [81]. Figures 49, 50, 51, and 52 summarize some of the facts. Figure 53 gives a graph theory analog of Figure 52 - the splitting of primes in a non-normal cubic cover.

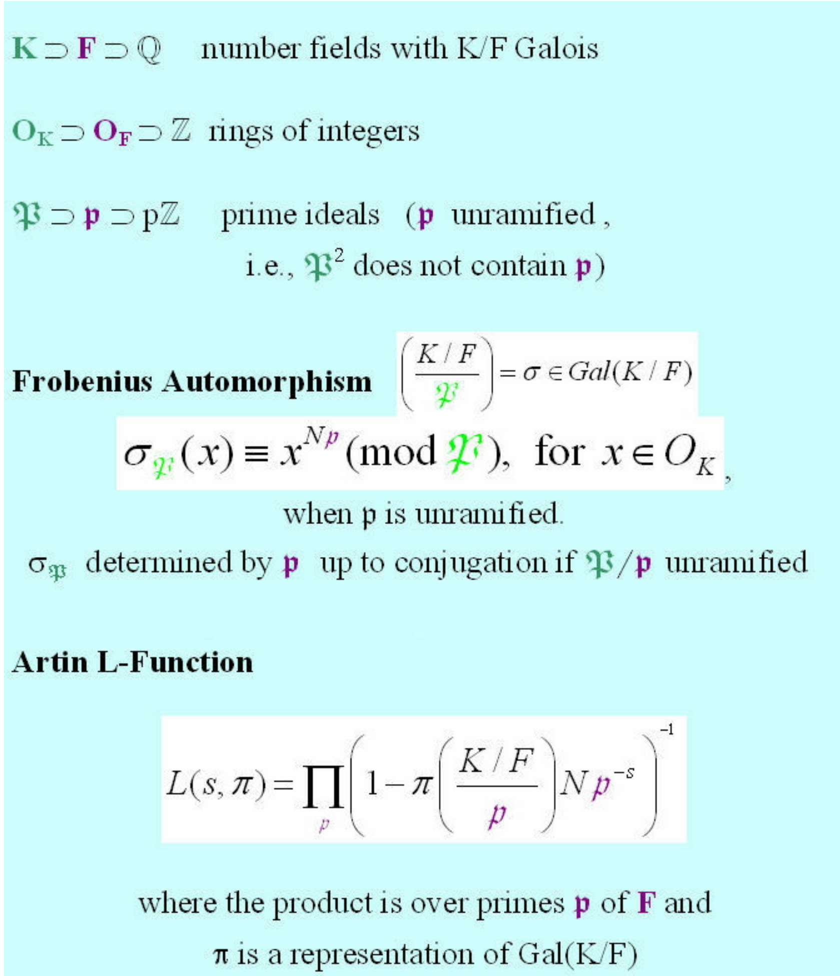

FIGURE 49. Definition of Frobenius symbol and Artin L-function of Galois extension of number fields.

⌘ Factorization

$$\zeta_K(s) = \prod_{\substack{\pi \\ \text{irreducible} \\ \text{degree } d_\pi}} L(s,\pi)^{d_\pi}$$

⌘ Chebotarev Density Theorem

∀ ∀ σ in Gal(K/F), ∃ ∞-ly many prime ideals $\mathfrak{p}$ of O_F such that ∃ $\mathfrak{P}$ in O_K dividing $\mathfrak{p}$ with Frobenius

$$\left(\frac{K/F}{\mathfrak{P}}\right) = \sigma$$

⌘ Artin Conjecture: L(s,π) entire for non-trivial irreducible rep π

⌘ Stark Conjectures: π not containing trivial rep

$$\lim_{s\to 0} s^a L(s,\pi) = \Theta(\pi) * R(\pi)$$

= algebraic number * determinant of a×a matrix in linear forms with alg. coeffs. of logs of units of K and its conjugate fields /ℚ.

References:

Stark's paper in *From Number Theory to Physics*, edited by Waldschmidt et al

Stark, Adv. in Math. papers on conjectures

Lang or Neukirch, *Algebraic Number Theory*

FIGURE 50. Applications of Artin L-functions of Number Fields

Chebotarev Density Theorem for K/$\mathbb{Q}$ normal.

For every conjugacy class C in G=Gal(K/$\mathbb{Q}$), the analytic density of the set of rational primes p such that C(p)=the conjugacy class of the Frobenius auto of a prime ideal $\mathfrak{P}$ of K dividing p is |C|/|G|.

Proof Sketch. **Sum the logs of the Artin L-functions × characters χ_π of all irreducible reps π of G.**
As $s \to 1+$, assuming the L-functions known to be well behaved

$$\log \frac{1}{s-1} \sim \sum_{\pi} \log L(s,\pi)\overline{\chi_\pi(C)}$$

$$\sim \sum_{\pi}\sum_{p} \chi_\pi(C(p))p^{-s}\,\overline{\chi_\pi(C)}$$

$$\sim \frac{|G|}{|C|} \sum_{\substack{p \\ C(p)=C}} p^{-s}$$

by the orthogonality relations of the characters of the irreducible representations π of G. Here C(p) denotes the conjugacy class of the Frobenius auto of the prime of K above p.

FIGURE 51. Chebotarev Density Theorem in Number Field Case

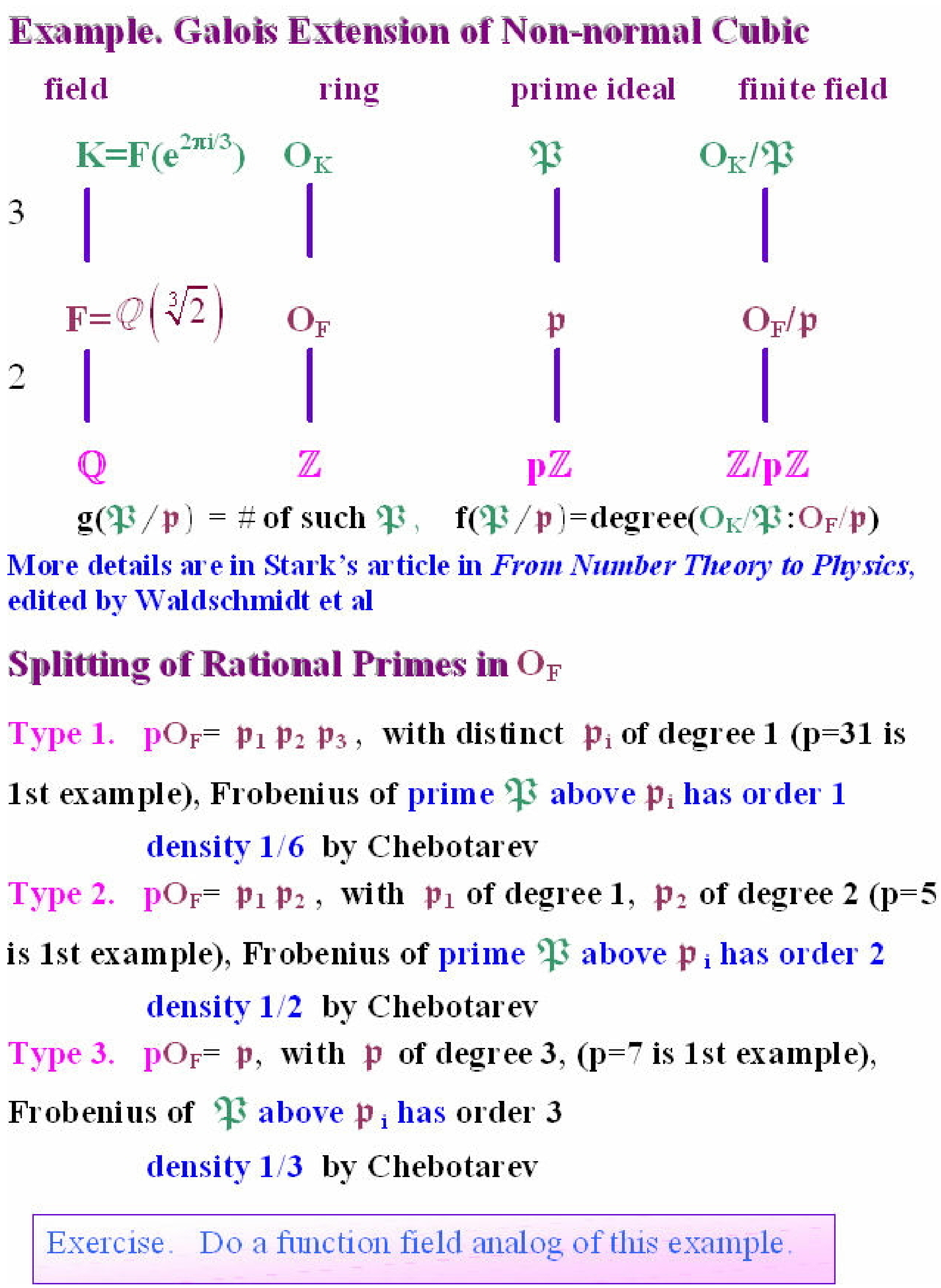

FIGURE 52. Example of splitting of unramified primes in non-normal cubic extension of the rationals.

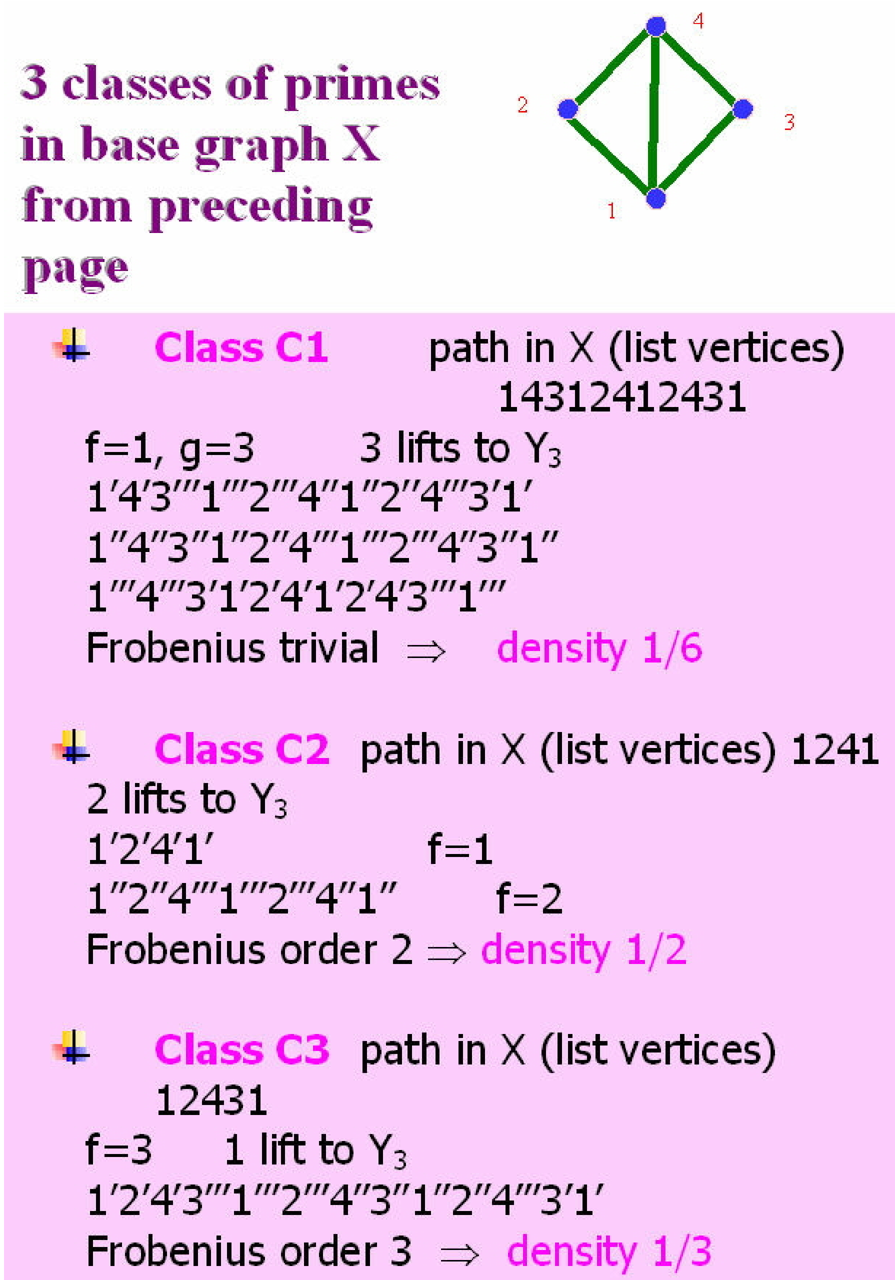

FIGURE 53. Splitting of primes in non-normal cubic cover of $K_4 - edge$ pictured in Figure 43.

The graph theory analog of this last example is found in Figure 43. Figure 53 gives examples of primes that split in various ways in the non-normal cubic intermediate field.

Suppose that Y is a normal unramified covering of X with Galois group $G = G(Y/X)$.

Definition 41. *If ρ is a representation of G with degree $d = d_\rho$, and u is a complex variable with $|u|$ sufficiently small, define the **vertex Artin-Ihara L-function** by*

$$L_V(u, \rho, Y/X) = L_V(u, \rho) = \prod_{[C]} \det\left(I - \rho\left([Y/X, D]\right) u^{\nu(C)}\right)^{-1},$$

where the product runs over primes $[C]$ *of* X *and* $[D]$ *is arbitrarily chosen from the primes in* Y *above* C. *Here* $[Y/X, D]$ *is the Frobenius automorphism of Definition 39. The subscript* V *stands for vertex* L*-function and* $\nu(C)$ *is the length of a curve* C *representing the prime* $[C]$.

When the representation ρ is trivial $(= 1)$, this is the Ihara zeta function of Definition 2

$$L_V(u, 1, Y/X) = \zeta_X(u). \tag{18.1}$$

For the representation theory of finite groups; e.g., the definition of induced representation needed in the next Proposition, see, for example, Terras [92]. You can copy the properties of Artin L-functions from Lang [53].

Proposition 8. ***Properties of the Vertex Artin L-Function***
Assume that Y/X *be a normal covering with Galois group* G.
1) $L_V(u, \rho_1 \oplus \rho_2) = L_V(u, \rho_1)L_V(u, \rho_2)$.
2) Suppose $\widetilde{X}$ *is intermediate to* Y/X *and assume* $\widetilde{X}/X$ *is normal,* $G = Gal(Y/X)$, $H = Gal(Y/\widetilde{X})$. *Let* ρ *be a representation of* $G/H \cong Gal(\widetilde{X}/X)$. *Thus* ρ *can be viewed as a representation of* G, *often called the* ***lift*** *of* ρ. *Then*

$$L_V(u, \rho, Y/X) = L_V(u, \rho, \widetilde{X}/X).$$

3) If $\widetilde{X}$ *is an intermediate cover to the normal cover* Y/X *and* ρ *is a representation of* $H = Gal(Y/\widetilde{X})$, *then let* $\rho^\# = Ind_H^G \rho$, *that is, the representation induced by* ρ *from* H *up to* G. *Then*

$$L_V(u, \rho^\#, Y/X) = L_V(u, \rho, Y/\widetilde{X}).$$

Proof. Only property 3) really requires some effort. We will postpone the proofs until the next section when we do the more general case of edge L-functions. □

Corollary 4. ***Factorization of the Ihara Zeta Function of an Unramified Normal Extension of Graphs*** *Suppose that* Y/X *is normal with Galois group* $G = G(Y/X)$. *Let* $\widehat{G}$ *be a complete set of inequivalent irreducible unitary representations of* G. *Then*

$$\zeta_Y(u) = L_V(u, 1, Y/Y) = \prod_{\rho \in \widehat{G}} L_V(u, \rho, Y/X)^{d_\rho}.$$

Proof. Use the fact that

$$Ind_{\{e\}}^G 1 = \sum_{\rho \in \widehat{G}}{}^{\oplus} d_\rho \, \rho. \tag{18.2}$$

□

We define some matrices associated to a representation ρ of $G(Y/X)$, where Y/X is a finite unramified normal covering of graphs.

Definition 42. *For* $\sigma, \tau \in G$ *and vertices* $a, b \in X$, *define the* $A(\sigma, \tau)$ ***matrix*** *to be the* $n \times n$ *matrix given by setting the entry* $A(\sigma, \tau)_{a,b} =$ *the number of directed edges in* Y *from* (a, σ) *to* (b, τ). *Here every undirected edge of* Y *has been given both directions.*

Except when (a, σ) and (b, τ) are the same vertex on Y (i.e., $a = b$ and $\sigma = \tau$), and even then if there is no loop at $(a, \sigma) = (b, \tau)$, $A(\sigma, \tau)_{a,b}$ is simply the number of undirected edges on Y connecting (a, σ) to (b, τ). However if there is a loop at $(a, \sigma) = (b, \tau)$, then it is counted in both directions and thus the undirected loop is counted twice. It follows from the Exercise below that we can write

$$A(\sigma, \tau) = A(1, \sigma^{-1}\tau) = A(\sigma^{-1}\tau). \tag{18.3}$$

Exercise 59. *Show that*

$$A(\sigma, \tau) = A(1, \sigma^{-1}\tau).$$

Definition 43. *If* ρ *is a representation of* $G(Y/X)$ *and* $A(\sigma, \tau)$ *is given by Definition 42 and Formula (18.3), define the* A_ρ ***matrix*** *by*

$$A_\rho = \sum_{\sigma \in G} A(\sigma) \otimes \rho(\sigma).$$

Also set

$$Q_\rho = Q \otimes I_d,$$

where $Q =$ *the* $|X| \times |X|$ *diagonal matrix with diagonal entry corresponding to* $a \in X$ *given by* $q_a = (\text{degree } a) - 1$ *and* d *is the degree of* ρ.

Theorem 21. ***Block Diagonalization of the Adjacency Matrix of a Normal Cover.*** *Suppose that Y/X is normal with Galois group $G = G(Y/X)$. Let $\widehat{G}$ be a complete set of inequivalent irreducible unitary representations of G. Then one can block diagonalize the adjacency matrix of Y as with diagonal blocks A_ρ, each listed d_ρ times as ρ runs through $\widehat{G}$.*

Proof. The adjacency operator on Y may be viewed as a coming from the representation $Ind_{\{e\}}^G 1$ with the decomposition in formula (18.2). List the vertices of Y as (x, τ), $x \in X$, $\tau \in G$. This decomposes A_Y into $n \times n$ blocks, where $n = |X|$, with blocks given by Definition 42 $A(\sigma, \tau) = A(\sigma^{-1}\tau)$, using formula (18.3) for $\sigma, \tau \in G$. This means $\sigma \in G$ is acting on the function $A : G \to \mathbb{R}$ via $\lambda(\sigma)A(\tau) = A(\sigma^{-1}\tau)$, with $\sigma, \tau \in G$. Then λ is the left regular representation of G. This is equivalent to $Ind_{\{e\}}^G 1$. It follows from formula (18.2) that A_Y has block decomposition into blocks A_ρ corresponding to $\rho \in \widehat{G}$, each listed d_ρ times. □

Now we can generalize Theorem 1.

Theorem 22. ***Ihara Theorem for Vertex Artin L-Function.***
With the hypotheses and definitions above, we have

$$L_V(u, \rho, Y/X)^{-1} = (1 - u^2)^{(r-1)d} \det(I - A_\rho\, u + Q_\rho u^2).$$

Here r is the rank of the fundamental group of X.

Proof. We postpone the proof until the next section where we give the L-function version of Bass's proof of Theorem 1. For this we will need edge Artin L-functions. □

Example 20. ***The Cube over the Tetrahedron.***

See Figure 34, where the action of the group $G = G(Y/X) = \{1, \sigma\}$ on Y is denoted with primes; i.e., $x' = (x, 1)$ and $x'' = (x, \sigma)$, for $x \in X$. In this case the representations of G are the trivial representation $\rho_0 = 1$ and the representation ρ defined by $\rho(1) = 1$, $\rho(\sigma) = -1$. So $Q_\rho = 2I_4$. There are two cases.

Case 1. The representation $\rho_0 = 1$.
Here $A_1 = A(1) + A(\sigma) = A$, where

$$A(1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A(\sigma) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and A is the adjacency matrix of X.

Case 2. The representation ρ.
Here we find

$$A_\rho = A(1) - A(\sigma) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

Now we proceed to check our formulas for this case. We know by the Corollary to Proposition 8 that

$$\zeta_Y(u) = L_V(u, 1, Y/Y) = L_V(u, 1, Y/X)L_V(u, \rho, Y/X) = \zeta_X(u)L_V(u, \rho, Y/X). \tag{18.4}$$

Ihara's Theorem 1 implies that

$$\zeta_X(u)^{-1} = (1 - u^2)^2(1 - u)(1 - 2u)(1 + u + 2u^2)^3$$

and

$$\zeta_Y(u)^{-1} = (1 - u^2)^2(1 + u)(1 + 2u)(1 - u + 2u^2)^3\zeta_X(u)^{-1}.$$

Then Theorem 22 implies (since $r = 3$)

$$\begin{aligned} L_V(u, \rho, Y/X)^{-1} &= (1 - u^2)^2 \det(I_4 - A'_\rho u + 2u^2 I_4) \\ &= (1 - u^2)^2(1 + u)(1 + 2u)(1 - u + 2u^2)^3. \end{aligned}$$

It is noteworthy that here $L_V(u, \rho, Y/X) = \zeta_X(-u)$, although this is not instantly obvious from the determinant formula where $-A \neq A_\rho$.

Note. Theorem 22 implies that equation (18.4) is a factorization of an 8×8 determinant as a product of 4×4 determinants:

$$det(I_8 - A_Y u + 2I_8 u^2) = det(I_4 - A_X u + 2I_4 u^2) \cdot det(I_4 - A'_\rho u + 2I_4 u^2).$$

Exercise 60. *Compute the spectra of the adjacency matrices of the cube and the tetrahedron. Are they Ramanujan?*

It is perhaps worthwhile to state the case of 2-coverings separately. We will also give an example of this proposition below (the cube over the tetrahedron).

Proposition 9. *If Y/X is a 2-covering, then the adjacency matrix A_Y has block decomposition with 2 blocks: one block being A_X (the adjacency matrix of X) and the other block being A_-. The matrix A_- is defined by having entry corresponding to two vertices a, b of X given by:*

$$(A_-)_{a,b} = \begin{cases} +1, & a \text{ and } b \text{ joined by edge } e \text{ in } X \text{ which lifts to edge of } Y \text{ that does not change sheets;} \\ -1 & a \text{ and } b \text{ joined by edge } e \text{ in } X \text{ which lifts to edge of } Y \text{ that changes sheets;} \\ 0 & a \text{ and } b \text{ not joined by edge } e \text{ in } X. \end{cases}$$

Note the following Conjecture made in Hoory et al [39].

Conjecture 1. *Every d-regular graph X has a 2 covering Y such that if A_Y is the adjacency matrix of Y, then*

$$Spectrum(A_Y) - Spectrum(A_X) \subset [-2\sqrt{d-1}, 2\sqrt{d-1}].$$

This conjecture would allow one to construct families of Ramanujan graphs of arbitrary degree with number of vertices going to infinity by taking repeated 2-covers.

Exercise 61. *1) For Example 20 all edges of K_4 not in the chosen spanning tree were lifted to start at sheet 1 and end at sheet 2. What happens if you only lift 1 edge?*
2) Check Conjecture 1 for spectra of 2-covers of K_4.

Exercise 62. *Experiment with Conjecture 1 concerning spectra of 2-covers to see whether any k-regular graph does have a 2-cover such that the spectrum of A_- lies in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. For example, you could look at all 2-coverings of the torus graph X obtained by taking a product of a 3-cycle and a 5-cycle. Are any of the 2-covers of X Ramanujan?*

There are commands in Mathematica to do most of this. First load the discrete math. package.

```
<< DiscreteMath`Combinatorica`
```

Then define the torus graph.

```
X=GraphProduct[Cycle[3],Cycle[10]]
```

Then compute the adjacency matrix - assuming the version of Mathematica is recent.

```
ToAdjacencyMatrix[X]
```

To get the new eigenvalues of the adjacency matrix A for a 2-cover Y, fix a spanning tree T in X, decide which edges not in T you are lifting from sheet 1 of Y to sheet 2, and put -1 for 1 in the corresponding places in A. Find the spectrum of this new symmetric matrix B.

```
Eigenvalues[B].
```

Example 21. ***The Cube over a Dumbbell.***

The covering we consider in this example is Y/X in Figure 37. The covering group $G(Y/X)$ is the integers mod 4 denoted $\mathbb{Z}_4 = \{0, 1, 2, 3 \pmod 4\}$. We label the sheets as follows:

$$x_1' = (x, 0(mod\ 4)),\ x_2' = (x, 1(mod\ 4)),\ x_1'' = (x, 2(mod\ 4)),\ x_2'' = (x, 3(mod\ 4)).$$

The irreducible representations are all one - dimensional and may be written $\chi_\nu(j) = \exp\left(\frac{2\pi i \nu j}{4}\right) = i^{\nu j}$, for $j, \nu \in \mathbb{Z}_4$. Note that although X has loops, Y does not. It follows that

$$A(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A(1) = A(3) = I_2,\ A(2) = 0.$$

Thus

$$A_{\chi_0} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ A_{\chi_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A_{\chi_3},\ A_{\chi_2} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The corresponding L-functions are

$$\begin{aligned} L(u,\chi_0,Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1-2u+2u^2 & -u \\ -u & 1-2u+2u^2 \end{pmatrix} \\ &= (1-u^2)(1-u)(1-2u)(1-u+2u^2); \end{aligned}$$

$$\begin{aligned} L(u,\chi_1,Y/X)^{-1} &= L(u,\chi_3,Y/X) = (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u \\ -u & 1+2u^2 \end{pmatrix} \\ &= (1-u^2)(1+u+2u^2)(1-u+2u^2) \end{aligned}$$

$$\begin{aligned} L(u,\chi_2,Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u+2u^2 & -u \\ -u & 1+2u+2u^2 \end{pmatrix} \\ &= (1-u^2)(1+u)(1+2u)(1+u+2u^2). \end{aligned}$$

One sees again that as in the Corollary to Proposition 8

$$\zeta_Y(u)^{-1} = L(u,\chi_0,Y/X)L(u,\chi_1,Y/X)L(u,\chi_2,Y/X)L(u,\chi_3,Y/X).$$

Note. Again you can view the preceding equality as a factorization of the determinant of an 8×8 matrix as a product of 4 determinants of 2×2 matrices.

Example 22. ***An*** S_3 ***Cover.***

Now consider the example in Figure 43. Here view the group S_3 as the dihedral group D_3. Thus it consists of motions of a regular triangle and is generated by F a flip and R a rotation.

$$A(I)=\begin{pmatrix} 0&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\end{pmatrix},\quad A(FR^2)=\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$A(FR)=\begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix},\quad A(R^2)=0,\quad A(R)=0,\quad A(F)=0.$$

Next we need to know the representations of S_3. See Terras [92], Chapters 16 and 17. The non-trivial 1-dimensional representation of S_3 has the values $\chi_1(FR)=-1$ and $\chi_1(FR^2)=-1$. The 2-dimensional representation ρ has the values

$$\rho(FR)=\begin{pmatrix} 0&\omega^2\\ \omega&0\end{pmatrix},\ \text{and}\ \rho\left(FR^2\right)=\begin{pmatrix} 0&\omega\\ \omega^2&0\end{pmatrix},\ \text{where}\ \omega=e^{2\pi i/3}.$$

Now we can compute the matrices in our L-functions:

$$\begin{aligned} A_{\chi_0} &= A,\ A_{\chi_1}=\begin{pmatrix} 0&1&1&1\\1&0&0&-1\\1&0&0&-1\\1&-1&-1&0\end{pmatrix}, \\ A_\rho &= A_1(I)\otimes\rho(I)+A_1(FR)\otimes\rho(FR)+A_1(FR^2)\otimes\rho(FR^2) \\ &= \begin{pmatrix} 0&1&1&1&0&0&0&0\\ 1&0&0&0&0&0&0&\omega^2\\ 1&0&0&0&0&0&0&\omega\\ 1&0&0&0&0&\omega^2&\omega&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&\omega&1&0&0&0\\ 0&0&0&\omega^2&1&0&0&0\\ 0&\omega&\omega^2&0&1&0&0&0 \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} L_V(u,\chi_0,Y_6/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2&-u&-u&-u\\-u&1+u^2&0&-u\\-u&0&1+u^2&-u\\-u&-u&-u&1+2u^2\end{pmatrix} \\ &= (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3); \end{aligned}$$

$$\begin{aligned} L_V(u,\chi_1,Y_6/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2&-u&-u&-u\\-u&1+u^2&0&u\\-u&0&1+u^2&u\\-u&u&u&1+2u^2\end{pmatrix} \\ &= (1-u^2)(1+u)(1+u^2)(1-u+2u^2)(1-u^2+2u^3); \end{aligned}$$

$$\begin{aligned} L_V(u,\rho,Y_6/X)^{-1} &= (1-u^2)^2\det(I_8 - A_\rho u + u^2 Q_\rho). \\ &= (1-u^2)^2(1+u+2u^2+u^3+2u^4)(1+u+u^3+2u^4) \\ &\quad \times(1-u+2u^2-u^3+2u^4)(1-u-u^3+2u^4). \end{aligned}$$

Putting all our results together, using Theorem 22, we have:

$$\zeta_X(u)^{-1} = L_V(u,\chi_0,Y_6/X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3);$$

$$\begin{aligned} \zeta_{Y_2}(u)^{-1}\zeta_X(u) &= L_V(u,\chi_1,Y_2/X)^{-1} = L_V(u,\chi_1,Y_6/X)^{-1} \\ &= (1-u^2)(1+u)(1+u^2)(1-u+2u^2)(1-u^2+2u^3); \end{aligned}$$

$$\begin{aligned} \zeta_{Y_3}(u)^{-1}\zeta_X(u) &= L_V(u,\rho,Y_6/X)^{-1} \\ &= (1-u^2)^2(1+u+2u^2+u^3+2u^4)(1+u+u^3+2u^4) \\ &\quad \times(1-u+2u^2-u^3+2u^4)(1-u-u^3+2u^4); \end{aligned}$$

and

$$\zeta_{Y_6}(u) = L_V(u,\chi_0,Y_6/X)L_V(u,\chi_1,Y_6/X)L_V(u,\rho,Y_6/X)^2 = \zeta_X(u)\frac{\zeta_{Y_2}(u)}{\zeta_X(u)}\left[\frac{\zeta_{Y_3}(u)}{\zeta_X(u)}\right]^2.$$

As a consequence, we find that

$$\zeta_X(u)^2\zeta_{Y_6}(u) = \zeta_{Y_2}(u)\zeta_{Y_3}(u)^2.$$

This is analogous to an example of zeta functions of pure cubic extensions of number fields that goes back to Dedekind. See Stark [81].

Note. Again the last equality can be viewed as two different factorizations of determinants involving polynomials in u.

Example 23. *A Klein 4-Group Cover* $\mathbf{Y/X}$ *from Figure 38.*

Here we can identify the Galois group $G = G(Y/X)$ with $\mathbb{Z}_2^2$. The identification is given by: $x_1' = (x,(1,0))$, $x_1'' = (x,(1,1))$, $x_2' = (x,(0,0))$, $x_2'' = (x,(0,1))$.

The characters of G are $\chi_{r,s}(u,v) = (-1)^{ru+sv}$, for $r,s,u,v \in \mathbb{Z}_2$. We find that

$$\begin{aligned} L_V(u,\chi_{0,0},Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2 & -3u \\ -3u & 1+2u^2 \end{pmatrix} \\ &= Z_X(u)^{-1} = (1-u^2)(1-u)(1+u)(1-2u)(1+2u). \end{aligned}$$

Similarly

$$\begin{aligned} L_V(u,\chi_{0,1},Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2 & -u \\ -u & 1+2u^2 \end{pmatrix} = L_V(u,\chi_{1,1},Y/X)^{-1} \\ &= Z_X(u)^{-1} = (1-u^2)(1-u+2u^2)(1+u+2u^2). \end{aligned}$$

Also

$$\begin{aligned} L_V(u,\chi_{1,0}Y/X)^{-1} &= (1-u^2)\det\begin{pmatrix} 1+2u^2 & u \\ u & 1+2u^2 \end{pmatrix} \\ &= (1-u^2)(1-u+2u^2)(1+u+2u^2). \end{aligned}$$

Thus all 3 L-functions with non-trivial characters are equal. This happens here because all 3 intermediate quadratic covers of X are isomorphic as abstract graphs (although not conjugate) and so they have equal zeta functions. Each intermediate zeta function is of the form $\zeta_{\widetilde{X}}(u) = \zeta_X(u)L_V(u,\chi,Y/X)$, where χ runs through the 3 non-trivial characters of G as $\widetilde{X}$ runs through the 3 intermediate quadratic covers of X. For $\zeta_Y(u)$ we have

$$\begin{aligned} \zeta_Y(u)^{-1} &= \prod_{\chi\in\widehat{G}} L_V(u,\chi,Y/X) \\ &= (1-u^2)^4(1-u)(1+u)(1-2u)(1+2u)(1-u+2u^2)^3(1+u+2u^2)^3. \end{aligned}$$

We also have

$$\zeta_X^2(u)\zeta_Y(u) = \zeta_{\widetilde{X}}(u)^3$$

which holds for all 3 intermediate quadratic covers $\widetilde{X}$ of X.

Example 24. *A Cyclic 6-Fold Cover* $\mathbf{Y/X}$ *from Figure 39.*

The covering group $G = G(Y/X) \cong \mathbb{Z}_6 = \{1, 2, 3, 4, 5, 6\ (mod\ 6)\}$, with identity element 6 (mod 6). Let $\omega = e^{2\pi i/6}$. The characters are $\chi_a(b) = \omega^{ab}$, for $a, b \in \mathbb{Z}_6$. Here the matrices $A(\tau)$ are 1×1. We obtain

$$A(6) = A(3) = 0, \quad A(1) = A(2) = A(4) = A(5) = 1.$$

We find that

$$\begin{aligned} A_{\chi_0} &= 4 = A = \text{adjacency matrix of } X; \\ A_{\chi_j} &= 0, \text{ for } j = 1, 3, 5; \\ A_{\chi_j} &= -2, \text{ for } j = 2, 4. \end{aligned}$$

Then

$$\begin{aligned} L_V(u, \chi_0, Y/X)^{-1} &= \zeta_X(u)^{-1} = (1-u^2)(1-u)(1-3u); \\ L_V(u, \chi_j, Y/X)^{-1} &= (1-u^2)(1+3u^2), \text{ for } j = 1, 3, 5; \\ L_V(u, \chi_j, Y/X)^{-1} &= Z_X(u)^{-1} = (1-u^2)(1+2u+3u^2), \text{ for } j = 2, 4. \end{aligned}$$

Set

$$m = \begin{pmatrix} 1+3u^2 & -u & -u & 0 & -u & -u \\ -u & 1+3u^2 & -u & -u & 0 & -u \\ -u & -u & 1+3u^2 & -u & -u & 0 \\ 0 & -u & -u & 1+3u^2 & -u & -u \\ -u & 0 & -u & -u & 1+3u^2 & -u \\ -u & -u & 0 & -u & -u & 1+3u^2 \end{pmatrix}.$$

By Ihara's formula

$$\begin{aligned} \zeta_Y(u)^{-1} &= (1-u^2)^6 \det(m) \\ &= (1-u^2)^6 (3u-1)(u-1)\left(3u^2+2u+1\right)^2 \left(1+3u^2\right)^3, \end{aligned}$$

which agrees with the product

$$\zeta_Y(u)^{-1} = \prod_{\chi \in \widehat{G}} L_V(u, \chi, Y/X).$$

Exercise 63. *Check whether the preceding graphs and covering graphs are Ramanujan graphs.*

19. Edge Artin L-Functions

Suppose that Y/X is a normal graph covering and recall Definition 23 of the edge matrix, Definition 24 of the edge zeta function, Definiton 39 of the Frobenius automorphism. We use these definitions to define the edge Artin L-function imitating the definitions from algebraic number theory.

Definition 44. *Given a path C in X, which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the* ***edge norm*** *of C is*

$$N_E(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{s-1} a_s} w_{a_s a_1}.$$

The ***edge Artin L- function*** *associated to a representation ρ of the Galois group $G(Y/X)$ and the edge matrix W is*

$$L(W, \rho) = L_E(W, \rho, Y/X) = \prod_{[C]} \det\left(I - \rho\left(\frac{Y/X}{D}\right) N_E(C)\right)^{-1},$$

where the product is over primes $[C]$ in X and $[D]$ is arbitrarily chosen from the primes in Y over C. Here W is the edge matrix of Definition 23 with variables $|w_{ef}|$ assumed sufficiently small, and $\left(\frac{Y/X}{D}\right)$ is the Frobenius automorphism of Definition 39.

Exercise 64. *Show that the determinant in the definition of the edge zeta function does not depend on the choice of D over C in Definition 44.*

Hint. *The various Frobenii $\left(\frac{Y/X}{D}\right)$ are conjugate to each other.*

For the factorization of edge zeta functions, we need a specialization of W matrices.

Definition 45. *Suppose that $\widetilde{X}$ is an unramified covering of X and that $\widetilde{W}$ and W are the corresponding edge matrices. Suppose that $\widetilde{e}$ and $\widetilde{f}$ are two edges of $\widetilde{X}$ with projections e and f in X using the covering map $\pi : \widetilde{X} \longrightarrow X$. If $\widetilde{e}$ feeds into $\widetilde{f}$ and $\widetilde{e} \neq \widetilde{f}^{-1}$, then e feeds into f and $e \neq f^{-1}$. Thus we can set the variable $\widetilde{w}_{\widetilde{e}\widetilde{f}} = w_{ef}$. When we do this for all the variables of $\widetilde{W}$, we call the* ***X-specialized matrix*** *$\widetilde{W}_{spec}$.*

Theorem 23. ***Main Properties of Edge Artin L-Functions.***
Assume $\widetilde{X}$ is a normal (unramified) cover of X.
1) The edge L-function at the trivial representation is the edge zeta function:

$$L_E(W, 1, \widetilde{X}/X) = \zeta_E(W, X),$$

2) The edge zeta function of $\widetilde{X}$ factors as a product of edge L-functions:

$$\zeta_E(\widetilde{W}_{spec}, \widetilde{X}) = \prod_{\rho \in \widehat{G}} L_E(W, \rho)^{d_\rho}.$$

Here the product is over all inequivalent irreducible unitary representations of the Galois group $Gal(\widetilde{X}/X)$. The matrix $\widetilde{W}_{spec}$ is the X-specialized multiedge matrix of Definition 45.

3)Let $m = |E|$ be the number of unoriented edges of X. If the representation ρ of G has degree d, define a $2dm \times 2dm$ matrix W_ρ with block form

$$W_\rho = \left(w_{ef}\,\rho(\sigma(e))\right),$$

where $\sigma(e)$ denotes the normalized Frobenius element of Definition 38 corresponding to edge e. Then

$$L_E(W, \rho, Y/X) = \det(I - W_\rho)^{-1}.$$

Part 1) follows from the definitions. Part 2) is proved using formula (18.2) and parts 2 and 4 of the next Theorem. For part 4) of the next proposition you need to know about representation π of G induced by a representation $\rho : H \to U(d_\rho) =$ {unitary $d_\pi \times d_\pi$ matrices} of a subgroup H of G. One defines the **induced representation** $\pi = Ind_H^G \rho$ as a linear transformation of the space of functions $\left\{ f : G \to \mathbb{C}^{d_\rho} \mid f(hx) = \rho(h) f(x), \forall h \in H, x \in G \right\}$. Then $\left[\pi(g) f\right](x) = f(xg), \forall x, g \in G$. See Terras [92] for more information.

The **direct sum** $\rho_1 \oplus \rho_2$ of 2 representations ρ_1, ρ_2 of G is defined to have block matrix decompositon $(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$.

Theorem 24. ***More Properties of Edge Artin L-Functions.*** *Assume that Y/X is an (unramified) normal cover. with Galois group G.*
1) If you specialize the non-zero w_{ij} to be u, then $L_E(W, \rho)$ specializes to $L_V(u, \rho)$.
2) $L_E(W, \rho_1 \oplus \rho_2) = L_E(W, \rho_1) L_E(W, \rho_2)$.
3) If $\widetilde{X}$ is intermediate to Y/X, $G = Gal(Y/X)$ and $H = Gal(Y/\widetilde{X})$. Assume that $\widetilde{X}/X$ is normal. Let ρ be a representation of $G/H \cong Gal(\widetilde{X}/X)$. Then ρ can be viewed as a representation of G, often called the ***lift*** *of ρ. Then*

$$L_E(W, \rho, Y/X) = L_E(W, \rho, \widetilde{X}/X).$$

4) Suppose H is any subgroup of $G = Gal(Y/X)$. Let $\widetilde{X}$ be the intermediate cover to Y/X corresponding to H by Theorem 16. Now we do not assume that H is a normal subgroup of G. Let ρ be a representation of H and let $\rho^{\#}$ denote the ***representation of** G **induced by*** *ρ. Then, using Definition 45 of $\widetilde{W}_{spec}$,*

$$L_E(\widetilde{W}_{spec}, \rho, Y/\widetilde{X}) = L_E(W, \rho^{\#}, Y/X).$$

Proof. Part 1) follows from the definitions. Part 2) is easily proved by rewriting the logarithm of the L-function as a sum involving traces of the representations since $Tr\,(\rho_1 \oplus \rho_2) = Tr\rho_1 + Tr\rho_2$. Part 3) follows from the definitions. Part 4) will be proved later. □

Example 25. ***The edge L-function of a Cube covering a Dumbbell.***

The edge L-functions for the representations of the Galois group of Y/X, which is $\mathbb{Z}_4$, require the matrix W which has entries w_{ij}, when edge e_i feeds into edge e_j. For the labeling of the edges of the dumbbell, see Figure 54. We find that the matrix W is:

$$W = \begin{pmatrix} w_{11} & w_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33} & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44} & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66} \end{pmatrix}.$$

Next we need to compute $\sigma(e_i)$ for each edge e_i where $\sigma(C)$ denotes the normalized Frobenius automorphism of Definition 38. We will write the Galois group $G(Y/X) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where $(x, \sigma_j) = x^{(j)}$, for $x \in X$. The identification of $G(Y/X)$ with $\mathbb{Z}_4$ sends σ_j to $(j - 1(mod\ 4))$. Then compute the Galois group elements associated to the edges: $\sigma(e_1) = \sigma_2$, $\sigma(e_2) = \sigma_1$, $\sigma(e_3) = \sigma_2$. The representations of our group are 1-dimensional, given by $\chi_a(\sigma_b) = i^{a(b-1)}$, for $a, b \in \mathbb{Z}_4$.

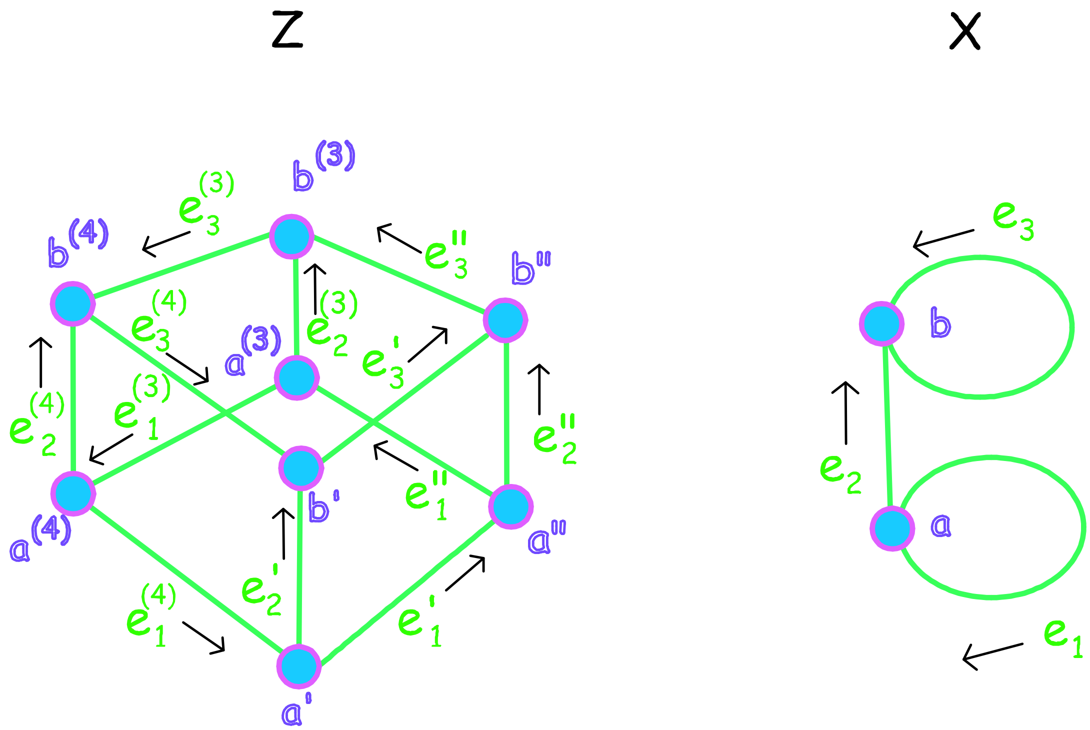

FIGURE 54. Edge Labelings for the Cube as a $\mathbb{Z}_4$ Covering of the Dumbbell.

So we obtain

$$L_E(W,\chi_0,Y/X)^{-1}=\zeta_E(W,X)^{-1}=det\begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{23} & 0 & 0 & w_{26}\\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0\\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0\\ w_{51} & 0 & 0 & w_{54} & -1 & 0\\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1\end{pmatrix};$$

$$L_E(W,\chi_1,Y/X)^{-1}=det(I-W_{\chi_1})=det\begin{pmatrix} iw_{11}-1 & iw_{12} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{23} & 0 & 0 & w_{26}\\ 0 & 0 & iw_{33}-1 & 0 & iw_{35} & 0\\ 0 & -iw_{42} & 0 & -iw_{44}-1 & 0 & 0\\ w_{51} & 0 & 0 & w_{54} & -1 & 0\\ 0 & 0 & 0 & 0 & -iw_{65} & -iw_{66}-1\end{pmatrix};$$

$$L_E(W,\chi_2,Y/X)^{-1}=det(I-W_{\chi_2})=det\begin{pmatrix} -w_{11}-1 & -w_{12} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{23} & 0 & 0 & w_{26}\\ 0 & 0 & -w_{33}-1 & 0 & -w_{35} & 0\\ 0 & -w_{42} & 0 & -w_{44}-1 & 0 & 0\\ w_{51} & 0 & 0 & w_{54} & -1 & 0\\ 0 & 0 & 0 & 0 & -w_{65} & -w_{66}-1\end{pmatrix};$$

$$L_E(W,\chi_3,Y/X)^{-1}=det(I-W_{\chi_3})=det\begin{pmatrix} -iw_{11}-1 & -iw_{12} & 0 & 0 & 0 & 0\\ 0 & -1 & w_{23} & 0 & 0 & w_{26}\\ 0 & 0 & -iw_{33}-1 & 0 & -iw_{35} & 0\\ 0 & iw_{42} & 0 & iw_{44}-1 & 0 & 0\\ w_{51} & 0 & 0 & w_{54} & -1 & 0\\ 0 & 0 & 0 & 0 & iw_{65} & iw_{66}-1\end{pmatrix}.$$

Note that, by part 2 of Theorem 23, the product of the preceding four 6×6 determinants has to be a 24×24 determinant:

$$det(I - \widetilde{W}_{spec}) = \prod_{i=0}^{3} det(I - W_{\chi_i}),$$

where $\widetilde{W}_{spec}$ is given by specializing the multiedge matrix of the cube as follows. For the edge variables of the dumbbell, we write

$$a = w_{11}, b = w_{12}, c = w_{23}, d = w_{26}, e = w_{33}, f = w_{35},$$
$$g = w_{42}, h = w_{44}, j = w_{51}, k = w_{54}, m = w_{65}, n = w_{66}.$$

Using the natural ordering of the edges of Y given by

$$e_1', e_1'', e_1^{(3)}, e_1^{(4)}, e_2', e_2'', e_2^{(3)}, e_2^{(4)}, e_3', e_3'', e_3^{(3)}, e_3^{(4)}, \text{ and the inverses in the same order,}$$

the matrix $\widetilde{W}_{spec}$ is the following

$$\begin{pmatrix}
0&a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
a&0&0&0&b&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&0&0&0&0&d\\
0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&d&0&0&0\\
0&0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&d&0&0\\
0&0&0&0&0&0&0&0&0&0&0&c&0&0&0&0&0&0&0&0&0&0&d&0\\
0&0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0\\
0&0&0&0&0&0&0&0&e&0&0&0&0&0&0&0&f&0&0&0&0&0&0&0\\
0&0&0&0&g&0&0&0&0&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0\\
0&0&0&0&0&g&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&g&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&g&0&0&0&0&0&0&h&0&0&0&0&0&0&0&0&0\\
j&0&0&0&0&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0\\
0&j&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0&0&0&0\\
0&0&j&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0&0&0\\
0&0&0&j&0&0&0&0&0&0&0&0&0&0&k&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&0&0&0&0&n\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&n&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&n&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&m&0&0&n&0
\end{pmatrix}.$$

One can prove part 3) of Theorem 23 using the same method as we gave for the analogous result for the edge Ihara zeta function. Let's begin this proof

Proof. of part 3) of Theorem 23.

We are trying to prove that

$$L_E(W, \rho, Y/X) = \det(I - W_\rho)^{-1}.$$

For this we imitate the proof of Theorem 11. with $\chi = Tr(\rho)$. So we start by applying the Euler differential operator to the logarithm of the L-function, obtaining

$$\begin{aligned}
\sum_{w_{ij}} w_{ij} \frac{\partial}{\partial w_{ij}} log(L_E(W, \rho, Y/X)) &= \sum_{[C]} \sum_{j=1}^{\infty} \nu(C) \chi(\sigma(C)^j) N_E(C^j) \\
&= \sum_{C} \sum_{j=1}^{\infty} \chi(\sigma(C)^j) N_E(C^j).
\end{aligned}$$

Now the sum is over paths C rather than classes $[C]$.

The block i_1, i_{n+1} entry of W_ρ^n is

$$\sum_{i_2,\ldots,i_n} w(i_1,i_2)\cdots w(i_n,i_{n+1})\rho(\sigma(i_1))\cdots\rho(\sigma(i_n)) = \sum_{\substack{C=i_1\cdots i_n \\ \nu(C)=n}} w(i_1,i_2)\cdots w(i_n,i_{n+1})\rho(\sigma(C)),$$

where the sum is over all paths C on X of length n with leading edge i_1.

The last sum may be restricted to those paths C whose initial edge is i_1 and whose terminal edge i_n feeds into i_{n+1} with the additional condition that i_{n+1} is not the inverse to i_n, since all remaining paths contribute 0 to the sum. Thus when $i_{n+1} = i_1$, we are talking about closed backtrackless, tailless paths.

Therefore

$$\begin{aligned}\sum_{w_{ij}} w_{ij}\frac{\partial}{\partial w_{ij}} log(L_E(W,\rho,Y/X)) &= Tr\left((I-W_\rho)^{-1}\right) \\ &= \sum_{w_{ij}} w_{ij}\frac{\partial}{\partial w_{ij}} log\left(\det\left((I-W_\rho)^{-1}\right)\right).\end{aligned}$$

The application of Lemma 3 completes the proof of part 3) of Theorem 23. □

Bass Proof of Ihara Theorem for Vertex Artin L-Functions.

Next we give the Bass proof of the Ihara Theorem 22 for vertex Artin L-functions. We must first generalize the S, T matrices in Proposition 4. For a representation ρ of the Galois group $G(Y/X)$, let d_ρ be its degree (i.e., the size of the matrices it goes into). When we write $B \otimes C$ for the $p \times p$ matrix B and the $r \times r$ matrix C, we mean the $pr \times pr$ matrix with block decomposition

$$B \otimes C = \begin{pmatrix} b_{11}C & \cdots & b_{1p}C \\ \vdots & \ddots & \vdots \\ b_{p1}C & \cdots & b_{pp}C \end{pmatrix}.$$

Definition 46. *With the definitions of S, T, J as in Proposition 4, set*

$$S_\rho = S \otimes I_{d_\rho},\ T_\rho = T \otimes I_{d_\rho},\ J_\rho = J \otimes I_{d_\rho}.$$

Definition 47. *We will also define the $2md_\rho \times 2md_\rho$ block diagonal matrix R_ρ to be*

$$R_\rho = \begin{pmatrix} \rho(\sigma(e_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}. \tag{19.1}$$

Here $e_1, \ldots, e_{2m}$ is our list of oriented edges of X, ordered in our ususal way. The last equality comes from the property of the normalized Frobenius saying $\sigma(e) = \sigma(e^{-1})$.

Recall from part 3) of Theorem 23 that if $m = |E|$, the matrix W_ρ is defined to be a $2md_\rho \times 2md_\rho$, matrix having $d_\rho \times d_\rho$ block corresponding to oriented edges $e.f$ given by $\left(W_\rho\right)_{e.f} = w_{e,f}\rho(\sigma(e))$. Define $\widetilde{W_\rho}$ by $\left(W_\rho\right)_{e.f} = \rho(\sigma(e))$. Note that there could be a conflict of notation if ρ is the trivial representation. We want W_1 to be the 0,1 edge matrix obtained by specializing the W matrix non-zero entries to be 1. Hopefully it will be clear from the context what we mean.

Finally recall that A_ρ, the adjacency matrix associated to ρ is as in Definition 43. With all the preceding definitions, we have the following proposition relating all the matrices.

Proposition 10. ***Formulas Involving ρ,Q,W,A,R,S,T,J.***

1) $\widetilde{W_\rho} = R_\rho\,(W_1 \otimes I_d)$.
2) $A_\rho = S_\rho R_\rho\,{}^tT_\rho$.
3) $S_\rho J_\rho = T_\rho,\ T_\rho J_\rho = S_\rho,\ Q_\rho + I_{nd_\rho} = S_\rho\,{}^tS_\rho = T_\rho\,{}^tT_\rho$.
4) $\widetilde{W_\rho} + R_\rho J_\rho = R_\rho\,{}^tT_\rho\,S_\rho$.
5) $(R_\rho J_\rho)^2 = I_{2|E|d}$.

Proof. 1) To see this, just multiply matrices in block form:

$$\left(\widetilde{W_\rho}\right)_{e,f} = (W_1)_{ef}\,\rho(\sigma(e)) = \left(\begin{pmatrix} \rho(\sigma(e_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix}\begin{pmatrix} w_{e_1,e_1}I_{d_\rho} & \cdots & w_{e_1,e_{2m}}I_{d_\rho} \\ \vdots & \ddots & \vdots \\ w_{e_{2m},e_1}I_{d_\rho} & \cdots & w_{e_{2m},e_{2m}}I_{d_\rho} \end{pmatrix}\right)_{e,f}.$$

2) Set $d = d_\rho$. Then we have

$$S_\rho R_\rho \, {}^t T_\rho = (S \otimes I_d) \begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} {}^t (T \otimes I_d)$$

$$= \begin{pmatrix} S_{11} I_d & \cdots & S_{1\,2m} I_d \\ \vdots & \ddots & \vdots \\ S_{n1} I_d & \cdots & S_{n\,2m} I_d \end{pmatrix} \begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} \begin{pmatrix} t_{11} I_d & \cdots & t_{n1} I_d \\ \vdots & \ddots & \vdots \\ t_{1\,2m} I_d & \cdots & t_{n\,2m} I_d \end{pmatrix}.$$

Then look at the block corresponding to vertices a, b of X and obtain

$$\begin{aligned} \left(S_\rho R_\rho \, {}^t T_\rho\right)_{a,b} &= \sum_e s_{a,e} \rho(\sigma(e)) t_{b,e} = \sum_{g \in G} \rho(g) \sum_{e, \sigma(e) = g} s_{a,e} t_{b,e} \\ &= \sum_{g \in G} (A(g))_{a,b} \rho(g). \end{aligned}$$

Here the last equality uses Definition 43 of $A(g)$. The result in part 2) follows.
3) The proof proceeds by the following computation:

$$\left(S_\rho J_\rho\right)_{v,e} = \left(\begin{pmatrix} s_{11} I_d & \cdots & s_{1\,2m} I_d \\ \vdots & \ddots & \vdots \\ s_{n1} I_d & \cdots & s_{n\,2m} I_d \end{pmatrix} \begin{pmatrix} 0 & I_m \otimes I_d \\ I_m \otimes I_d & 0 \end{pmatrix} \right)_{v,e} = (T)_{v,e}.$$

$$\left(S_\rho \, {}^t S_\rho\right)_{a,b} = \sum_e s_{a,e} I_d s_{b,e} I_d = (\#\text{ edges out of } a)\, \delta_{a,b} I_d = (Q + I)_{a,b} I_d$$

4)

$$\begin{aligned} \left(R_\rho \, {}^t T_\rho S_\rho\right)_{e,f} &= \left(\begin{pmatrix} \rho(\sigma(e_1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho(\sigma(e_{2m})) \end{pmatrix} \begin{pmatrix} t_{11} I_d & \cdots & t_{n1} I_d \\ \vdots & \ddots & \vdots \\ t_{12m} I_d & \cdots & t_{n\,2m} I_d \end{pmatrix} \begin{pmatrix} s_{11} I_d & \cdots & s_{1\,2m} I_d \\ \vdots & \ddots & \vdots \\ s_{n1} I_d & \cdots & s_{n\,2m} I_d \end{pmatrix} \right)_{e,f} \\ &= \sum_{\substack{v \\ \overset{e}{\to} v \overset{f}{\to}}} \rho(\sigma(e)) t_{ev} s_{vf} I_d = \rho(\sigma(e)) \, (W_1)_{e,f} + \rho(\sigma(e)) J_{e,f}. \end{aligned}$$

The last term is for the case that $f = e^{-1}$, when $(W_1)_{e,f} = 0$.
5) To prove this, just note that:

$$\left(\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right)^2 = \begin{pmatrix} 0 & U \\ U^{-1} & 0 \end{pmatrix}^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

□

Next we prove the main formulas in the Bass proof.

Proposition 11. ***Main Formulas in Bass Proof of Ihara's Theorem for Artin L-Functions.***
1)

$$\begin{pmatrix} I_{nd} & 0 \\ R_\rho \, {}^t T_\rho & I_{2md} \end{pmatrix} \begin{pmatrix} I_{nd}\left(1 - u^2\right) & S_\rho u \\ 0 & I_{2md} - u \widetilde{W_\rho} \end{pmatrix}$$
$$= \begin{pmatrix} I_{nd} - A_\rho u + Q_\rho u^2 & S_\rho u \\ 0 & I_{2md} + R_\rho J_\rho u \end{pmatrix} \begin{pmatrix} I_{nd} & 0 \\ R_\rho \, {}^t T_\rho - {}^t S_\rho u & I_{2md} \end{pmatrix}.$$

2)

$$I_{2md} + R_\rho J_\rho u = \begin{pmatrix} I_{md} & U u \\ U^{-1} u & I_{md} \end{pmatrix}.$$

3)

$$\begin{pmatrix} I_{md} & 0 \\ -U^{-1} u & I_{md} \end{pmatrix} \left(I_{2md} + R_\rho J_\rho u\right) = \begin{pmatrix} I_{md} & U u \\ 0 & I_{md}\left(1 - u^2\right) \end{pmatrix}.$$

Proof. The proofs are an **Exercise** in block multiplication of matrices using Proposition 10. □

Proof. of Theorem 22.

We are trying to show that

$$L_V(u,\rho,Y/X)^{-1} = (1-u^2)^{(r-1)d}\det\left(I_{nd} - A_\rho u + Q_\rho u^2\right).$$

First recall that

$$L_V(u,\rho,Y/X) = L_E(W,\rho,Y/X)|_{\substack{non-0\\ w_{ef}\text{ set }=u}}$$

and

$$L_E(W,\rho,Y/X)^{-1} = \det(I-W).$$

As a consequence, we need to show that

$$\det\left(I_{2md} - u\widetilde{W_\rho}\right) = \left(1-u^2\right)^{(r-1)d}\det\left(I_{nd} - A_\rho u + Q_\rho u^2\right).$$

We see, upon taking determinants of the formula in part 1) of the last Proposition, that

$$\left(1-u^2\right)^{nd}\det\left(I_{2md} - u\widetilde{W_\rho}\right) = \det\left(I_{nd} - A_\rho u + Q_\rho u^2\right)\det\left(I_{2md} + R_\rho J_\rho u\right).$$

Then parts 2) and 3) of the last Proposition imply that

$$\det\left(I_{2md} + R_\rho J_\rho u\right) = \left(1-u^2\right)^{md}.$$

The result follows from the fact that $m-n=r-1$. □

Remarks on Zeta Functions of Weighted Graphs.

Suppose that X is a weighted graph with weight function L.If one replaces Definitions 46 and 47 with $S_L = S, T_L = T, J_L = J, Q_L = Q, R_L = 2m \times 2m$ diagonal matrix with entry corresponding to edge e given by $r(e)$. No tensoring is involved in this case ($d=1$). Part 4) of Proposition 10 morphs into $(R_L\, {}^tTS)_{ef} = r(e)\,(W_1)_{ef}$. We want to multiply this by u and obtain $u^{L(e)}\,(W_1)_{ef}$. This requires $r(e)u = u^{L(e)}$. Thus we want $r(e) = u^{L(e)-1}$. Since we need $r(e^{-1}) = r(e)^{-1}$, we need $L(e^{-1}) = 2 - L(e)$. This is certainly not the usual condition.

This produces a correct specialization of the W matrix to get the zeta function of the weighted graph. But it is a very weird assignment of weights.

What adjacency matrix does it yield? Consider the proof of part 2 of Proposition 10. Assume our graph has no loops and multiple edges. Then, writing $a \overset{e}{\to} b$ to mean that edge e has starting vertex a and terminal vertex b,

$$\begin{aligned}\left(S_L R_L\, {}^tT_L\right)_{a,b} &= \sum_e r(e)s_{a,e}t_{b,e} = \sum_e u^{L(e)-1}s_{a,e}t_{b,e}\\ &= u^{L(e)-1},\ \text{if } a \overset{e}{\to} b.\end{aligned}$$

This "adjacency matrix" is a function of u! Let's call it $A_L(u)$. We find that if $a \overset{e}{\to} b$, then

$$A_L(u)_{b,a} = u^{L(e^{-1})-1} = A_L(u)_{a,b}^{-1}.$$

If $|u|=1$, then the matrix $A(u)$ would be Hermitian. If all $L(e)=1$, then the adjacency matrix is the usual one.

We have nevertheless a weighted determinant formula with the weighted zeta function from Definition 10:

$$\begin{aligned}\zeta_X(u,L)^{-1} &= \det(I-W)|_{non-0\ \ w_{ab}=u^{L(a)}}\\ &= (1-u^2)^{(r-1)}\det(I - A_L(u)\,u + Q_L\,u^2).\end{aligned}$$

Exercise 65. *Look at some weighted graph examples. Where does the zeta function converge? Where are the poles?*

The induction property of the edge Artin L-function (part 4) of Theorem 24 is the next thing for us to prove. To imitate the proof of the analogous number theory fact, one needs the following Lemma.

Lemma 9. *Suppose Y/X is normal with Galois group G and H is the subgroup of G corresponding to an intermediate covering $\widetilde{X}$. Let $\chi = Tr\,\rho$ be a character of H and $\chi^{\#} = Tr\left(Ind_H^G\rho\right)$ be the corresponding induced character of G. For any prime $[C]$ of X, we have*

$$\sum_{j=1}^{\infty}\frac{1}{j}\chi^{\#}(\sigma(C))\,N_E(C)^j = \sum_{[\widetilde{C}]|[C]}\sum_{j=1}^{\infty}\frac{1}{j}\chi\left(\widetilde{\sigma}\left(\widetilde{C}\right)^j\right)N_E\left(\widetilde{C}\right)^j_{spec}.$$

Here $\sigma(C) \in G$ is the normalized Frobenius automorphism for C in X and $\widetilde{\sigma}(\widetilde{C}) \in H$ is the normalized Frobenius corresponding to $\widetilde{C}$ in $\widetilde{X}$.

Proof. Let D_1 be the prime of Y above C starting on sheet 1. Then $\sigma(C) = [Y/X, D_1]$. Using the Frobenius formula for the induced character, we have

$$\sum_{j=1}^{\infty} \frac{1}{j} \chi^{\#}\left(\sigma(C)^j\right) N_E(C)^j = \sum_{j=1}^{\infty} \sum_{\substack{g \in G \\ \left(g\sigma(C)g^{-1}\right)^j \in H}} \frac{1}{j|H|} \chi\left(\left(g\sigma(C)g^{-1}\right)^j\right) N_E(C)^j.$$

Each distinct prime $[D]$ of Y above C has the form $D = g \circ D_1$ and occurs for $f = f(D, Y/X)$ elements of G, where f is the residual degree of Definition 35. From Proposition 7 we see that

$$\sum_{j=1}^{\infty} \frac{1}{j|H|} \sum_{\substack{g \in G \\ \left(g\sigma(C)g^{-1}\right)^j \in H}} \chi\left((g\sigma g^{-1})^j\right) N(C)^j = \sum_{[D]|[C]} \sum_{\substack{j \geq 1 \\ [Y/X, D]^j \in H}} \frac{f}{j|H|} \chi\left([Y/X, D]^j\right) N(C)^j.$$

Group the various D over C into those over a fixed $\widetilde{C}$ and then sum over the $\widetilde{C}$. For a fixed $\widetilde{C}$, all D dividing $\widetilde{C}$ have the same minimal power $j = f_1 = f(\widetilde{C}, \widetilde{X}/X)$ such that $[Y/X, D]^j \in H$. This power gives the Frobenius automorphism of D with respect to $Y/\widetilde{X}$ by Theorem 19. Thus the last double sum is

$$\sum_{[\widetilde{C}]|[C]} \sum_{[D]|[\widetilde{C}]} \sum_{j \geq 1} \frac{f}{f_1 j|H|} \chi\left([Y/\widetilde{X}, D]^j\right) N(C)^{f_1 j}.$$

For all $[D]|[\widetilde{C}]$, the $[Y/\widetilde{X}, D]$ are conjugate to each other in H and there are g_2 such D where $g_2 f_2 = |H|$. Here $f_2 = f(D, Y/\widetilde{X})$ and $g_2 = g(D, Y/\widetilde{X})$. If we pick one fixed D above $\widetilde{C}$, we therefore get

$$\begin{aligned}
\sum_{[D]|[\widetilde{C}]} \sum_{j \geq 1} \frac{f}{f_1 j|H|} \chi\left([Y/\widetilde{X}, D]^j\right) N_E(C)^{f_1 j} &= \sum_{j \geq 1} \frac{f g_2}{f_1 j|H|} \chi\left([Y/\widetilde{X}, D]^j\right) N(C)^{f_1 j} \\
&= \sum_{j \geq 1} \frac{1}{j} \chi\left([Y/\widetilde{X}, D]^j\right) N(C)^{f_1 j}.
\end{aligned}$$

The proof is completed by putting the chain of equalities together, since

$$N(C)^{f_1} = N(\widetilde{C})_{spec}.$$

□

The following Corollary will be needed for our discussion of graphs that are isospectral but not isomorphic.

Corollary 5. *If Y/X is normal with Galois group G and H is the subgroup of G corresponding to an intermediate cover $\widetilde{X}$. Let $\chi_1^{\#}$ be the character of G induced from the trivial character 1 of H. Then the number of primes $\left[\widetilde{C}\right]$ of $\widetilde{X}$ above a prime $[C]$ of X with lengths $\nu\left(\widetilde{C}\right) = \nu(C)$ is $\chi_1^{\#}(\sigma(C))$, where $\sigma(C)$ denotes the normalized Frobenius automorphism of Definition 38. This means that $\chi_1^{\#}(\sigma(C))$ is the number of primes of $\widetilde{X}$ above $[C]$ with residual degree 1.*

Proof. Set $\chi = \chi_1$ in Lemma 9 and set each non-zero edge variable $w_{ij} = u$. This makes $N_E(C) = u^{\nu(C)}$ and $N_E(\widetilde{C})_{spec} = u^{\nu(\widetilde{C})}$. Look at the $u^{\nu(C)}$ term on both sides of equation (**??**). The coefficient of $u^{\nu(C)}$ on the left side comes from the $j = 1$ term and it is $\chi^{\#}(\sigma(C))$. The coefficient of the $u^{\nu(C)}$ term on the right is the number of $[\widetilde{C}]$ above $[C]$ with $\nu(\widetilde{C}) = \nu(C)$. □

Proof. of the Induction Property of Edge L-Functions.

By the definition of the edge L-function for Y/X, we have

$$log(L_E(W, \rho^{\#}, Y/X)) = \sum_{[C]} \sum_{j=1}^{\infty} \frac{1}{j} \chi^{\#}(\sigma(C)^j) N(C)^j.$$

Apply Lemma 9 to see that the right side is

$$\sum_{[\widetilde{C}]} \sum_{j=1}^{\infty} \frac{1}{j} \chi\left(\widetilde{\sigma}(\widetilde{C})^j\right) N_E(\widetilde{C})^j_{spec},$$

where the sum is over all primes $\widetilde{C}$ of $\widetilde{X}$ and $\widetilde{\sigma}(\widetilde{C})$ is the corresponding normalized Frobenius automorphism in H. The proof is completed using the definition of the edge L-function for $Y/\widetilde{X}$. □

reps

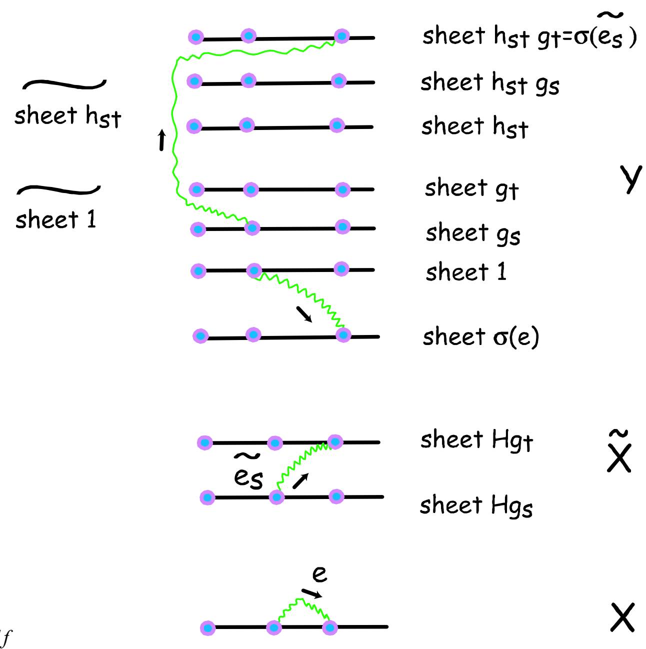

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FIGURE 55. Proving the induced representations property of edge L-functions

We could also give a purely combinatorial proof of the induction property - noting that the two determinants arising from part 3) of Theorem 23 are the same size. Using the definition of induced representations, one can see that the two determinants are the same. See Stark and Terras [86].

Exercise 66. *Find a combinatorial proof of the induction property of the edge L-function,*

$$L_E(\widetilde{W}_{spec}, \rho, Y/\widetilde{X}) = L_E(W, \rho^{\#}, Y/X),$$

by looking at the formulas $L_E(W, \rho, Y/X) = \det(I - W_\rho)^{-1}$ *and the analogous result for* $L_E(W, \rho^{\#}, Y/X)$, *with* $\rho^{\#} = Ind_H^G \rho$.

Look at Figure 55. You need to split the $\widetilde{W}_\rho$ *matrix into* $2m \times 2m$ *blocks indexed by oriented edges* e, f *of* X. *The e block row comes from directed edges of* $\widetilde{X}$ *projecting to e. Edge e lifts to each sheet of* $\widetilde{X}$. *These sheets are labeled by cosets* Hg_k. *Write an edge of* $\widetilde{X}$ *as* $\widetilde{e_s}$ *if it projects to e and has initial vertex on sheet* Hg_s. *Then we claim*

$$\left(\widetilde{W}_{\rho,spec}\right)_{e,f} = \rho^{\#}(\sigma(e))\, w_{ef}.$$

Suppose edge $\widetilde{f_t}$ *has initial vertex on sheet* Hg_t. *If* $\widetilde{e_s}$ *feeds into* $\widetilde{f_t}$, *we see that e lifts to edge of Y starting on sheet* g_s *and ending on sheet* $h_{st}g_t$ *with*

$$g_s\sigma(e) = h_{st}g_t = \widetilde{\sigma}(\widetilde{e_s}) \in Gal(Y/\widetilde{X}).$$

So we find

$$\left(\widetilde{W}_{\rho,spec}\right)_{e,f} = \left(\rho\left(g_s\sigma(e)g_t^{-1}\right)\right) w_{ef}$$

by the formula for the matrix entries of an induced representation, setting ρ *equal to 0 outside of* H.

20. The Path Artin L-Function of a Covering.

There is one final kind of Artin L-function - the path L-function invented by Stark which generalizes the path zeta function discussed earlier. Recall Definitions 26 and 27 of the path matrix Z, path norm and path zeta function.

Definition 48. *Assume Y/X normal with Galois group G. Given a representation ρ of G and path matrix Z with $|z_{ef}|$ sufficiently small, the **path Artin L-function** is defined by*

$$L_P(Z,\rho)=\prod_{\substack{[C]\ prime\\ in\ X}}\det\left(1-\rho\left(\frac{Y/X}{D}\right)N_P(C)\right)^{-1}.$$

Here $\left(\frac{Y/X}{D}\right)$ is from Definition 39, the path matrix Z is from Definition 26, the path norm $N_P(C)$ is from Definition 27, and the product is over primes $[C]$ of X, with $[D]$ any prime of Y over $[C]$.

The path Artin L-function has analogous properties to the edge L-function. You just have to replace E with P in Theorems 23 and 24.

Proposition 12. ***Some Properties of the Path Artin L-Function.***

1) $L_P(Z,1,Y/X)=\zeta_P(Z,X)$.

2) $L_P(Z,\rho_1\oplus\rho_2,Y/X)=L_P(Z,\rho_1,Y/X)L_P(Z,\rho_2,Y/X)$.

*3) Let Y/X be normal with Galois group G, and $\widetilde{X}$ be intermediate to Y/X and normal with Galois group H. Let ρ be a representation of $G/H\cong G(\widetilde{X}/X)$. View ρ as a representation of G (the **lift** of ρ). Then*

$$L_P(Z,\rho,Y/X)=L_P(Z,\rho,\widetilde{X}/X).$$

Theorem 25. ***The path Artin L-Function is the inverse of a polynomial.***

The path L- function satisfies

$$L_P(Z,\rho,Y/X)=\det\left(I-Z_\rho\right)^{-1}.$$

where $Z_\rho=(z_{ef}\rho\left(\sigma\left(e\right)\right)$ and I is the $2dr\times 2dr$ identity matrix, where d is the degree of ρ.

Proof. The proof is like that of part 3) of Theorem 23 for the edge L-function. Choose D to represent $[Y/X,D]$ by $\sigma(C)$,the normalized Frobenius. Note that a path C on X corresponds to a sequence of non-tree edges $a_1,...,a_s$, then $\sigma(C)=\sigma\left(a_1\right)\cdots\sigma\left(a_s\right)$, since all the rest of the edges of C are edges in a spanning tree of X and for any edge b on the tree we have $\sigma(b)=1$. □

Just as with the path zeta functions the variables of the path L-function can be specialized to obtain the edge L-function. Suppose that $e_1,...,e_r$ are the (oriented) edges left out of a spanning tree T of X. This specialization was given in Formula (12.1).

Via this specialization, we find that

$$L_P(Z(W),\rho)=L_E(W,\rho). \tag{20.1}$$

Example 26. ***Path Artin L-functions for Cyclic Cover of 2 Loops with Extra Vertex on 1 Loop.***

Consider the base graph of 2 loops with an extra vertex on 1 loop. Now for our n-cyclic cover we lift edge a up 1 sheet and keep edge b in the same sheet. See Figure 56.

The path matrix of the Artin L-function for an n-cyclic cover of two loops with an extra vertex on one loop as in Figure 56 is 4×4 and we can compute the L-functions for the n-cyclic cover by hand or use Scientific Workplace. So the L-function for the cyclic n-cover of 2 loops with an extra vertex on 1 loop in Figure 56, with $\rho=e^{2\pi ia/n}$ and $s=2\cos(2\pi a/n)$, is

$$\begin{aligned}L(u,\chi_a)^{-1} &= det\begin{pmatrix}\rho u-1 & \rho u & 0 & \rho u^2\\ u^2 & u^2-1 & u^2 & 0\\ 0 & \rho^{-1}u & \rho^{-1}u-1 & \rho^{-1}u^2\\ u & 0 & u & u^2-1\end{pmatrix}\\ &= \left(u^2-1\right)\left(-3u^4+su^3+su-1\right).\end{aligned}$$

We also compute the Artin L-functions of this cover using Matlab and the W_1-matrix. We have decided to plot the eigenvalues of W_1 which are the reciprocals of the poles of the Ihara zeta of the cover. The result is in Figure 57. It should be compared with figures of Angel, Friedman and Hoory [2] for random covers of simple base graphs such as K_4-edge. What is amazing about Figure 57 is that it implies that the non-real poles of the Ihara zeta function of the cover lie on the Riemann hypothesis circle (the reciprocal of the green circle).

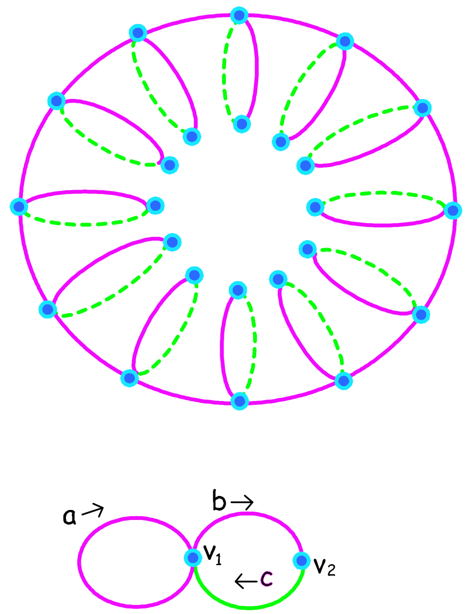

FIGURE 56. A 12-cyclic cover of the base graph with 2 loops and 2 vertices. The spanning tree in the base graph is green dashed. The sheets of the cover above are also green dashed.

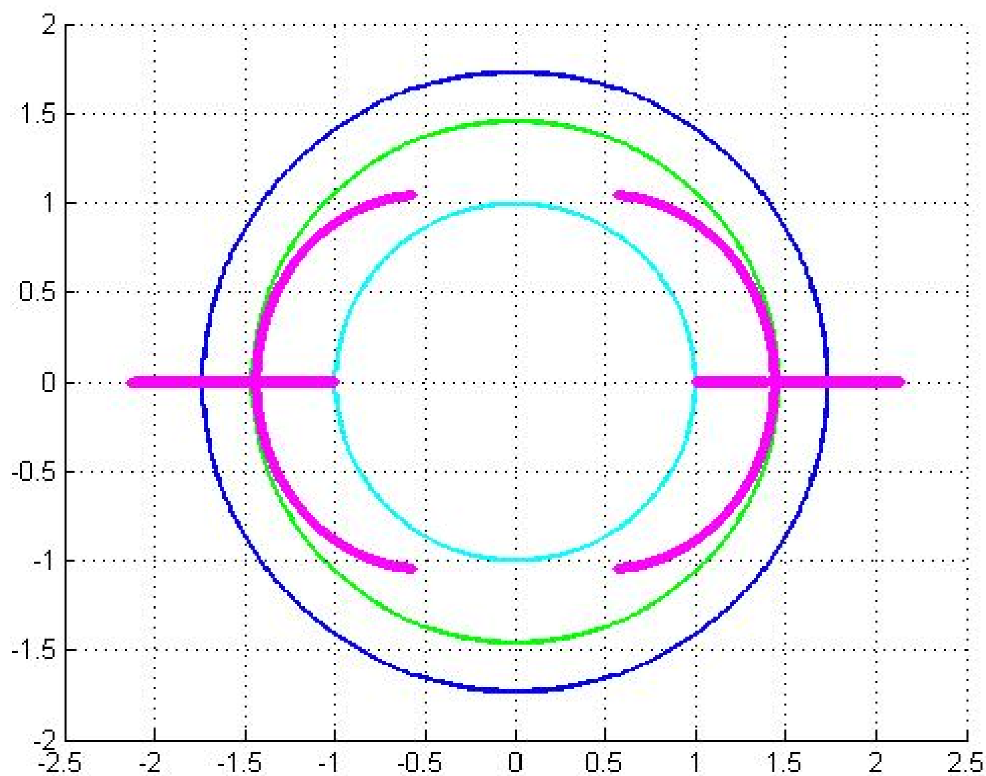

FIGURE 57. The pink points are the eigenvalues of the W_1 matrix for the 10001 cyclic cover of 2 loops with an extra vertex on 1 loop analogous to the cover in Figure . These are the reciprocals of the poles of the Ihara zeta function for the covering graph. The circles are centered at the origin and have radii $\sqrt{p}$, $1/\sqrt{R}$, $\sqrt{q}$. Here $p = 1$, $1/R \cong 2.1304$, $q = 3$.

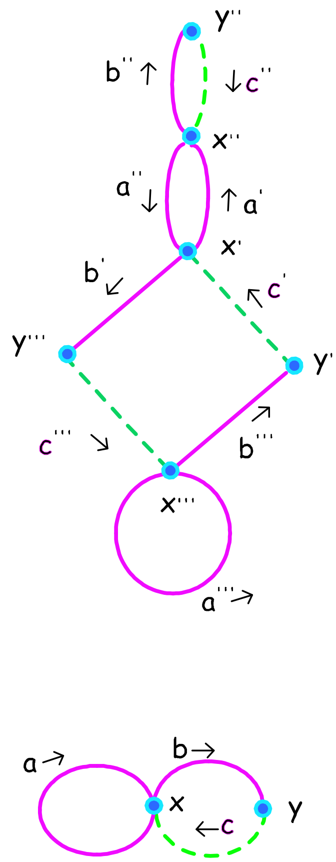

FIGURE 58. The random 3-cover of 2 loops with and extra vertex with the lift of a corresponding to the permutation (12) and the lift of b corresponding to the permutation (13). The green dashed line in the base graph is the spanning tree and the green dashed lines in the cover are the sheets of the cover.

Exercise 67. *Do the analogue of the previous example but instead of keeping the lifts of edge b in the same sheet lift them down 1 sheet.*

Exercise 68. *Do the analogue of the previous examples but replace the base graph with* K_4 − one edge.

To produce a figure such as 57 for a random cover Y of the same base graph X consisting of 2 loops with an extra vertex on one of them, we can use the formula for the edge matrix W_1 of Y in terms of the start matrix S and the terminal matrix T from Proposition 4. It is also convenient to write $A = (MN)$, $T = (NM)$ where M and N have $m = |E|$ columns. We used this fact in the proof of the Kotani & Sunada theorem in the same section as Proposition 4. It follows that

$$W_1 = \begin{pmatrix} {}^tNM & {}^tNN - I \\ {}^tMM - I & {}^tMN \end{pmatrix}.$$

Now we arrange the columns of M so that the columns corresponding to lifts of a given edge of the base graph are listed in order of the sheet on which the lift starts. And the lifts of a given vertex of the base graph are also listed together in the order of the sheets where they live. Then

$$M = \begin{pmatrix} I_n & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix}, \text{ and } N = \begin{pmatrix} A & 0 & I_n \\ 0 & B & 0 \end{pmatrix},$$

where A and B are permutation matrices. Suppose $n = 3$ and the lift of edge a correxponds to the permutation (12) while the lift of edge b corresponds to the permutation (13). Then we get the graph in Figure 58.

We used Matlab to plot the eigenvalues of W_1 for covers in which A and B are random permutation matrices (found using the command randperm in Matlab). When $n = 407$, we obtain the spectrum of Figure 59. If we compare this with the picture

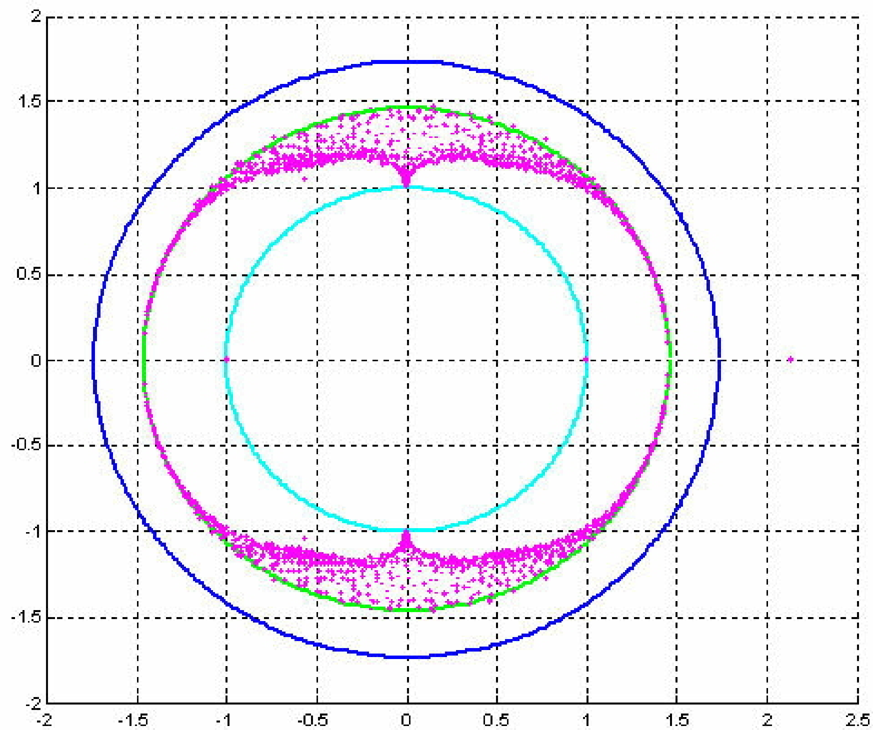

FIGURE 59. A random 407-cover of 2 loops with an extra vertex on 1 loop. The pink points are the eigenvalues of the W_1 matrix. These are the reciprocals of the poles of the Ihara zeta for the cover. The circles are centered at the origin and have radii $\sqrt{p}, 1/\sqrt{R}, \sqrt{q}$. Here $p = 1, R = 1/2, q = 3$. The Frobenius eigenvalue is approximately 2.1304.

found by Angel, Friedman, and Hoory [2] for random covers of the base graph $K_4 - edge$, we see that there is much similarity, though their Frobenius eigenvalue is 1.5 while ours is approximately 2.1304.

Next we want to discuss the induction property of the path L-functions. For this, if $\widetilde{X}$ is a covering of X, we need to specialize the path matrix $\widetilde{Z}$ of $\widetilde{X}$ to the variables in the path matrix Z of X. This must be done in such a way that if $\widetilde{C}$ is a reduced cycle in its conjugacy class of the fundamental group of $\widetilde{X}$, then under the specialization, $N_E(\widetilde{C})$ becomes $N_E(C)$ where C is the projected cycle of $\widetilde{C}$ in X.

Specialization Rule.

First we need a contraction rule.

In X, we contract the base tree T to a point. See Figure 60. This gives a graph $B(X)$ which is a bouquet of loops, pictured on the right in the Figure. Graphs X and $B(X)$ have the same fundamental group. The path and edge zeta functions of $B(X)$ are the same. In the cover $\widetilde{X}$, we also contract each sheet (the connected inverse images of T) to a point. This gives a graph we call $C\left(\widetilde{X}\right)$ pictured on the right at the top of Figure 60. The lifts of the r generating paths of X to $\widetilde{X}$ give the edges of $C(\widetilde{X})$. What makes this interesting is that if $\widetilde{X}$ is an d-fold covering of X, then $d-1$ of the lifted edges from the $B(X)$ must be used in the tree of $\widetilde{X}$. The remaining $dr-(d-1) = d(r-1)+1$ non-tree edges of the $C\left(\widetilde{X}\right)$ give the generators of the fundamental group of $\widetilde{X}$. The specialization algorithm needs to take account of the tree edges.

First specialize variables in the path matrix $\widetilde{Z}\left(\widetilde{X}\right)$ to the edge variables on the contracted graph $C\left(\widetilde{X}\right)$. This turns the path norm into the edge norm on the contracted graph $C\left(\widetilde{X}\right)$. Then specialize the edge variables of the contracted $C\left(\widetilde{X}\right)$ to the edge variables of the contracted base graph $B(X)$ in our usual manner from from the induction theorem for the edge Artin L-function. This turns the edge norm on $\widetilde{X}$ into the edge norm on X which is the same as the path on $B(X)$. This is the desired specialization. Call it $\widetilde{Z}_{spec}$.

Example 27. ***Contracted Covers.*** *The contracted versions of X and Y_3 from Figure 43 are shown in Figure 61.*

and specialize

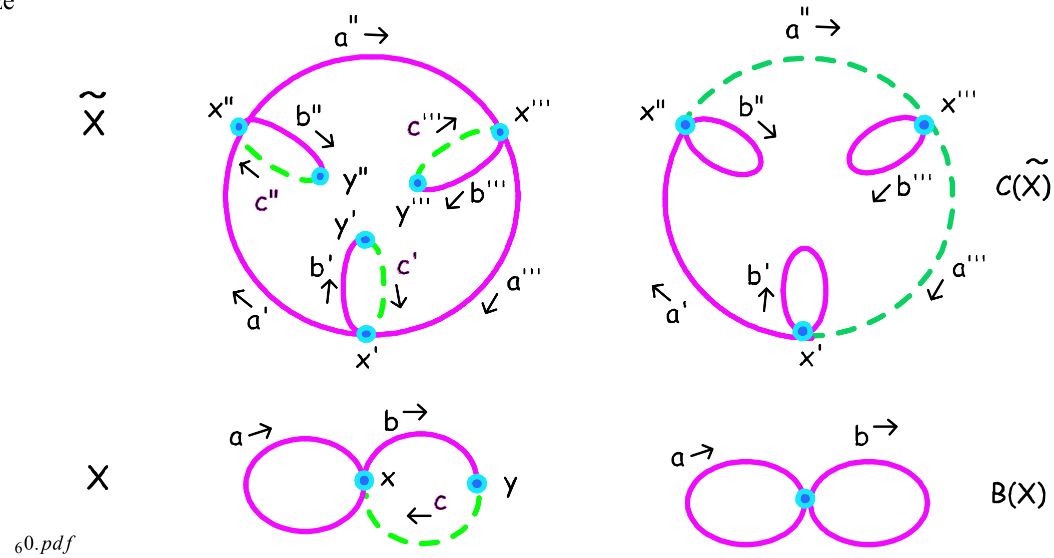

FIGURE 60. Illustration of the contraction of sheets of a cover corresponding to the spanning trees contracted below. Here on the left we have a $d = 3$-cyclic cover $\widetilde{X}$ of X. A spanning tree of X is shown on the left below with green dashed lines. When we contract the spanning tree of X below, we get the bouquet of loops $B(X)$ on the right. On the right at the top, the graph $C(\widetilde{X})$ is obtained by contracting the sheets of $\widetilde{X}$. In $C(\widetilde{X})$ the new spanning tree is shown with green dashed edges and it will have $d - 1 = 2$ edges.

covers

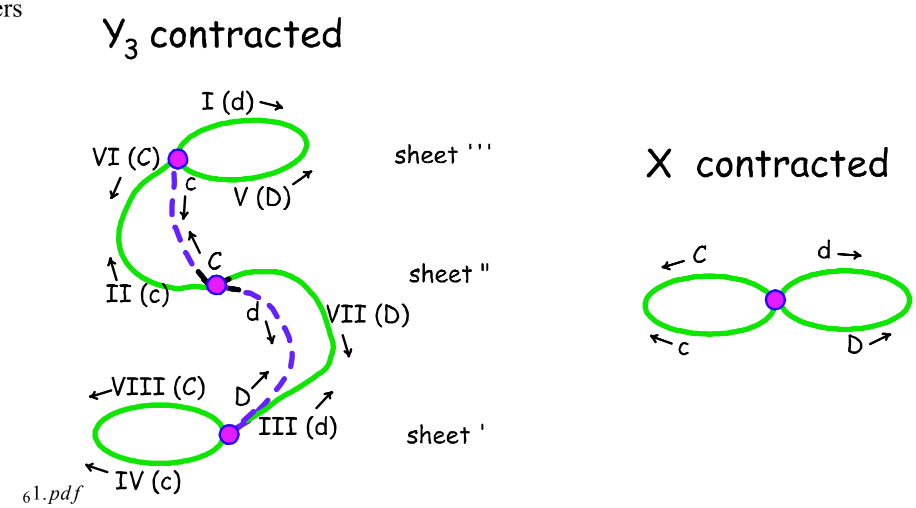

FIGURE 61. Contracted versions of X and Y_3 from Figure 43. Solid edges are the non-tree edges generating the fundamental group.

The tree $\widetilde{T}$ of Y_3 is completed with one of the lifts of the cC pair between the top two sheets of Y_3 and one of the lifts of the dD pair between the bottom two sheets. The remaining four undirected edges of the contracted Y_3 give rise to the fundamental group of Y_3 and the resulting 8×8 path matrix $\widetilde{Z}$. We give these edges directions projecting to either c or d, rather than C or D, and labels I, II, III, IV, as shown. The inverse edges, projecting to C and D, are given labels $V, VI, VII, VIII$, as shown. The rows and columns of $\widetilde{Z}$ are then labeled by the Roman numerals $I - VIII$.

Following our specialization algorithm described above, the resulting specialized matrix $\widetilde{Z}_{spec}$ is then

$$\begin{pmatrix}
z_{dd} & z_{dc}z_{cc} & z_{dc}z_{cd}z_{dd} & z_{dc}z_{cd}z_{dc} & 0 & z_{dC} & z_{dc}z_{cD} & z_{dc}z_{cd}z_{dC} \\
z_{cd} & z_{cc}z_{cc} & z_{cc}z_{cd}z_{dd} & z_{cc}z_{cd}z_{dc} & z_{cD} & 0 & z_{cc}z_{cD} & z_{cc}z_{cd}z_{dC} \\
z_{dC}z_{Cd} & z_{dc} & z_{dd}z_{dd} & z_{dd}z_{dc} & z_{dC}z_{CD} & z_{dC}z_{CC} & 0 & z_{dd}z_{dC} \\
z_{cD}z_{DC}z_{Cd} & z_{cD}z_{Dc} & z_{cd} & z_{cc} & z_{cD}z_{DC}z_{CD} & z_{cD}z_{DC}z_{CC} & z_{cD}z_{DD} & 0 \\
0 & z_{Dc}z_{cc} & z_{Dc}z_{cd}z_{dd} & z_{Dc}z_{cd}z_{dc} & z_{DD} & z_{DC} & z_{Dc}z_{cD} & z_{Dc}z_{cd}z_{dC} \\
z_{CC}z_{Cd} & 0 & z_{Cd}z_{dd} & z_{Cd}z_{dc} & z_{CC}z_{CD} & z_{CC}z_{CC} & z_{CD} & z_{Cd}z_{dC} \\
z_{DD}z_{DC}z_{Cd} & z_{DD}z_{Dc} & 0 & z_{Dc} & z_{DD}z_{DC}z_{CD} & z_{DD}z_{DC}z_{CC} & z_{DD}z_{DD} & z_{DC} \\
z_{CD}z_{DC}z_{Cd} & z_{CD}z_{Dc} & z_{Cd} & 0 & z_{CD}z_{DC}z_{CD} & z_{CD}z_{DC}z_{CC} & z_{CD}z_{DD} & z_{CC}
\end{pmatrix}$$

For example, the IV, I entry $\widetilde{Z}_{IV,I}$ follows directed edge IV (projecting to c), through two edges of $\widetilde{T}$ (projecting to D and C consecutively) to edge I (projecting to d) resulting in the specialized value $z_{cD}z_{DC}z_{Cd}$. This agrees with the fact that any path on Y_3 going through consecutive cut edges IV and I must project to a path on X going consecutively through c, D, C, d.

Exercise 69. *Work out* $\widetilde{Z}_{spec}$ *for the example in Figure 60.*

Theorem 26. ***Induction Property for Path L-functions.***
Suppose Y/X is normal with Galois group G. If H is a subgroup of G corresponding to the intermediate covering $\widetilde{X}$, ρ is a representation of H, and $\rho^{\#}$ is the representation of G induced by ρ, then assuming the variables of the path matrix $\widetilde{Z}l$ for $Y/\widetilde{X}$ are specialized according to

$$L(\widetilde{Z}_{spec}, \rho, Y/\widetilde{X}) = L(Z, \rho^{\#}, Y/X).$$

Proof. Contract each copy of the tree T to a point, both in X and in $\widetilde{X}$. Then both sides of the equality in this theorem become edge L-functions attached to a graph with one vertex and r loops and the corresponding covering of it. Since the induction theorem has been proved in our proof of part 4) Theorem 24 for edge L-functions, we are done. □

Remark. From Theorem 25, the equality of Theorem 26 becomes

$$det(I - \widetilde{Z}_{\text{spec},\rho}) = det(I - Z_{\rho^{\#}}).$$

Unlike the analogous equality for the edge L-functions obtained from combining Theorem 23 and Theorem 24, here these determinants have different sizes!

Corollary 6. ***Factorization of the Path Zeta Function.*** *Suppose Y/X is normal with Galois group G. Then the path zeta function, once the variables are specialized, factors into products of Artin L-functions:*

$$\zeta_P\left(\widetilde{Z}_{spec}, Y\right) = \prod_{\rho\in\widehat{G}} L_P(Z, \rho, Y/X)^{d_\rho}.$$

Proof. The proof is the same as that for the analogous property of the edge Artin L-function. □

Example 28. ***Factorization of the path zeta function of a non-normal cubic cover Y_3 over X from Figure 43.***

This is analogous to the example from zeta functions of number fields which goes back to Dedekind (see Section 3.3 of [81]).

Here we re-consider the last example in light of the factorization theorem. Set $\omega = e^{2\pi i/3}$ and

$$\begin{aligned}
u_1 &= z_{cc}, u_2 = z_{cd}, u_3 = z_{cD}, u_4 = z_{dc}, u_5 = z_{dd}, u_6 = z_{dC}, \\
u_7 &= z_{Cd}, u_8 = z_{CC}, u_9 = z_{CD}, u_{10} = z_{Dc}, u_{11} = z_{DC}, u_{12} = z_{DD}.
\end{aligned}$$

By the Corollary to Theorem 26, the product of

$$det\begin{pmatrix}
u_1 - 1 & u_2 & 0 & u_3 \\
u_4 & u_5 - 1 & u_6 & 0 \\
0 & u_7 & u_8 - 1 & u_9 \\
u_{10} & 0 & u_{11} & u_{12} - 1
\end{pmatrix}$$

and

$$det\begin{pmatrix} -1 & \omega^2 u_1 & 0 & \omega^2 u_2 & 0 & 0 & 0 & \omega^2 u_3 \\ \omega u_1 & -1 & \omega u_2 & 0 & 0 & 0 & \omega u_3 & 0 \\ 0 & \omega u_4 & -1 & \omega u_5 & 0 & \omega u_6 & 0 & 0 \\ \omega^2 u_4 & 0 & \omega^2 u_5 & -1 & \omega^2 u_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 u_7 & -1 & \omega^2 u_8 & 0 & \omega^2 u_9 \\ 0 & 0 & \omega u_7 & 0 & \omega u_8 & -1 & \omega u_9 & 0 \\ 0 & \omega u_{10} & 0 & 0 & 0 & \omega u_{11} & -1 & \omega u_{12} \\ \omega^2 u_{10} & 0 & 0 & 0 & \omega^2 u_{11} & 0 & \omega^2 u_{12} & -1 \end{pmatrix}$$

must equal the determinant of the matrix $\widetilde{Z}_{spec} - I =$

$$\begin{pmatrix} u_5-1 & u_4u_1 & u_4u_2u_5 & u_4u_2u_4 & 0 & u_6 & u_4u_3 & u_4u_2u_6 \\ u_2 & u_1u_1-1 & u_1u_2u_5 & u_1u_2u_4 & u_3 & 0 & u_1u_3 & u_1u_2u_6 \\ u_6u_7 & u_4 & u_5u_5-1 & u_5u_4 & u_6u_9 & u_6u_8 & 0 & u_5u_6 \\ u_3u_{11}u_7 & u_3u_{10} & u_2 & u_1-1 & u_3u_{11}u_9 & u_3u_{11}u_8 & u_3u_{12} & 0 \\ 0 & u_{10}u_1 & u_{10}u_2u_5 & u_{10}u_2u_4 & u_{12}-1 & u_{11} & u_{10}u_3 & u_{10}u_2u_6 \\ u_8u_7 & 0 & u_7u_5 & u_7u_4 & u_8u_9 & u_8u_8-1 & u_9 & u_7u_6 \\ u_{12}u_{11}u_7 & u_{12}u_{10} & 0 & u_{10} & u_{12}u_{11}u_9 & u_{12}u_{11}u_8 & u_{12}u_{12}-1 & u_{11} \\ u_9u_{11}u_7 & u_9u_{10} & u_7 & 0 & u_9u_{11}u_9 & u_9u_{11}u_8 & u_9u_{12} & u_8-1 \end{pmatrix}.$$

21. Non-Isomorphic Regular Graphs Without Loops or Multiedges Having the Same Ihara Zeta Function.

Algebraic number fields K_1, K_2 can have the same Dedekind zeta functions without being isomorphic. See Perlis [69]. The smallest examples have degree 7 over $\mathbb{Q}$ and come from Artin L-functions of induced representations from subgroups of $G = GL(3, \mathbb{F}_2)$, the simple group of order 168. An analogous example of 2 graphs (each having 7 vertices) which are isospectral but not isomorphic was given by P. Buser. These graphs are found in Figure 62 below. See Buser [17] or Terras [92], Chapter 22. Buser's graphs ultimately lead to 2 planar isospectral drums which are not obtained from each other by rotation and translation, answering the question raised by M. Kac in [46]: Can you hear the shape of a drum? See Gordon et al [30] who show that there are (non-convex) planar drums that cannot be heard using the same basic construction.

Buser's graphs are not simple. That is, they have multiple edges as well as loops. We can use our theory to obtain examples of simple regular graphs with 28 vertices which are isospectral but not isomorphic. See Figure 63. The graphs in Figure 63 are constructed using the same group G and subgroups H_j as in Buser's examples. Sunada [89] shows how to apply the method from number theory to obtain isospectral compact connected Riemannian manifolds that are not isometric.

Define $G = GL(3, \mathbb{F}_3)$, which is a simple group of order 168. Two subgroups H_j of index 7 in G are:

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \text{ and } H_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.$$

Exercise 70. *Show that H_1 and H_2 are not conjugate in G.*

This fact also follows from the fact that we will construct two non-isomorphic intermediate graphs corresponding to these subgroups H_j of G. One can show that these two groups give rise to equivalent permutation representations of G (i.e., the representation we have called $Ind_{H_i}^G 1$). The same argument as we used in Terras [92] , for Buser's graphs says that the representations $\rho_j = Ind_{H_j}^G 1$ are equivalent because the subgroups H_j are almost conjugate (i.e. $|H_1 \cap \{g\}| = |H_2 \cap \{g\}|$, for every conjugacy class $\{g\}$ in G). This implies that we have equality of the corresponding characters $\chi_{\rho_1} = \chi_{\rho_2}$. Therefore we will get graphs with the same zeta functions (using the induction property of vertex L-functions):

$$\zeta_{\widetilde{X_1}}(u) = L_V(u, \rho_1) = L_V(u, \rho_2) = \zeta_{\widetilde{X_2}}(u)$$

See Terras [92] for more information.

Given $g \in G$, all elements of $H_1 g$ have the same first row. The 7 possible non-zero first rows correspond naturally to the numbers 1-7 in binary. Thus order the 7 right cosets $H_1 g_j$ by the numbers represented by the first rows in binary. For example, the first row of g_6 is (110); and $H_1 g_4$ is the identity coset. For any g, it is easy to figure out what coset $H_1 g_j g$ is, as the first row of the product $g_j g$ depends only on the first row of g_j. So for $g \in G$, it is we find the permutation $\mu(g)$ corresponding to multiplying the 7 cosets $H_1 g_j$ by g on the right; i.e., $H_1 g_j g = H_1 g_{\mu(j)}$.

We need the permutations $\mu(A)$ and $\mu(B)$ for Buser's matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{21.1}$$

Computation shows that

$$\mu(A) = (1436)(2)(57) \text{ and } \mu(B) = (132)(4)(576). \tag{21.2}$$

Exercise 71. *Check these formulas. For example, to find $H_1 g_3 A$, we want the first row of*

$$\begin{pmatrix} 0 & 1 & 1 \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \in H_1 g_6$$

and so $\mu(A)$ takes 3 to 6.

We have to do the same permutation calculation with the matrices A and B acting on the right cosets of H_2. It might appear that the right cosets of H_2 would be more difficult to deal with. But there is a very useful automorphism of G to help. It is $\varphi(g) = {}^t g^{-1}$, where ${}^t g$ denotes the transpose of $g \in G$. This map φ is an automorphism of G such that $\varphi(H_1) = H_2$. If we apply φ to the right cosets $H_1 g_j$, we get G as a union of the 7 right cosets $H_2 \, {}^t g_j^{-1}$. To figure out how $g \in G$ permutes the cosets, it suffices to consider the action of ${}^t g^{-1}$ on the $H_1 g_j$. Note that

$${}^t A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } {}^t B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

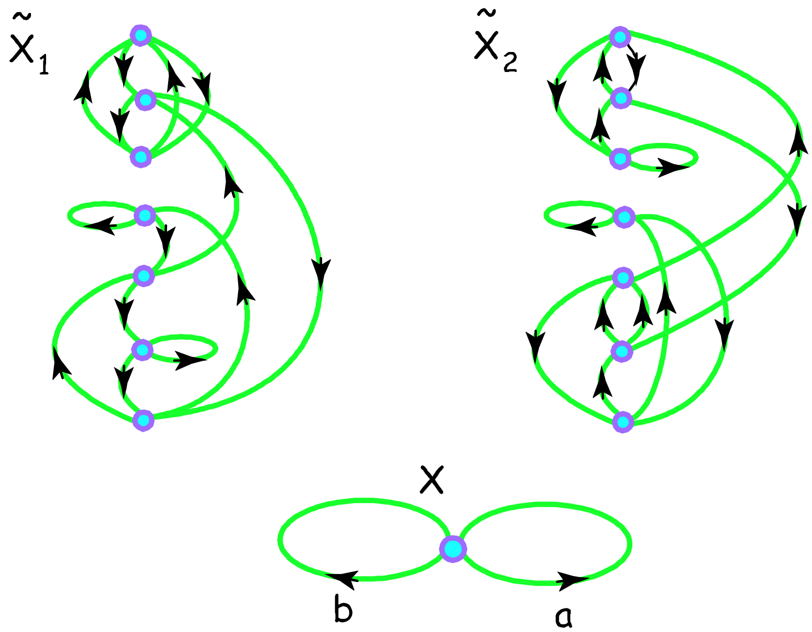

FIGURE 62. Buser's Isospectral non-Isomorphic Schreier Graphs. See Buser [17]. The sheets of $\widetilde{X_1}$ and $\widetilde{X_2}$ are numbered 1 to 7 from bottom to top. Lifts of a are on the right in each graph; lifts of b are on the left.

Therefore the action of ${}^tA^{-1}$ and ${}^tB^{-1}$ on the right cosets H_1g_j is given by the permutations

$$\mu({}^tA^{-1}) = (14)(2376)(5) \text{ and } \mu({}^tB^{-1}) = (123)(4)(567). \tag{21.3}$$

These same permutations give the actions of A and B on the right cosets $H_2\,{}^tg_j^{-1}$.

Exercise 72. *Check these formulas for $\mu({}^tA^{-1})$ and $\mu({}^tB^{-1})$.*

Exercise 73. *Prove that the matrices A and B in formula (21.1) generate the group G.*

Buser [17] used the matrices A and B to construct 2 Schreier graphs corresponding to the 2 subgroups H_1 and H_2. Using the Galois theory we have worked out in preceding sections, this means find coverings $\widetilde{X_1}$ and $\widetilde{X_2}$ of X, where X is the graph consisting of a single vertex and a double loop. Direct each loop resulting in two directed edges a and b, say. Assign the normalized Frobenius elements $\sigma(a) = A$ and $\sigma(b) = B$. The resulting normal cover of X is the Cayley graph of G corresponding to the generators A and B. We want two intermediate graphs $\widetilde{X_1}$ and $\widetilde{X_2}$ corresponding to the subgroups H_1 and H_2 by Theorem 16; which are Schreier graphs. The permutations $\mu(A)$ and $\mu(B)$ that we just found tell us how to lift the edges a and b. This tells us how to draw the graphs $\widetilde{X_1}$ and $\widetilde{X_2}$. See Figure 62

There are many ways to prove that the 2 graphs in Figure 62 are not isomorphic - even as undirected graphs. Look at triple edges; look at double edges; look at distances between loops, etc. Therefore H_1 and H_2 are not conjugate in G. Both graphs are 4-regular; they have the same zeta function and their adjacency matrices have the same spectrum.

Constructing the Graphs in Figure 63 - Simple, 3-Regular Isospectral but not Isomorphic.

Next we construct graphs like Buser's that have no loops or multiple edges. Use the same G, H_1 and H_2, but take X to be a tetrahedron K_4. Thus X is 3-regular and has a fundamental group of rank 3. Take the cut or deleted edges (directed as in Figure 63) to be a, b, c. Choose the normalized Frobenius automorphisms to be

$$\sigma(a) = A, \ \sigma(b) = \sigma(c) = B.$$

Take 7 copies of the tree of X to be the sheets of $\widetilde{X_1}$ and again for $\widetilde{X_2}$. On $\widetilde{X_1}$, we lift a, b, c using the permutations $\mu(A)$ and $\mu(B)$ from formula (21.2) above. On $\widetilde{X_2}$ we lift a, b, c using the permutations $\mu({}^tA^{-1})$ and $\mu({}^tB^{-1})$ from formula (21.3). This produces graphs $\widetilde{X_1}$ and $\widetilde{X_2}$ shown in Figure 63.

Both graphs are 3-regular; they have the same zeta function and their adjacency matrices have the same spectrum. The proof that

$$\zeta_{\widetilde{X_1}}(u) = L_V(u, \rho_1) = L_V(u, \rho_2) = \zeta_{\widetilde{X_2}}(u)$$

is the same argument that we used above and in Terras [92] for Buser's graphs.

More Discussion of the Construction.

The edge c goes from vertex 2 to vertex 3 in X and has the normalized Frobenius automorphism $\sigma(c) = B$. The lifts of c to $\widetilde{X_1}$ are determined by the permutation $\mu(B) = (132)(4)(576)$. This means that c in X lifts to an edge in $\widetilde{X_1}$ from $2'$ to $3^{(3)}$, an edge from $2^{(3)}$ to $3^{(2)}$, an edge from $2''$ to $3'$ and then (beginning a new cycle) to an edge from $2^{(4)}$ to $3^{(4)}$, etc. The edge b lifts in exactly the same manner as c. Similarly, for $\widetilde{X_1}$, the edge a in X corresponds to the permutation $(1436)(2)(57)$. This means that edge a in X lifts to an edge in $\widetilde{X_1}$ from $3^{'}$ to $4^{(4)}$, an edge from $3^{(4)}$ to $4^{(3)}$, an edge from $3^{(3)}$ to $4^{(6)}$ etc.

To see that graphs $\widetilde{X_1}$ and $\widetilde{X_2}$ in Figure 63 are not isomorphic, proceed as follows. There are exactly 4 triangles in each graph (shown by very thick solid lines in Figure 63) and they are connected in pairs in both graphs. This distinguishes in each pair the 2 vertices not on common edges (starred vertices). In $\widetilde{X_1}$ we can go in 3 steps (via dotted lines) from a starred vertex in one pair to a starred vertex in the other pair and, in fact, in 2 different ways. This cannot be done at all in $\widetilde{X_2}$.

We said each $\widetilde{X_i}$ has 4 triangles (up to equivalence and choice of direction). Why? Since X has no loops or multiedges, any triangle on $\widetilde{X_1}$ or $\widetilde{X_2}$ projects to a triangle on X. We saw back in Chapter 1 that (up to equivalence and choice of direction) $X = K_4$ has 8 primes of length 3 and therefore 4 triangles.

Let χ_1 be the trivial character on H_1 or H_2. The induced character $Ind_{H_i}^{G} 1 = \chi_1^{\#}$ on G is the same for $i = 1, 2$. By Corollary 5, for any directed triangle C on X, there are $\chi_1^{\#}(\sigma(C))$ directed triangles above C on $\widetilde{X_1}$ and also above C on $\widetilde{X_2}$. Reversing the direction of C reverses the direction of the covering triangles. We choose the most convenient direction for each triangle.

Three of the triangles on X have 2 edges on the tree of X with normalized Frobenius elements = 1 automatically. Thus, with appropriate choice of direction in each case, $\sigma(C) = A, B, B$, for each triangle. The fourth triangle may be taken to be the path $ab^{-1}c$ whose normalized Frobenius is $\sigma(a)\sigma(b)^{-1}\sigma(c) = AB^{-1}B = A$. For $g \in G$, $\chi_1^{\#}(g)$ is simply the number of 1 cycles in the permutation $\mu(g)$. In particular, $\chi_1^{\#}(A) = \chi_1^{\#}(B) = 1$ (the same for both H_1 and H_2). Thus each of the 4 triangles of X has precisely 1 triangle of $\widetilde{X_j}$ above it for $j = 1$ or 2. Thus the triangles shown in Figure 63 are all triangles on $\widetilde{X_1}$ and $\widetilde{X_2}$ as we claimed.

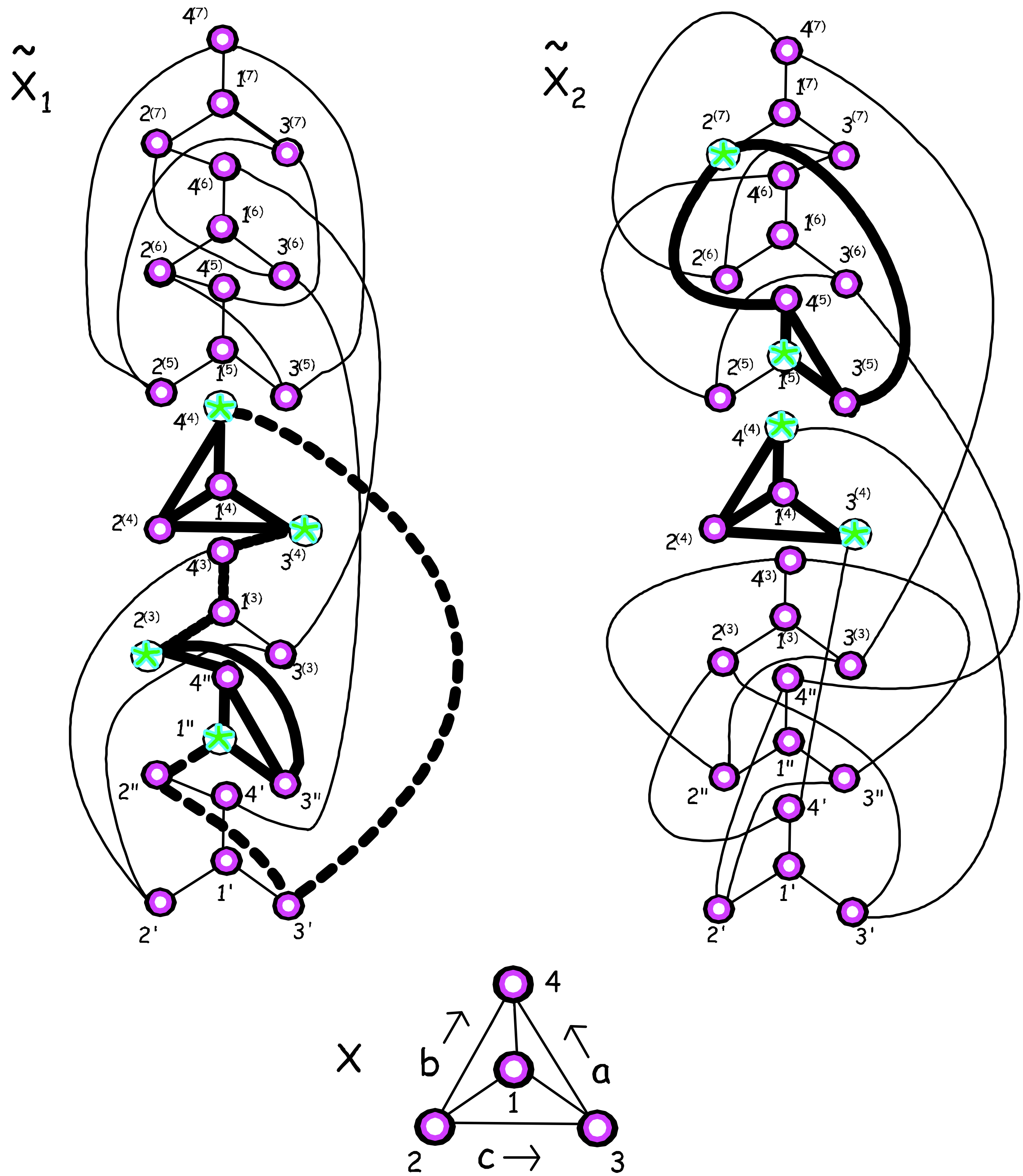

FIGURE 63. Non-Isomorphic Graphs Without Loops or Multiedges Having the Same Ihara Zeta Functions. The superscripts number the sheets of $\widetilde{X_1}$ and $\widetilde{X_2}$. The lifts of a are on the right side of each graph, lifts of b are on the left, and lifts of c cross from the left to the right.

Other Results on Isospectral Graphs.

See [57] for infinite towers of isospectral graphs coming from finite simple groups.

Other questions can be asked. See Gnutzmann et al [7] for the question of whether a count of nodal domains for the eigenvectors resolves the isospectrality of our examples. The question is "Can one count the shape of a drum?" Classically a nodal domain is a maximally connected region where the eigenfunction ψ of the $-\Delta$ for a bounded region $D \subset \mathbb{R}^n$ (ψ satisfying Dirichlet boundary conditions on ∂D, meaning ψ vanishes on ∂D, or Neumann boundary conditions, meaning the normal derivative of ψ vanishes on ∂D) has a constant sign. If $n = 1$, Sturm's oscillation theorem states that the n^{th} eigenfunction has exactly n nodal domains. Here the eigenfunctions are ordered by increasing eigenvalues. In higher dimensions Courant showed that the number of nodal domains of the n^{th} eigenfunction is less than or equal to n.

In [7], the conjecture is stated that nodal counts resolve isospectrality of isospectral quantum graphs. Quantum graphs are weighted graphs which have a Schödinger operator which is the 1 D Laplacian on an edge. There are boundary conditions (say Neumann) at the vertices. A **wavefunction** is a function on each edge (continuous at the vertices and satisfying boundary conditions). Let S_i be the set of edges from vertex i. The wavefunction ψ_b with wave number k can be written if vertex i and vertex j are connected by edge b of length L_b and with coordinate x_b along the edge:

$$\psi_b(x_b) = \frac{1}{\sin(kL_b)}\left(\phi_i \sin k\,(L_b - x_b) + \phi_j \sin kx_b\right),$$

$$\sum_{b\in S_i} \frac{d}{dz_b}\psi_b(x_b)\bigg|_{x_b=0} = 0.$$

Here the wave function ψ_b takes the values ϕ_i and ϕ_j at vertex i, j, respectively. Substitute the 1st equation into the 2nd and obtain equations for the ϕ_j given by

$$\sum_{j=1}^{|E|} A_{i,j}\left(L_1, .., L_{|E|}; k\right)\phi_j = 0, \quad \text{for all } 1 \le i \le |V|.$$

The spectrum $\{k_n\}$ is a discrete, positive, unbounded sequence - the zero set of the determinant of the matrix of coefficients $A_{i,j}\left(L_1, .., L_{|E|}; k\right)$. Then one must regularize the determinant function.

There are 2 ways to define the nodal domains for quantum graphs. The discrete way says: a nodal domain is a maximal set of connected interior (meaning degree ≥ 3) vertices where the vertex eigenfunctions ϕ_i have the same sign. This definition is modified if any of the ϕ_i vanishes.

It has been shown that isospectral pairs of quantum graphs must have rationally dependent edge lengths.

22. The Chebotarev Density Theorem

The Chebotarev density theorem for algebraic number fields was proved in 1922. There are discussions in Stevenhagen and Lenstra [87] and Stark [81]. The Stark version is sketched in Figure 51. The Chebotarev density theorem generalizes the Dirichlet theorem saying that there are an infinite number of primes in an arithmetic progression of the form $\{a + nb \,|n \in \mathbb{Z}\}$, when a, b are relatively prime. It also generalizes a theorem of Frobenus concerning a polynomial $f(x) \in \mathbb{Z}[x]$ with non-0 discriminant $\Delta(f)$ and the list of exponents of the irreducible factors mod p (called the decomposition type mod p) for primes p not dividing $\Delta(f)$. Let K be the extension field of $\mathbb{Q}$ obtained by adjoining all the roots of $f(x)$ and let G be the Galois group of $K/\mathbb{Q}$. The Frobenius theorem says that the (analytic) density of such p for given decomposition type $e_1, ..., e_t$ is

$$\frac{\#\{\sigma \in G \,|\text{cycle pattern of } \sigma \text{ is } e_1, ..., e_t\}}{\#(G)}.$$

In order to prove the graph theory analog of the Chebotarev density theorem we will need some information on the poles of zeta and L-functions of graph coverings. We begin with a Lemma for which you need to recall our notation: the largest circle of convergence R_X from Definition 3 and the 0,1 edge matrix $W_1 = W_1(X)$ from Definition 8.

Lemma 10. *Suppose Y is an n-sheeted covering of X. The maximal absolute value of an eigenvalue of $W_1(X)$ is the same as that for $W_1(Y)$. This common value is $R_Y^{-1} = R_X^{-1} = \omega_Y = \omega_X$.*

Proof. First note that from Stark and Terras [83], we know $\zeta_X^{-1}(u)$ divides $\zeta_Y^{-1}(u)$. See Proposition 5. It follows that $R_Y \le R_X$.

Then a standard estimate from the theory of zeta functions of number fields works for graph theory zeta functions as well. For all real $u \ge 0$ such that the infinite product for $\zeta_X(u)$ converges, we have

$$\zeta_Y(u) \le \zeta_X(u)^n, \tag{22.1}$$

with n equal to the number of sheets of the covering. Thus $R_X \le R_Y$.

We take the idea of the proof of formula (22.1) from Lang [53] (p. 160). One begins with the product formula for $\zeta_Y(u)$ and the behavior of primes in coverings.

So for real u such that $R_Y > u \geq 0$, we have a product over primes $[D]$ of Y giving

$$\zeta_Y(u) = \prod_{[D]} \left(1 - u^{\nu(D)}\right)^{-1}.$$

Rewrite this as a product over primes $[C]$ of X such that $[D]$ is above $[C]$:

$$\zeta_Y(u) = \prod_{[C]} \prod_{i=1}^{g} \left(1 - u^{\nu(D_i)}\right)^{-1}.$$

Recall that the primes above C have the form $D_i = \widetilde{C^{f_i}}$, a closed path obtained by lifting C a total of f_i times. This means

$$\zeta_Y(u) = \prod_{[C]} \prod_{i=1}^{g} \left(1 - u^{f_i \nu(C)}\right)^{-1}.$$

We know (see the section on behavior of primes in coverings) that $n = \sum_{i=1}^{g} f_i$. It follows that

$$\zeta_Y(u) \leq \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-g} \leq \zeta_X(u)^n.$$

□

Next let us define the analytic density.

Definition 49. *If S is a set of primes in X, define the* ***analytic density*** *$\delta(S)$ to be*

$$\delta(S) = \lim_{u \to R-} \frac{\sum\limits_{[C]\in S} u^{\nu(C)}}{\sum\limits_{[C]} u^{\nu(C)}} = \lim_{u \to R-} \frac{\sum\limits_{[C]\in S} u^{\nu(C)}}{\log \zeta_X(u)} = \lim_{u \to R-} \frac{\sum\limits_{[C]\in S} u^{\nu(C)}}{-\log\left(R_X - u\right)}.$$

Here the sums are over primes $[C]$ in X.

Question: Why does $\sum\limits_{[C]} u^{\nu(C)}$ blowup at $u = R_X$ like $\log \zeta_X(u)$?

To answer this, recall that for $0 \leq u < R_X$

$$\zeta_X(u) = \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-1} = \det\left(I - W_1(X)\right).$$

Take the logarithm and obtain:

$$\begin{aligned} \log \zeta_X(u) &= -\sum_{[C]} \log\left(1 - u^{\nu(C)}\right) = \sum_{[C]} \sum_{m \geq 1} \frac{1}{m} u^{m\nu(C)} \\ &= \sum_{[C]} u^{\nu(C)} + H(u), \end{aligned}$$

where

$$H(u) = \sum_{[C]} \sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)}.$$

The amazing thing is that $H(u)$ is bounded up to $u = R_X$ and beyond. Thus $\sum\limits_{[C]} u^{\nu(C)}$ must account for the blowup of $\log \zeta_X(u)$ at $u = R_X$.

To see that $H(u)$ is bounded up to $u = R_X$, we do a few estimates:

$$\sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)} \leq \sum_{m \geq 2} u^{m\nu(C)} = \frac{u^{2\nu(C)}}{1 - u^{\nu(C)}} \leq \frac{u^{2\nu(C)}}{1 - u}.$$

using the fact that $0 \leq u \leq R_X \leq 1$. It follows that

$$H(u) \leq \frac{1}{1 - u} \sum_{[C]} u^{2\nu(C)},$$

which converges up to $u^2 = R_X$ or $u = \sqrt{R_X}$, as $\log \zeta_X(u^2)$ converges up to $u = \sqrt{R_X}$.

What if $R_X = 1$? Then the graph must be a cycle. Why?

Note that $\zeta_X(u)^{-1} = \det(I - W_X) = (1 - u^n)$, by the Perron-Frobenius theorem, as the roots must be equally spaced.

Theorem 27. ***Graph Theory Chebotarev Density Theorem.***

Suppose the graph X is not a cycle graph. If Y/X is normal and $\{g\}$ is a fixed conjugacy class in the Galois group $G = G(Y/X)$

$$\delta\{\,[C] \text{ prime of } X \mid \sigma(C) = \{g\}\} = \frac{|\{g\}|}{|G|}.$$

Here $\sigma(C)$ is the normalized Frobenius for C.

Proof. We imitate the proof sketched by Stark in [81]. This means we sum the terms $\overline{\chi_\pi(g)} \log L(u, \pi, Y/X)$over all irreducible representations π of G. Here $\chi_\pi = Tr(\pi)$. This gives the following asymptotic formula as u approaches R_X from below:

$$\log\left(\frac{1}{R_X - u}\right) \underset{u \to R_X-}{\sim} \log \zeta_X(u) = \sum_{\pi \in \widehat{G}} \overline{\chi_\pi(g)} \log L(s, \pi).$$

Here we use Lemma 10 and the fact that

$$\zeta_Y(u) = \prod_{\rho \in \widehat{G}} L_V(u, \rho, Y/X)^{d_\rho}.$$

It follows from the Euler product definition of the L-functions that

$$\log\left(\frac{1}{R_X - u}\right) \underset{u \to R_X-}{\sim} \sum_{\pi \in \widehat{G}} \sum_{\substack{[C] \\ \text{prime of } X}} \chi_\pi(\sigma(C)) u^{\nu(C)} \overline{\chi_\pi(g)}$$

$$+ \sum_{\pi \in \widehat{G}} \sum_{\substack{[C] \\ \text{prime of } X}} \sum_{m \geq 2} \frac{1}{m} \chi_\pi(\sigma(C^m)) u^{m\nu(C)} \overline{\chi_\pi(g)}.$$

The second term in the sum is holomorphic as $u \to R_X-$. To see this, note that the second term can be written as

$$\sum_{\substack{[C] \\ \text{prime of } X}} \sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)} \sum_{\pi \in \widehat{G}} \chi_\pi(\sigma(C^m)) \overline{\chi_\pi(C)}.$$

Then, using the orthogonality relations for characters of G, we find that this last sum is for $0 \leq u < R_X$

$$\begin{aligned} \frac{|G|}{|\{g\}|} \sum_{\substack{[C] \\ \{\sigma(C^m)\} = \{g\}}} \sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)} &\leq |G| \sum_{[C]} \sum_{m \geq 2} \frac{1}{m} u^{m\nu(C)} \\ &\leq |G| \sum_{[C]} \frac{u^{2\nu(C)}}{1 - u} \leq \frac{|G|}{1 - u} \sum_{[C]} u^{2\nu(C)}. \end{aligned}$$

This is holomorphic up to $u = \sqrt{R_X}$.

Thus, we have shown that $\log\left(\frac{1}{R_X - u}\right)$

$$\begin{aligned} &\underset{u \to R_X-}{\sim} \sum_{\pi \in \widehat{G}} \sum_{\substack{[C] \\ \text{prime of } X}} \chi_\pi(\sigma(C)) u^{\nu(C)} \overline{\chi_\pi(g)} \\ &= \sum_{\substack{[C] \\ \text{prime of } X}} \sum_{\pi \in \widehat{G}} \chi_\pi(\sigma(C)) u^{\nu(C)} \overline{\chi_\pi(g)} \\ &= \frac{|G|}{|\{g\}|} \sum_{\substack{[C] \\ \{\sigma(C)\} = \{g\}}} u^{\nu(C)}. \end{aligned}$$

The theorem follows. □

An example of our result can be found in Figure 53. The example concerns the splitting of primes in a non-normal cubic covering Y_3/X, where $X = K_4 - edge$. Thus one must consider what happens in the normal cover for which Y_3/X is intermediate.

Exercise 74. *Fill in the details concerning the Example in Figure 53 by imitating Stark's arguments for the corresponding example in* [81].

23. Siegel Poles

In number theory, there is a known zero free region of a Dedekind zeta function which can be explicitly given except for the possibility of a single first order real zero within this region. This possible exceptional zero has come to be known as a **"Siegel zero"**and is closely connected with the Brauer-Siegel Theorem on the growth of the class number times the regulator with the discriminant. There is no known example of a Siegel zero for Dedekind zeta functions. In number fields, a Siegel zero "deserves" to arise already in a quadratic extension of the base field. This has now been proved in many cases (see Stark [82]).

$\zeta_X(u)^{-1}$ is a polynomial with a finite number of zeros. Thus there is an $\epsilon > 0$ such that any pole of $\zeta_X(u)$ in the region $R_X \leq |u| < R_X + \epsilon$ must lie on the circle $|u| = R_X$. This gives us the graph theoretic analog of a **"pole free region"**, $|u| < R_X + \epsilon$; the only exceptions lie on the circle $|u| = R_X$. We will show that $\zeta_X(u)$ is a function of u^δ with $\delta = \delta_X$ a positive integer from Definition 51 below. This implies there is a $\delta-$fold symmetry in the poles of $\zeta_X(u)$; i.e., $u = \varepsilon_\delta R$ is also a pole of $\zeta_X(u)$, for all δ^{th} roots of unity ε_δ. Any further poles of $\zeta_X(u)$ on $|u| = R$ will be called **"Siegel poles"** of $\zeta_X(u)$. Thus if $\delta = 1$, any pole $u \neq R$ of $\zeta_X(u)$ with $|u| = R$ will be called a Siegel pole.

Definition 50. *A vertex of X having degree $\geq$ 3 is called a* ***node****of X.*

A graph X of rank $\geq$ 2 always has at least one node.

Definition 51. *If X has rank $\geq$ 2 define* ***small delta*** *to be*

$$\delta_X = g.c.d.\left\{\nu(P) \;\middle|\; \begin{array}{c} P = \text{backtrackless path in } X \text{ such that the} \\ \text{initial and terminal vertices are both nodes} \end{array}\right\}.$$

When a path P in the definition of δ_X is closed, the path will be backtrackless but may have a tail. Later we give an equivalent definition of δ_X not involving paths with tails. The relation between δ_X and our earlier Δ_X from Definition 7 is given by the following result.

Theorem 28. *Suppose X has rank $\geq$ 2. Then either $\Delta_X = \delta_X$ or $\Delta_X = 2\delta_X$.*

It is easy to see that if Y is a covering graph of X (of rank $\geq$ 2) we have $\delta_Y = \delta_X$ since they are the $g.c.d.$s of the same set of numbers. Therefore δ_X is a covering invariant. Because of this, Theorem 28 give a useful Corollary.

Corollary 7. *If Y is a covering of a graph X of rank ≥ 2, then*

$$\Delta_Y = \Delta_X \quad \text{or} \quad 2\Delta_X.$$

For a cycle graph X the ratio Δ_Y/Δ_X can be arbitrarily large. The general case, of the theorem about Siegel poles can be reduced to the more easily stated case where $\delta_X = 1$, where any pole of $\zeta_X(u)$ on $|u| = R$ other than $u = R$, is a **Siegel pole**.

Theorem 29. ***Siegel Poles when $\delta_X = 1$.*** *Suppose X has rank $\geq$ 2, and $\delta_X = 1$. Let Y be a connected covering graph of X and suppose $\zeta_Y(u)$ has a Siegel pole μ. Then we have the following facts.*

1. *The pole μ is a first order pole of $\zeta_Y(u)$ and $\mu = -R$ is real.*
2. *There is a unique intermediate graph X_2 to Y/X with the property that for every intermediate graph $\widetilde{X}$ to Y/X (including X_2), μ is a Siegel pole of $\zeta_{\widetilde{X}}(u)$ if and only if $\widetilde{X}$ is intermediate to Y/X_2.*
3. *X_2 is either X or a quadratic (i.e., 2-sheeted) cover of X.*

Before proving Theorem **28.**, we need a Lemma.

Lemma 11. *The invariant δ of Definition 51 equals δ' defined by*

$$\delta' = g.c.d.\left\{\nu(P) \;\middle|\; \begin{array}{c} P \text{ is backtrackless and the initial and terminal vertices of } P \text{ are} \\ \text{(possibly equal) nodes and no intermediate vertex is a node} \end{array}\right\}.$$

Proof. Clearly $\delta|\delta'$.
To show $\delta'|\delta$, note that anything in the length set for δ is a sum of elements of the length set for δ'. □

Proof of Theorem 28.

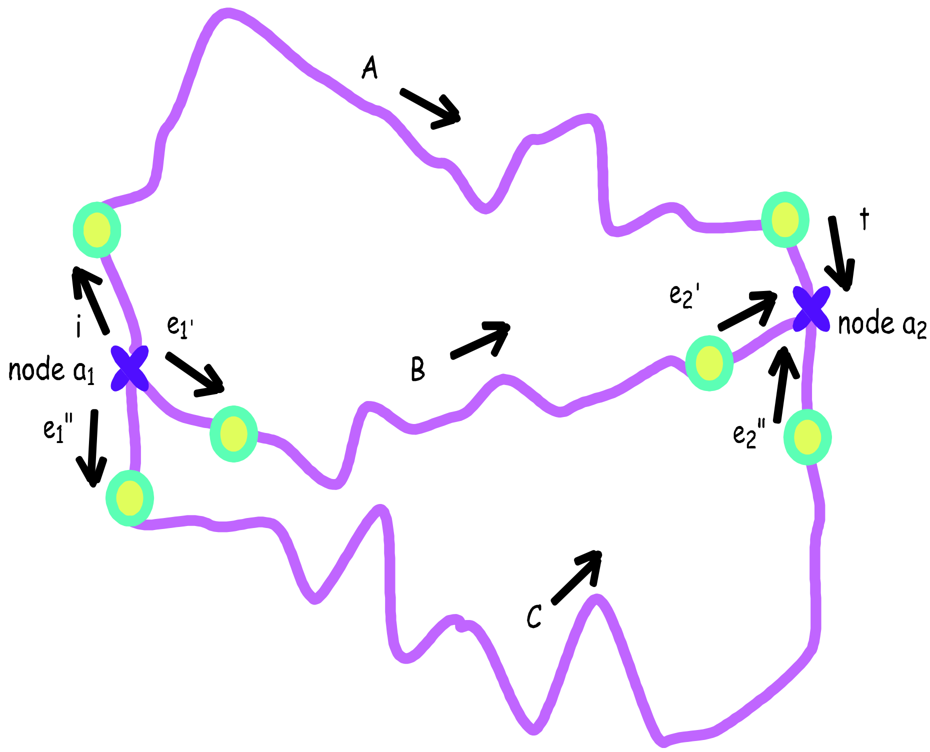

FIGURE 64. The paths in the proof of Theorem **28** for the case when the nodes are different.

Proof. Theorem 28 says that if Δ_X is odd then $\Delta_X = \delta_X$ and otherwise either $\Delta_X = \delta_X$ or $\Delta_X = 2\delta_X$. First note that $\delta|\Delta$ since every cycle in a graph X of rank ≥ 2 has a node (otherwise X would not be connected). Second we show that $\Delta|2\delta$.

If X has a loop, then the vertex of the loop is a node (if the rank is ≥ 2) and thus $\Delta = \delta = 1$. **So assume X loopless for the rest of the proof .**

By Lemma 11 we may consider only backtrackless paths A between arbitrary nodes α_1 and α_2 without intermediate nodes. There are two cases.

Case 1. $\alpha_1 \neq \alpha_2$.

Let e_1' be an edge out of α_1 not equal to the initial edge i of A (or i^{-1} since there are no loops) and e_2' be an edge into α_2 not equal to the terminal edge t of A (or t^{-1}). Let $B = P(e_1', e_2')$ from Lemma 4.

Suppose e_1'' is another edge out of α_1 such that $e_1'' \neq i$, $e_1'' \neq e_1'$ (or their inverses). Likewise suppose e_2'' is another edge into α_2 such that $e_2'' \neq t$, $e_2'' \neq e_2'$ (or their inverses). Let $C = P(e_1'', e_2'')$ from Lemma 4. See Figure 64.

Then AB^{-1}, AC^{-1}, BC^{-1} are backtrackless tailless paths from α_1 to α_1.

We have

$$\begin{aligned} \Delta|\nu(AB^{-1}) &= \nu(A) + \nu(B), \\ \Delta|\nu(AC^{-1}) &= \nu(A) + \nu(C), \\ \Delta|\nu(BC^{-1}) &= \nu(B) + \nu(C). \end{aligned}$$

It follows that Δ divides $2\nu(A)$ since

$$2\nu(A) = (\nu(A) + \nu(B)) + (\nu(A) + \nu(C)) - (\nu(B) + \nu(C)).$$

Case 2. $\alpha_1 = \alpha_2$.

Then A is a backtrackless path from α_1 to α_1 without intermediate nodes. This implies that A has no tail, since then the other end of the tail would have to be an intermediate node. Therefore Δ divides $\nu(A)$ and hence Δ divides $2\nu(A)$. Thus, in all cases, Δ divides $2\nu(A)$ and hence $\Delta|2\delta$. □

Lemma 12. $\zeta_Y(u) = f(u^d)$ *if and only if d divides* Δ_Y.

Proof. Clearly $\zeta_Y(u)$ is a function of u^{Δ_Y} and therefore of u^d for all divisors d of Δ_Y.

Conversely suppose $\zeta_Y(u)$ is a function of u^d. In the power series for $\zeta_Y(u)$, d divides n for all n with $a_n > 0$. But if P is a prime cycle of Y with length $\nu(P) = n$, then $a_n \geq 1$ and hence for all prime cycles P, d divides $\nu(P)$. Therefore, by Definition 7, d divides Δ_Y. □

Proof of Theorem 29.

Proof. First reduce the Theorem to the case that Y/X is normal with Galois group G. To see that this is possible, let $\widetilde{Y}$ be a normal cover of X containing Y. Since $\zeta_{\widetilde{Y}}(u)^{-1}$ is divisible by $\zeta_Y(u)^{-1}$ and both graphs have the same R (by Lemma 10), as well as the same δ. Therefore a Siegel pole of $\zeta_Y(u)$ is a Siegel pole of $\zeta_{\widetilde{Y}}(u)$. Once the Theorem is proved for normal covers of X, the graph X_2 which we obtain will be contained in Y as well as in every graph intermediate to Y/X whose zeta function has the Siegel pole and we will be done. From this point on, assume Y/X is normal.

Recall Corollary 2 , $\zeta_X(u)^{-1} = \det(I - W_X u)$ and Definition 23 of the 0,1 edge matrix W_X. Poles of $\zeta_X(u)$ are reciprocal eigenvalues of W_X. Note that for graphs of rank ≥ 2 the edge matrix W_X satisfies the hypotheses of the Perron-Frobenius theorem, namely that W_X is irreducible.

By Lemma 10, the Perron-Frobenius Theorem 13 now says that if there are d poles of $\zeta_Y(u)$ on $|u| = R_Y = R_X = \frac{1}{\omega}$, then these poles are equally spaced first order poles on the circle and further $\zeta_Y(u)$ is a function of u^d. By Lemma 12, Δ_Y has to be divisible by d. But $\delta = \delta_X = \delta_Y = 1$ implies $\Delta_Y = 1$ or 2. Therefore $d = 1$ or 2. If there is a Siegel pole, $d > 1$. Thus if there is a Siegel pole, $d = 2$, $\Delta_Y = 2$ and the equal spacing result says the Siegel pole is $-R_X$ and it is a pole of order one.

Theorem 4 says

$$\zeta_Y(u) = \prod_{\pi \in \widehat{G}} L(u, \pi)^{d_\pi}. \tag{23.1}$$

Therefore $L(u, \pi)$ has a pole at $-R_X$ for some π and $d_\pi = 1$. Moreover π must be real or $L(u, \overline{\pi})$ would also have a pole at $-R_X$.

So either π is trivial or it is first degree and $\pi^2 = 1$, $\pi \neq 1$. Then we say π **is quadratic.**

Case 1. π is trivial.

Then $\Delta_X = 2$ just like $\Delta_Y = 2$. Every intermediate graph then has poles at $-R_X$ as well.

Case 2. $\pi = \pi_2$ is quadratic.

No other $L(u, \pi)$ has $-R_X$ as pole since it is a first order pole of $\zeta_Y(u)$. Let

$$H_2 = \{x \in G \,|\, \pi_2(x) = 1\} = \ker \pi_2.$$

Then $|G/H_2| = 2$ which implies there is a graph X_2 corresponding to H_2 by Galois theory. Moreover X_2 is a quadratic cover of X.

Consider the diagram of covering graphs with Galois groups indicated next to the covering lines in Figure 65. Then

$$\zeta_{\widetilde{X}}(u) = L(u, Ind_H^G 1) = \prod_{\kappa \in \widehat{G}} L(u, \kappa)^{m_\kappa}.$$

$L(u, \kappa)$ appears m_κ times in the factorization and Frobenius reciprocity says

$$m_\kappa = \left\langle \chi_{Ind_H^G 1}, \kappa \right\rangle = \langle 1, \kappa|_H \rangle \leq \deg \kappa.$$

Let $\kappa = \pi_2$, which has $\deg \kappa = 1$. This implies $\zeta_{\widetilde{X}}(u)$ has $-R_X$ as a (simple) pole if and only if $\pi_2|_H =$ identity. Note that $-R_X$ is not a pole of any $L(u, \pi)$, for $\pi \neq \kappa$. We have $\pi_2|_H =$identity if and only if $H \subset H_2 = \ker \pi_2$, which is equivalent to saying $\widetilde{X}$ covers X_2.

X_2 is unique as each version of X_2 would cover the other. □

Note that if $\widetilde{X}$ is intermediate to Y/X in Theorem 29, then $\Delta(\widetilde{X}) = 1$ or 2 and the Perron-Frobenius Theorem says $\zeta_{\widetilde{X}}(u)$ is a function of u^d, where d is the number of poles of $\zeta_{\widetilde{X}}(u)$ on the circle $|u| = \omega^{-1}$. Thus the $\widetilde{X}$ with $\Delta(\widetilde{X}) = 2$ are exactly the $\widetilde{X}$ with $\zeta_{\widetilde{X}}(u)$ having $-\omega^{-1}$ as a Siegel pole and these are the $\widetilde{X}$ which cover X_2. Since $\Delta(\widetilde{X}) = 2$ is the condition for $\widetilde{X}$ to be bipartite, this says that $\widetilde{X}$ is bipartite. $\widetilde{X}$ is not quadratic unless $\widetilde{X} = X_2$. All remaining intermediate graphs $\widetilde{X}$ to Y/X have $\Delta_{\widetilde{X}} = 1$.

Every graph X of rank ≥ 2 has a covering Y with zeta function having a Siegel pole as we will see. This is probably not the case for algebraic number fields.

Corollary 8. *Under the hypotheses of Theorem 29 with X_2 the unique graph defined in that Theorem, the set of intermediate bipartite covers to Y/X is precisely the set of graphs intermediate to Y/X_2.*

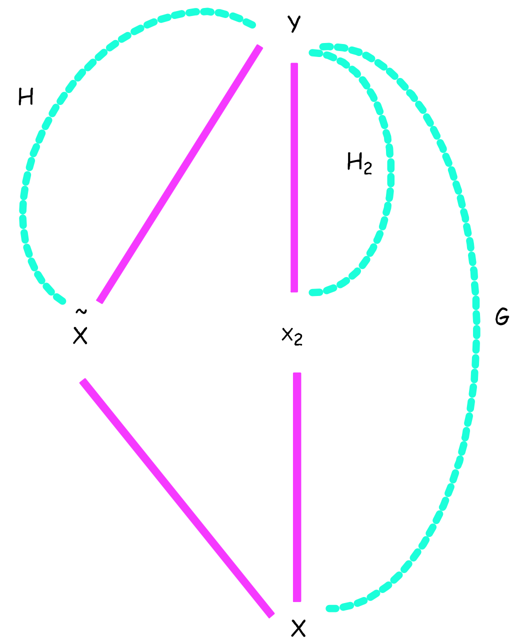

FIGURE 65. The covering appearing in Theorem 29. Galois groups are indicated with dashed lines.

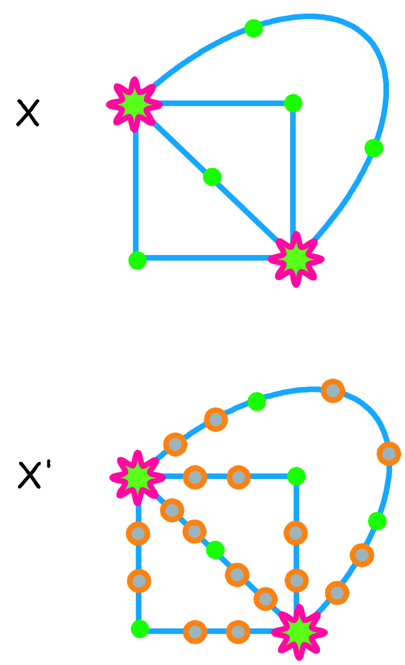

FIGURE 66. X' is the inflation of X increasing the length of paths by a factor of 3.

For the next result, we need some definitions.

Definition 52. *The **inflation** $I^\delta(X)$ is defined by putting $\delta - 1$ vertices on every edge of X.*

Definition 53. *The **deflation** $D_\delta(X)$ is obtained from X by collapsing δ consecutive edges between consecutive nodes to one edge.*

See Figure 66 for examples.

Theorem 30. ***Siegel Poles in General.*** *Suppose X is connected, not a cycle, with no danglers and $\delta = \delta_X = \Delta_X$. Suppose that Y covers X and Y is connected with $\Delta_Y = 2\Delta_X = 2\delta$. Then the follow results hold.*

(1) *There is a unique intermediate quadratic cover X_2 to Y/X such that $\Delta_{X_2} = 2\delta$.*
(2) *Let $\widetilde{X}$ be any graph intermediate to Y/X. Then $\Delta_{\widetilde{X}} = 2\delta$ if and only if $\widetilde{X}$ is intermediate to Y/X_2.*

Proof. When $\delta > 1$, this is proved by deflation. The deflated graph $D_\delta(X) = X'$ contains all the information on X and its covers. This graph X' has $\delta_{X'} = 1$ and $\zeta_X(u) = \zeta_{X'}(u^\delta)$. Every single Y/X has a corresponding Y' covering X' such that

$$\zeta_Y(u) = \zeta_{Y'}(u^{\delta_X}).$$

There is also a relation between all the Artin L-functions

$$L_{Y/X}(u,\pi) = L_{Y'/X'}(u^{\delta_X},\pi).$$

where π is a representation of $Gal(Y/X) = Gal(Y'/X')$. Theorem 30 now follows from Theorem 29 which contains the case $\delta = 1$ of Theorem 30. □

Note that in Theorem 30 if $\delta = 1$, the $\widetilde{X}$ with $\Delta_{\widetilde{X}} = 2\delta$ are the bipartite covering graphs intermediate to Y/X. and, in particular, X_2 is bipartite. Even when $\Delta_X = \delta_X = \delta$ is odd, the $\widetilde{X}$ with $\Delta_{\widetilde{X}} = 2\delta$ are precisely the bipartite covering graphs intermediate to Y/X. But, if $\Delta_X = \delta_X = \delta$ is even, then every graph intermediate to Y/X, including X itself, is bipartite, and thus being bipartite does not determine which quadratic cover of X is X_2. Note also that when the rank of X is ≥ 2, we have proved the following purely graph theoretic equivalent theorem.

Theorem 31. ***The Story of Bipartite Covers.***
Suppose X is a finite connected graph of rank ≥ 1 and that Y is a bipartite covering graph of X. Then we have the following facts.

(1) *When X is bipartite, every intermediate covering $\widetilde{X}$ to Y/X is bipartite.*
(2) *When X is not bipartite, there is a unique quadratic covering graph X_2 intermediate to Y/X such that any intermediate graph $\widetilde{X}$ to Y/X is bipartite if and only if $\widetilde{X}$ is intermediate to Y/X_2.*

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