The Computational Complexity of Some Classical Problems from Statistical Physics

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1. Introduction

The motivation of this paper is to attempt to review and classify the difficulty of a range of problems, arising in the statistical mechanics of physical systems and to which I was introduced by J.M. Hammersley in the early sixties. Their common characteristics at the time were that they all seemed hard and there was little existing mathematical machinery which was of much use in dealing with them. Twenty years later the situation has not changed dramatically; there do exist some mathematical techniques which appear to be tools in trade for this area, subadditive functions and transfer matrices for example, but they are still relatively few and despite a great deal of effort the number of exact answers which are known to the many problems posed is extremely small. Below we shall attempt to explain why this should be so by showing how the problems originally studied are special cases of a wide range of problems which can, in a well defined sense, be regarded as the most intractable enumeration problems that can sensibly be posed.

We do this by relating the problems to their position in the hierarchy of computational complexity theory. While concepts such as P (polynomial time) and NP (nondeterministic polynomial time) have rapidly become commonplace ideas in mathematics since their introduction via Cook's theorem in 1970, the counting analogue of NP, denoted by #P, introduced by Valiant (1979a) and like NP also having *complete* or hardest problems has received less attention. As we shall see, most of the natural problems arising in statistical physics can be described in this framework and as first pointed out by Valiant (1979a,b) and Jerrum (1981, 1987) this goes a long way towards explaining their apparent intractability. There are however what can almost be described as 'pockets of resistance', for example some of these problems do have 'exact' solutions for some 2-dimensional lattices, and whether or not this is a phenomenon of dimension or planarity or the very special nature of the lattice is an interesting and unanswered question to which we return at the conclusion.

2. The Statistical Physics Problems

In this section I define slightly generalised versions of the main problems from statistical physics which we shall be considering. By 'generalised' I mean that instead of formulating them as problems on one of the standard lattices the underlying structure will be a general graph. The graph theoretic notation will be standard (see Bondy and Murty 1976).

Percolation Theory

As originally propounded by Broadbent and Hammersley (1957) this is concerned with the spread of blight through a medium in which the elements of the medium independently permit or fail to permit passage. More formally this can be described as follows.

Let G be an arbitrary undirected graph and let p, $0 \leq p \leq 1$ be fixed. Suppose now that each edge of G is, independently of each other edge, removed with probability $q = 1-p$. Denote the resulting (random) subgraph of G by ω and let $P(G;p)$ denote the probability that ω is spanning, that is that in ω it is possible to move from any vertex of G to any other. We call $P(G;p)$ the *percolation probability* of G. It is clearly a measure of the reliability or vulnerability of G regarded as a communication network and has the standard S-shaped curve as p varies between 0 and 1, and is called the *all terminal reliability* by Provan and Ball (1983).

When G is a lattice we let $P(p)$ denote the probability that the component of ω which contains the origin is infinite. It is easy to see that there exists a *critical probability* p_c defined by

$$p_c = \inf\{p : P(p) > 0\}.$$

Determining p_c exactly is extraordinarily difficult; see for example the proof by Kesten (1980) that $p_c = \frac{1}{2}$ for the square lattice. As far as I am aware exact results are known only for some 2-dimensional lattices. For further details we refer to the monographs of Kesten (1982) and Grimmett (1989).

The Ising Model

This is a problem of long standing and can be defined for a general graph as follows. Let σ be an assignation of *positive* $(+1)$ and *negative* (-1) spins to the vertices of a graph G. The *interaction energy* $E(\sigma)$ is defined by

$$E(\sigma) = -J \sum_{i \sim j} \sigma_i \sigma_j - H \sum_i \sigma_i$$

where the first summation is only over i, j which are adjacent in G, H is the *external magnetic field* and J is the *coupling constant*. The *partition* *function* Z is then given by

$$Z(G) = \sum_{\sigma} \exp(-\beta E(\sigma))$$

where the sum is over all possible spin configurations. The fundamental problem is to determine $Z(G)$, though as far as the applications to physics are concerned it would suffice to find the thermodynamic limit of Z as G moved through an increasing sequence of subgraphs of the d-dimensional lattice. The Onsager solution (see Percus 1971) for the case $d = 2$ with zero external magnetic field is the classic result of this area. As yet there is no extension known in higher dimensions.

Self Avoiding Walks

The basic question about self avoiding walks on a lattice is to determine $f(n)$, the number of paths starting at the origin, having n edges, and visiting no point more than once. Hammersley (1957) used subadditivity to prove the existence of a constant θ such that

$$\lim_{n\to\infty} [f(n)]^{1/n} = \theta.$$

The constant θ clearly depends on the lattice, but even for the 2-dimensional square lattice its value is not known exactly; the best exact bounds give only $2.58\ldots \leq \theta \leq 2.72\ldots$. The natural generalisation of this to a general graph G is to let $W_n(G)$ be the number of paths in G of length n and which pass through each vertex at most once. When $n+1$ equals the number of vertices of G this is the well known problem of counting the Hamiltonian paths of G.

Animals or Polyominoes

Conceptually very close to self avoiding walks, counting animals or polyominoes has been for a long time a popular if frustrating sport, see for example the article by Whittington in this volume. Although usually defined for lattices we can define an *animal* of *size* n on an arbitrary graph G to be any subset X of the vertex set $V(G)$ such that the subgraph of G induced by X is connected and $|X| = n$. We let $a_n(G)$ denote the number of such animals. When G is a lattice $\mathcal{L}$ it is easy to use subadditivity to prove that

$$\lim_{n\to\infty} [a_n(\mathcal{L})]^{1/n} = a(\mathcal{L})$$

exists, but determining this limit exactly, or merely obtaining close bounds, again seems to be extraordinarily difficult, even for the square lattice.

The Monomer-Dimer Problem

This arises in the study of physical systems involving diatomic molecules (*dimers*). For a general graph G a formal description of the problem is to ask for the number of arrangements of N_1 dimers and N_2 monomers on the edges and vertices of G such that each dimer is placed on an edge, each monomer on a vertex, and each vertex of G is either occupied by exactly one monomer or is the endvertex of exactly one dimer. Clearly for this to be possible

$$2N_1 + N_2 = |V(G)| = N.$$

The ratio N_2/N is called the *monomer density*.

When there are no monomers we have what is known as the *dimer problem*, it can be rephrased as counting the number of perfect matchings of a graph (a *matching* is a set of edges no two of which share a common vertex, it is a *perfect matching* of G if each vertex of G is the endpoint of one edge of the matching).

Similarly the *monomer dimer problem* is exactly the problem of counting the number of matchings of a given size in a graph.

Ice-Type Models

The simplest ice-type model can be described as follows. Let G be any regular 4-valent graph and let Z_{ice}, the partition function, count the number of orientations of the edges of G which satisfy the rule that the number of arrows into each vertex equals the number of arrows out. In graph theoretic terminology Z_{ice} counts the number of Eulerian orientations of G. Details of the physical motivation for this and a description of a range of ice-type models can be found in Baxter (1982). A remarkable result about the ice model is that of Lieb (1967) who showed that if $Z_{\text{ice}}(m, n)$ denotes the ice partition function on the $m \times n$ section of the square lattice then

$$\lim_{m,n\to\infty} [Z_{\text{ice}}(m, n)]^{1/mn} = (4/3)^{3/2}.$$

No extension of this to higher dimensions is known.

The q-State Potts Model

This is naturally defined for any graph G and positive integer q as follows. A *state* σ of the vertex set of G is a function which assigns to each vertex i of G a *spin* σ_i, where $\sigma_i \in \{1, 2, \ldots, q\}$. The *energy* associated with state σ is defined to be

$$E(\sigma) = -J \sum \delta(\sigma_i, \sigma_j)$$

where the summation is over all distinct $i, j \in V(G)$ which are joined by an edge and δ is the usual delta function taking values 1 and 0 depending on whether σ_i equals σ_j or not.

The *partition function* $Z(G)$ is then defined by

$$Z(G) = \sum_{\sigma} \exp\{-KE(\sigma)/J\}$$

where J, K are physical constants, see Baxter (1982). For $q = 2$ it is just the Ising model.

3. Computational Complexity

The basic notions of computational complexity are now familiar concepts in most branches of mathematics. One of the main purposes of the theory is to classify and explain the gap that seems to separate tractable computational problems from the apparently intractable. Deciding whether or not P = NP is probably the most important problem in theoretical computer science. The extension of these ideas to enumeration problems has received less attention and we will briefly review the main concepts here.

We regard a computational (enumeration) problem as a function mapping inputs to solutions, (graphs to the number of their 3-vertex colourings for example). A problem is *polynomial time computable* if there exists an algorithm which computes the function in a length of time (number of steps) bounded by a polynomial in the size of the problem instance. The class of such problems we denote by P. If A and B are two problems we say that A is *polynomial time Turing reducible* to B, written $A \propto B$, if it is possible with the aid of a subroutine for problem B to solve A in polynomial time, in other words the number of steps needed to solve A (apart from calls to the subroutine for B) is polynomially bounded.

The class #P can be described informally as the class of enumeration problems in which the structures being counted are recognisable in polynomial time. In other words there is an algorithm which runs in polynomial time and which will verify that a given structure has the form needed to be included in the count. For example counting hamiltonian paths in a graph is in #P because it is easy to check in polynomial time that a given set of edges is a hamiltonian path.

Like NP, #P has a class of 'hardest' problems called the #P-*complete* problems. They can be formally described by, problem A belonging to #P is #P-complete if for any other problem $B \in$ #P, we have $B \propto A$. The classic example of a #P-complete problem is counting truth assignments of a Boolean function. This consists of

INPUT: A Boolean formula ϕ in variables $x_1, x_2, \ldots, x_n$ and the connectives $\vee$, $\wedge$, $\neg$.

QUESTION: How many distinct assignments of truth values to the x_1, $x_2, \ldots, x_n$ make ϕ true?

The #P-complete problems tend to be the enumerative counterparts of NP-complete problems though it has to be emphasized that there is no exact formulation of this remark.

As with NP, we define a problem to be #P-*hard* if any problem in #P is polynomial time reducible to it. In other words A is #P-hard if the existence of a polynomial time algorithm for A would imply the existence of a polynomial time algorithm for any problem in #P.

It is clear from this that describing a problem as #P-hard or #P-complete is very strong evidence of its inherent intractability. There are now several thousand problems known to be #P-complete. A polynomial time algorithm for any one of them would imply #P = P and this in turn would imply NP = P. For a more precise formulation we refer to Garey and Johnson (1979).

4. The Complexity of the Physical Problems

We now turn to an examination of the status in the complexity hierarchy of the previously discussed physical problems.

Self Avoiding Walks

As defined, counting the number of self avoiding walks of n steps on a graph G of $n+1$ vertices is exactly the problem of counting the number of Hamiltonian paths in G. Recall that a path is *Hamiltonian* if it visits each vertex exactly once. This is one of the classical #P-complete problems and is known to be #P-complete even when restricted to planar graphs with maximum degree 3.

It is not surprising therefore that no exact result about self avoiding walks seems to be known except for tree like structures such as Bethe lattices.

The Dimer Problem

For a general graph this is exactly the problem of counting perfect matchings. As far as complexity is concerned it is probably the most intriguing of the problems discussed in that there is a clear cut distinction between planar and nonplanar structures. This is because of the following statements which are partial restatements of classic theorems of Kasteleyn (1967) and Valiant (1979a).

(1) Counting perfect matchings in a planar graph can be done in polynomial time.

(2) Counting perfect matchings in a general graph is a #P-complete problem.

The difference between the two statements (1) and (2) is partially explained by the following observation.

(3) For planar graphs, counting perfect matchings reduces to evaluating

the Pfaffian of a matrix and this is equivalent to evaluating a determinant. For general graphs the problem is equivalent to evaluating the permanent of a 0–1 matrix.

The *permanent* of a matrix A is just the expansion of the determinant of A with all terms having positive signs. Paradoxically (at least at the naive level) this makes it hard to compute.

The method of dealing with planar graphs is an extension of the method developed by Kasteleyn (1961) and Temperley and Fisher (1961) to show that on the 2-dimensional square lattice, if $f(N)$ denotes the number of dimer coverings of an $N \times N$ section, then

$$\lim_{n\to\infty}[f(N)]^{1/N^2} = e^{2G/\pi} = 1.791622\ldots$$

where G is Catalan's constant given by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

The fact that in higher dimensions the lattices are nonplanar would suggest in view of (2), that it will be exceedingly difficult to obtain any such exact result.

The Monomer-Dimer Problem

Even for planar graphs the general version of the monomer dimer problem is #P-complete. This was first shown by Jerrum (1981, 1987), who showed that counting the total number of matchings in a graph is #P-complete. Its apparent intractability goes someway towards explaining the paucity of exact results. As far as I am aware there have not been significant improvements for the 2-dimensional square lattice over the rather weak bounds given in Bondy and Welsh (1966) and Hammersley and Menon (1970).

We next turn to the remaining problems under discussion, namely percolation, Ising, Potts and the ice problem. It turns out that they can all be regarded as specific evaluations of a well known graph polynomial. We treat this briefly first.

The Dichromate or Whitney-Tutte Polynomial

A crucial concept in what follows is the following graph polynomial introduced by Tutte (1947). It is closely related to the rank generating function introduced by Whitney (1932) and has a natural extension to vector spaces and matroids. In this context it has interpretations as the weight enumerator of a linear code (see Welsh 1976) and has recently been observed to have considerable significance in the theory of knots, see for example Kauffman (1987) or Lickorish (1988).

However in this article we will restrict attention to graphs and then it can be fairly simply defined as follows.

Let G be a graph with edge set E. For any subset A of E we define the *rank* $r(A)$ by

$$r(A) = v(A) - k(A)$$

where $v(A)$ is the number of vertices of G incident with A and $k(A)$ is the number of (connected) components of the subgraph spanned by A. Then define the *Tutte polynomial* of G to be the 2-variable polynomial

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}. \tag{4.1}$$

Hence if I is an isthmus and L denotes a loop

$$T(I; x, y) = x, \ T(L; x, y) = y. \tag{4.2}$$

This, together with the recursion formulae

$$T(G; x, y) = T(G'_e; x, y) + T(G''_e; x, y) \tag{4.3}$$

whenever e is not an isthmus or a loop, effectively determines T uniquely. Here G'_e and G''_e are the graphs obtained from G by respectively deleting and contracting the edge e. When e is an isthmus or loop replacing (4.2) by

$$T(G; x, y) = \begin{cases} xT(G'_e; x, y) & e \text{ an isthmus} \\ yT(G'_e; x, y) & e \text{ a loop} \end{cases} \tag{4.4}$$

gives a complete recursion formula for calculating T for any graph G.

A more striking property of the Tutte polynomial is the following. A function f defined on the set of all graphs is an *invariant* if whenever G_1 and G_2 are isomorphic $f(G_1) = f(G_2)$. A special case of the main result of Oxley and Welsh (1979) is the following.

THEOREM 1. *Let f be a graph invariant taking values in a commutative ring R satisfying for some $a, b \in R$, the relation*

$$f(G) = af(G'_e) + bf(G''_e) \tag{4.5}$$

when e is not a loop or isthmus, and

$$f(G) = f(\{e\})f(G'_e)$$

when e is a loop or isthmus. Then f is given by

$$f(G) = a^{|E|-|V|+1} b^{|V|-1} T(G; x/b, y/a). \tag{4.6}$$

Using this theorem it is now easy to prove that all the remaining problems of statistical physics described in the last section can be reduced to evaluating the Tutte polynomial of the graph along particular curves in the x, y–plane. The proof technique is just to verify that the quantity in question, be it partition function, probability, or enumeration satisfies a recursive formula of the type (4.5) for suitable a and b.

This method gives the following interpretations of T.

Percolation

The percolation probability $P(G; p)$ is given by

$$P(G; p) = q^{|E|-|V|+1} p^{|V|-1} T(G; 1, q^{-1})$$

for any connected graph G and where $q = 1 - p$.

The Ising and Potts Models

In the absence of an external magnetic field the Ising model is the special case of the Potts model defined with $q = 2$. It is straightforward to use the recursion formula (4.5) to verify that the general partition function Z is given by an evaluation of the Tutte polynomial T along the hyperbola $(x-1)(y-1) = 2$. It is perhaps easier to see this if Z is reparameterised in the following form. Let $A(\sigma)$ denote the sets of edges of G which have both endpoints the same sign under σ. Let $B(\sigma)$ be the complementary set of edges, then the generalised partition function

$$Z(G; \theta, \phi) = \theta^{|E|-|V|+1} (\theta - \phi)^{|V|-1} T\left(G; \frac{\theta + \phi}{\theta - \phi}, \frac{\theta}{\phi}\right) \qquad (4.7)$$

where

$$\theta = e^{\beta}, \phi = e^{-\beta}, \beta = J/kT$$

where J is the interaction strength, T is temperature and k is Boltzmann's constant.

Again using the recursion formula (4.5) it is straightforward to check that for the q-state Potts model, the partition function is given by

$$Z(G; q, v) = q^n T\left(G; \frac{q + v}{v}, v + 1\right) \qquad (4.8)$$

where n is the number of vertices, and v is the parameter defined by $v + 1 = \exp(-1/kT)$.

In other words the partition function of the Potts model is, up to an easily determined constant, the Tutte polynomial of G evaluated along the hyperbola $H_q \equiv (x-1)(y-1) = q$.

The relation between the above models and the Tutte polynomial seems to have been first noticed by Fortuin and Kasteleyn (1972), though their proofs are different from the method indicated above and they were using an equivalent (up to a change of variable) polynomial due to H. Whitney (1932).

The Ice Model

Lenard (see Lieb 1967) showed that determining the partition function $Z_{\text{ice}}(G)$ for any planar 4-valent graph G was equivalent to counting the 3 colourings of the faces of G in such a way that no two faces with a common edge are given the same colour. But then by a standard result linking colourings and evaluations of the Tutte polynomial we have

$$Z_{\text{ice}}(G) = T(G; 0, -2) \tag{4.9}$$

and in fact it is easy to prove directly from the recursion formula (4.3) that (4.9) holds for all (not necessarily planar) 4-valent graphs.

It follows from the above observations that whenever the evaluation of the Tutte polynomial is 'easy' then so are each of the above problems. However, for general graphs determination of the Tutte polynomial or even evaluation at a particular point has been proved to be #P-hard except in very special cases

This follows from results of Jaeger, Vertigan, and Welsh (1989), a special case of which is the following theorem.

THEOREM 2. *Evaluating the Tutte polynomial of a graph at a particular point of the complex plane is* #P*-hard except when either*
(a) the point lies on the hyperbola $(x-1)(y-1) = 1$*, or*
(b) the point is one of the special points $(1, 1), (-1, 0), (0, -1), (-1, -1), (i, -i), (-i, i), (j, j^2), (j^2, j)$ *where* $j = e^{2\pi i/3}$.
In the special cases the evaluation can be carried out in polynomial time.

As far as the physical problems are concerned the special points and special hyperbola seem to have no significance. There are combinatorial interpretations of $T(G; x, y)$ at each of the points, the most interesting being at $(1, 1)$ where T counts the number of spanning trees of the graph for which there is the well known Kirchhoff determinantal formula.

5. Approximations, Monte Carlo Methods and Randomised Algorithms

One result of the evidence of intractability of most of these problems as propounded in the last section is that good approximation techniques assume even greater importance. Monte Carlo methods have long been a favoured approach to many of these; see for example Hammersley and Handscomb (1964). However, until very recently there has been very little known about the rate, or accuracy of convergence of what can loosely be described as probabilistic methods of approximation.

Very recently, however, there has been a notable breakthrough by Jerrum and Sinclair (1988) on one specific problem, and the methods used have the capability of extension to other #P-hard problems.

We now describe what we mean by a randomised approximation scheme. First, for any real numbers a, $\hat{a}$ and $r \geq 1$ we say that $\hat{a}$ *approximates a within ratio* r if

$$\hat{a}/r \leq a \leq \hat{a}r.$$

A *fully polynomial randomised approximation scheme*, abbreviated to fpras for a function $f : \Sigma^* \to \mathbb{N}$ is a randomised algorithm which when presented with a string $x \in \Sigma^*$ and a real number $\epsilon > 0$ runs in time which is polynomial in $|x|$ and ϵ^{-1} and with probability at least $\frac{1}{2} + \delta$ $(\delta > 0)$ its output approximates $f(x)$ within ratio $1 + \epsilon$. (By $|x|$ we mean the number of elements in or length of the string x, and as usual Σ^* is the set of strings of symbols from the finite alphabet Σ.)

It is not difficult to see that the existence of a fpras means the existence (in a precise mathematical sense) of a fast, good approximation algorithm, and what Jerrum and Sinclair have done is to show the existence of such an algorithm for determining the partition function of the monomer-dimer problem. We now sketch the ideas of their method.

Let $(X_t : 0 \leq t < \infty)$ be a finite state, ergodic, time homogeneous Markov chain M with transition matrix $P = (p_{ij})$ and having stationary distribution $\pi = (\pi_i)$.

The *relative pointwise distance* $\Delta(t)$ is defined by

$$\Delta(t) = \max_{i,j} \frac{|p_{ij}(t) - \pi_j|}{\pi_j}$$

where as usual $p_{ij}(t)$ represents the t–step transition probability. Thus $\Delta(t)$ is a measure of the rate of convergence of the Markov chain to its stationary distribution and the ideal is a situation where $\Delta(t)$ converges to zero exponentially fast as a function of time.

Now suppose that the ergodic chain M is time reversible so that it satisfies the balance condition $p_{ij}\pi_i = p_{ji}\pi_j$, $\forall i, j$. We associate with M an undirected weighted graph $G(M)$ in which the vertices are the states of M, the edges join all pairs of states with $p_{ij} > 0$ and the *weight* w_{ij} of the edge (i, j) is given by

$$w_{ij} = p_{ij}\pi_i = p_{ji}\pi_j.$$

A measure of the rate at which the Markov chain can move around its state space is the **conductance** Φ defined as follows. For any set A of vertices of $G(M)$ let ∂A denote the set of edges having exactly one endpoint in A and then let $\omega(\partial A)$ be the sum of the weights of these edges. Then

$$\Phi = \min\left(\omega(\partial A) \Big/ \sum_{i \in A} \pi_i\right)$$

where the minimum is taken over all sets A of states for which

$$0 < \sum_{i \in A} \pi_i \leq \tfrac{1}{2}.$$

The main result of Sinclair and Jerrum (1988) is the following.

THEOREM 3. *If M is a time reversible ergodic chain with $p_{ii} \geq \frac{1}{2}$ for each i, then the relative pointwise distance $\Delta(t)$ satisfies*

$$\Delta(t) \leq (1 - \Phi^2/2)^t / \pi_{\min}$$

where $\pi_{\min}$ is the minimum of the stationary state probabilities.

There are a few points to note about Theorem 3. First, the condition $p_{ii} \geq \frac{1}{2}$ though strange, is technical, and can be introduced into any chain by replacing P by $(I + P)/2$, this leaves the stationary distribution unchanged and reduces the conductance by a factor of $\frac{1}{2}$. This is immaterial in the sort of situations in which it is used, namely to prove:

THEOREM 4. *There exists a fully polynomial randomised approximation scheme for counting the number of weighted matchings in a graph.*

In other words the partition function of the monomer dimer problem can be approximated accurately, quickly.

The basic idea underlying the proof of Theorem 4 is to set up an appropriate Markov chain which can be proved to be rapidly mixing. For the monomer dimer (or counting matchings) problem on a graph G the Markov chain M will have as its states the matchings of G and transitions between states are carried out according to the following rules:

Let I be a particular matching or state of M and let e be any edge of G. The pair (I, e) determines a new state J by the formulae:

(a) $e \in I, J = I \backslash e$,
(b) if $e \notin I$ and $I \cup e$ is a matching then $J = I \cup e$,
(c) if $e \notin I$ but exactly one endpoint of e is covered by an edge e' of I then $J = (I \backslash e') \cup e$.

The Markov chain M is of the Metropolis type with transition probabilities determined by choosing edges of G at random and then adopting the change from I to J with probability $\frac{1}{2}$. When modified to make the self loop probabilities no smaller than $\frac{1}{2}$ as indicated, Jerrum and Sinclair show that its conductance Φ satisfies

$$\Phi \geq \frac{1}{8|E(G)|}$$

and hence using Theorem 1 the chain M is rapidly mixing.

The broad idea of the counting algorithm is given by the steps A1–3.

A1: Let the Markov chain M run for time long enough to generate a random sample of 'approximately random' members of the stationary distribution of matchings.

A2: Use the fact that for a specific edge e the partition function $Z(G)$ can be written as

$$Z(G) = Z(G^+) + Z(G^-)$$

where G^-, is the graph obtained from G by deleting $e = (u, v)$ and G^+ is the graph obtained from G by removing e, u, v and all edges incident with u, v.

A3: Let z^+, z^- be the number of members of the sample which contain the specific edge e and use these to estimate $Z(G^+)$, $Z(G^-)$. From these we can recursively estimate $Z(G)$.

The important point is that because the chain is rapidly mixing the method works in the sense that to obtain a final estimate which approximates $Z(G)$ within a ratio $1 + \epsilon$ with probability at least $\frac{3}{4}$ the sample size required is only $O(|E|^3\epsilon^{-2})$ where $E = E(G)$.

6. Conclusion

As far as I am aware all the exact results which have been proved for any of the physical problems have been for some of the 2-dimensional planar lattices. Accordingly one might suspect that it is planarity which makes things easier. However, from the viewpoint of complexity this cannot be the case, and further work extending Theorem 2 by Vertigan (1989) shows that except at a few very special points the Tutte polynomial of planar graphs is #P-hard to compute. Hence if one believes the thesis that exact results about #P-hard problems are in general almost impossible to obtain one is led to ask what additional properties of the 2-dimensional square lattices makes possible the exact results obtained for the Ising, ice, dimer and percolation problems on this particular lattice. It is doubtful if there is an easy answer to this problem. In this context it should be emphasized that calculating the asymptotic limit of a particular sequence of graph functions may be a much easier problem than the exact evaluation problem.

We close with the following:

Problem: Is there any way of extending the Jerrum-Sinclair randomised approximation approach to any of the other physical problems?

As far as I am aware the only other problems to which the method has so far been applied successfully is to estimating the volume of convex bodies (Dyer, Frieze, and Kannan 1988) and generating random graphs (Jerrum and Sinclair 1988b). Ideally we would like to be able to prove that the Metropolis type Monte Carlo methods developed in Hammersley and Handscomb (1964) for example are based on rapidly mixing Markov chains. This may be the case but proving it could be very difficult.

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