

# RANDOM WALKS ON INFINITE GRAPHS AND GROUPS — A SURVEY ON SELECTED TOPICS

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## 1. Introduction

Random walks on graphs, finite or infinite, have attracted considerable interest in both pure and applied mathematics. The body of literature ranges from topics such as the type problem of Riemannian manifolds (see, for example, Kanai [118]) to modelling phenomena such as the spread of cancer (Blumenson [21]). For random walks on integer lattices, an important body of work is due to Chung, Spitzer, Kesten and others, culminating in the comprehensive and influential monograph by Spitzer [205] (first edition in 1964). In spite of this, a lot of the later work has been done by isolated researchers or small research groups in various countries, without a permanent exchange or the forming of bigger 'schools'. On the other hand, the beautiful little book by Doyle and Snell [63] has undoubtedly contributed to (re-)popularizing the subject, and one can observe increasing research activities concerning random walks in the last few years.

The purpose of this paper is to review a few selected topics concerning random walks on infinite graphs and (as a special case) groups. The definition of a random walk adopted here is that of a time-homogeneous Markov chain whose transition probabilities are adapted in some way (which has to be specified more precisely) to a graph structure of the underlying discrete state space. Of course, a graph can be associated with any time-homogeneous Markov chain on a denumerable state space, so that one could say that this notion of random walks coincides with arbitrary Markov chains. However, our point of view is reversed: what we have in mind is to start with a graph (or some other combinatorial, geometric or algebraic structure) and study the interplay between properties of random walks on this object on the one hand and the underlying structure itself on the other.

Besides the probabilistic aspect, the material presented here also contains a lot of (discrete) potential theory. The emphasis is on concrete quantitative and qualitative results, not the abstract theory. In this context we mention the beautiful set of lectures by T. Lyons [154], which discusses (concrete) potential theory, mostly in the continuous setting, in the context of its interplay with probability, geometry and other fields.

Good general references for the theory of Markov chains are the books by Kemeny, Snell and Knapp [126] (first edition in 1966) and by Revuz [187]. In the present survey we do not go into details concerning the role of random walks in mathematical physics or other branches of applied mathematics; see the book by Barber and Ninham [15] (which—besides Spitzer [205]—also is an important source concerning random walks in Euclidean setting, discrete or nondiscrete), the long surveys by Hughes and Prager [106] and by Weiss and Rubin [236] and the bibliography by Liyange, Gulati and Hill [148]. Also, here we do not deal with topics such as intersections of random walks, selfavoiding random walks, relations with percolation theory or random walks on random graphs: see, for example, Lawler [142], Alm and Janson [3], R. Lyons [150] and Grimmett [96] respectively, to obtain a flavour of results and further references in these directions. Random walks on topological groups will be discussed only marginally in direct connection with graphs. See Guivarc'h, Keane and Roynette [102] for the 'state of the art' in 1977; of course, there is also a huge body of more recent work on random walks on topological (mostly Lie) groups, to a large extent by French probabilists.

Besides those listed above, many other topics would merit inclusion here. For example, the original intention to include a paragraph on particular aspects of

random walks on integer lattices (besides those covered by Spitzer [205] or by Barber and Ninham [15]) was abandoned in view of the length of the material accumulated already. To give an idea, we mention here a few references, chosen ‘at random’: Schinzel [194], Kesten [132], Erdős and Révész [69, 70].

We point out that the present survey (in particular Section 3) is related with the one by Mohar and Woess [162] concerning spectral theory of infinite graphs, and that the latter may serve as a complement to this survey. In addition, a long bibliography concerning random walks on *finite* graphs is available from D. Aldous [1]. Regarding the large body of more recent work on random walks on finite graphs, Diaconis and Stroock [56] may serve as a good source of information.

The author does not claim (nor intend) that the references given here are complete in every respect. In particular, it is impossible to draw a precise borderline between random walks and general Markov chain theory. Also, the author apologizes for the fact that almost certainly various references will have escaped his attention and/or memory. Several results and methods have been improved at various times—the citations will not always give the complete chain of respective papers, but sometimes only the most significant steps or even only the last one.

This paper grew out of the manuscript for a survey talk given at the 1990 Oberwolfach meeting on ‘Probability measures on groups’ (this is the reference Woess [246], which is a first version of Section 7 below). The author wishes to express his gratitude to Professor Herbert Heyer for this opportunity. Also, the author would like to acknowledge helpful comments of D. I. Cartwright, Th. Coulhon, P. de la Harpe, V. A. Kaimanovich, R. Lyons, T. Lyons, Y. Peres, P. Révész and N. Th. Varopoulos on the preliminary version of this survey.

## 2. Basic definitions and preliminaries

Throughout this paper,  $X$  denotes (the vertex set of) an *infinite graph*; we consider the nonoriented edge set as a symmetric subset of  $X \times X$  and write  $x \sim y$  if  $x$  and  $y$  are neighbours. The *degree* of  $x \in X$ , denoted by  $\deg(x)$ , is the number of neighbours of  $x$ . A *path of length  $n$*  from  $x$  to  $y$  is a sequence  $x = x_0, x_1, \dots, x_n = y$  of distinct vertices such that  $x_{i-1} \sim x_i$ . Unless stated otherwise, we *always assume* that  $X$  is *infinite, locally finite* ( $\deg(x) < \infty$  for all  $x \in X$ ) and *connected* (for every  $x, y \in X$  there is a finite path from  $x$  to  $y$ ). The *distance*  $d(x, y)$  between vertices  $x$  and  $y$  is the minimal length of a path connecting the two.

A *random walk* on  $X$  is given by a stochastic transition matrix

$$P = (p(x, y))_{x, y \in X}$$

which describes the one-step transitions of a Markov chain  $Z_n$ ,  $n = 0, 1, 2, \dots$ , with state space  $X$ . To model  $Z_n$ , consider the *trajectory space*  $\Xi = X^{\mathbb{N}_0}$ , equipped with the usual product sigma-algebra arising from the discrete one on  $X$ . Then  $Z_n$  is the  $n$ th projection  $\Xi \rightarrow X$ . This describes the random walk starting at  $x \in X$ , if  $\Xi$  is equipped with the probability measure  $\Pr_x$  given by

$$\Pr_x[Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n] = \delta_x(x_0) p(x_0, x_1) \dots p(x_{n-1}, x_n).$$

We denote

$$p^{(n)} = (x, y) = \Pr_x[Z_n = y].$$

This is the  $(x, y)$ -entry of the matrix power  $P^n$ , with  $P^0 = I$ , the identity matrix over  $X$ .

We shall refer to the random walk alternatively by writing  $(Z_n)$  or  $(X, P)$ . We shall always assume that all states communicate.

**BASIC ASSUMPTION.**  $(X, P)$  is irreducible, that is, for every  $x, y \in X$  there is some  $n \in \mathbb{N}$  such that  $p^{(n)}(x, y) > 0$ .

### A. Adaptedness to the graph structure

Irreducibility alone is, of course, not enough to say that the random walk is well adapted to the underlying graph structure, and so we now present a list of conditions and examples which will serve to meet this requirement in some form.

(NN) The random walk is of *nearest neighbour* type if

$$p(x, y) > 0 \text{ implies } y \sim x.$$

An example is the *simple random walk* (SRW) on  $X$ , given by

$$p(x, y) = \begin{cases} 1/\deg(x), & \text{if } y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

(UI) The random walk is called *uniformly irreducible* if there are  $\varepsilon_0 > 0$  and  $K < \infty$  such that

$$x \sim y \text{ implies } p^{(k)}(x, y) > \varepsilon_0 \text{ for some } k \leq K.$$

Note that (UI) implies that  $\deg(x) \leq K/\varepsilon$  for every  $x \in X$ .

(FR) The random walk has *finite range* if

$$\{y \mid p(x, y) > 0\} \text{ is finite for every } x \in X.$$

(BR) The random walk has *bounded range* if

$$\sup \{d(x, y) \mid x, y \in X, p(x, y) > 0\} < \infty.$$

The *distance* (or *step length*) *distribution* at  $x \in X$  is the probability on  $\mathbb{N}_0$  given by

$$\sigma_x(n) = \Pr_x[d(Z_1, Z_0) = n] = \sum_{y: d(y, x) = n} p(x, y).$$

(TD) The random walk has *tight distance distributions* if

$$\sup_{x \in X} \sigma_x([n, \infty)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(UM) The random walk satisfies a *uniform first moment condition* if

$$\bar{\sigma} = \bar{\sigma}(X, P) = \sum_{n=1}^{\infty} \sup_{x \in X} \sigma_x(n) < \infty.$$

### B. Reversible Markov chains

(R) The random walk is called *reversible* if there is a function  $c: X \rightarrow (0, \infty)$  such that

$$c(x)p(x, y) = c(y)p(y, x) \text{ for all } x, y \in X.$$

In this case,  $c(x, y) = c(x)p(x, y) = c(y, x)$  is called the *conductance* between  $x$  and  $y$ , and  $c(x)$  is the *total conductance* at  $x$ . Conversely, if  $c: X \times X \rightarrow [0, \infty)$  is a symmetric

function such that  $c(x) = \sum_y c(x, y)$  is positive and finite for every  $x$ , then  $p(x, y) = c(x, y)/c(x)$  defines a reversible Markov chain (random walk). In general, we shall deal with reversible random walks which are not necessarily nearest neighbour. However, even in this case it will be convenient to equip (the vertex set of)  $X$  with a—possibly new—edge set  $E_p$  such that for the resulting new graph,  $P$  is of nearest neighbour type:  $[x, y] \in E_p$  if and only if  $c(x, y) > 0$ . Note that  $(X, E_p)$  is not necessarily locally finite. In addition, for each—a priori unoriented—edge  $e \in E_p$  we specify an orientation and call  $e^-$  its initial vertex and  $e^+$  its terminal vertex. (None of the results described in the following sections depends on the particular orientation chosen.) The *resistance* of  $e \in E_p$  is  $r(e) = 1/c(e^-, e^+)$ . The triple  $\mathcal{N} = (X, E_p, r)$  is called an (*electrical*) *network*.

In particular, the SRW arises from the network where  $c(x, y) = 1$  if  $x \sim y$  (and  $c(x, y) = 0$  otherwise). We shall call this the *simple network*.

(SR) A reversible random walk  $(X, P)$  is called *strongly reversible* if there is a constant  $M \in (0, \infty)$  such that

$$M^{-1} \leq c(x) \leq M \quad \text{for all } x \in X.$$

The SRW on  $X$  is strongly reversible if and only if there is a finite upper bound on the vertex degrees in  $X$ .

### C. Random walks on groups

We now explain how random walks on groups fit into our graph theoretical setting. Let  $\Gamma$  be a finitely generated, discrete group. We shall denote the unit element by  $o$ . (The letter  $e$  is already in use for edges in networks. For general graphs,  $o$  will denote a fixed reference vertex; for groups we always use the identity when we need a reference point.) Choose a finite symmetric set  $A$  of generators of  $\Gamma$ . The *Cayley graph* of  $\Gamma$  with respect to  $A$  has vertex set  $X = \Gamma$ , and  $x \sim y$  if and only if  $x^{-1}y \in A$ . Denote by  $d_A$  the resulting graph metric (often called *word metric*). If  $B$  is another finite symmetric set of generators, then  $d_A$  and  $d_B$  are equivalent: there is a positive constant  $L$  such that

$$L^{-1}d_A \leq d_B \leq Ld_A.$$

If  $\mu$  is a probability measure on  $\Gamma$ , then it gives rise to the *random walk on  $\Gamma$  with law  $\mu$* :

$$p(x, y) = \mu(x^{-1}y), \quad \text{and} \quad p^{(n)}(x, y) = \mu^{(n)}(x^{-1}y),$$

where  $\mu^{(n)}$  denotes the  $n$ th convolution power of  $\mu$ . Irreducibility is now equivalent to the property that the support of  $\mu$  generates  $\Gamma$  as a semigroup:

$$\bigcup_{n=1}^{\infty} (\text{supp } \mu)^n = \Gamma.$$

In particular, (UI) and (TD) hold automatically, (FR) and (BR) are equivalent with  $\text{supp } \mu$  being finite, and (UM) means that  $\mu$  has finite first moment with respect to  $d_A$ , that is,  $\sum_x d_A(o, x) \mu(x) < \infty$  for some ( $\Leftrightarrow$  every) finite symmetric set  $A$  of generators. The random walk with law  $\mu$  is reversible if and only if there is an *exponential*  $\psi$  on  $\Gamma$  (that is, a homomorphism from  $\Gamma$  into the multiplicative group of positive reals) such that  $x \mapsto \psi(x) \mu(x)$  is symmetric. In this case,  $c(x) = 1/\psi(x)^2$ . (See, for example, Woess [240] for this easy exercise.) If  $\mu$  is symmetric, then we also say that the corresponding random walk is symmetric. The SRW on the Cayley graph of  $\Gamma$  with respect to  $A$  is the one whose law is  $\mu = \chi_A/|A|$ .

The theory of random walks on groups was initiated by Kesten [128]. For random walks on integer lattices, the book by Spitzer [205] is a detailed source. Various aspects of the theory of random walks on (infinite) groups are presented and completed in the influential paper by Kaimanovich and Vershik [115] which also contains a considerable number of references up to 1983.

#### D. Group-invariant random walks on graphs

Besides random walks on groups, one can study random walks with more general group-invariance properties. The *automorphism* (or *isometry*) *group* of a graph  $X$  is the group  $\text{Aut}(X)$  of self-isometries of  $(X, d)$ . With respect to pointwise convergence, it is a locally compact, totally disconnected Hausdorff group; see Trofimov [220] or Woess [245] for more details. A subgroup  $\Gamma \leq \text{Aut}(X)$  is called *transitive* if it acts transitively on (the vertex set of)  $X$ , and *quasi-transitive* if it acts with finitely many orbits. (Note that such a group may be quite far from being discrete and finitely generated.) If such a group exists, then also the graph is called *transitive* or *quasi-transitive*, respectively.

(GP) If  $\Gamma \leq \text{Aut}(X)$ , then we say that  $P$  is  $\Gamma$ -invariant if

$$p(\gamma x, \gamma y) = p(x, y) \quad \text{for all } x, y \in X \text{ and } \gamma \in \Gamma.$$

(GC) If  $\Gamma \leq \text{Aut}(X)$ , then we say that a reversible  $P$  has  $\Gamma$ -invariant conductances if

$$c(\gamma x, \gamma y) = c(x, y) \quad \text{for all } x, y \in X \text{ and } \gamma \in \Gamma.$$

Note that (GC) implies (GP), but (even in the reversible case) not vice versa.

#### E. Harmonic and superharmonic functions

To conclude this section, we introduce the spaces of harmonic and superharmonic functions. The transition matrix acts on functions  $f: X \rightarrow \mathbb{R}$  by

$$Pf(x) = \sum_{y \in X} p(x, y)f(y),$$

whenever this sum converges for every  $x$ . (This is, of course, the case for every  $f$  when (FR) holds.) A function  $f$  is called *superharmonic* if  $Pf \leq f$  and *harmonic* if  $Pf = f$ . We define the spaces

$$\mathcal{S} = \mathcal{S}(X, P) = \{f \mid Pf \leq f\}, \quad \mathcal{H} = \mathcal{H}(X, P) = \{h \mid Ph = h\},$$

the cones  $\mathcal{S}^+$  and  $\mathcal{H}^+$  of the respective nonnegative functions, and the linear space  $\mathcal{H}^\infty$  of *bounded harmonic functions*. Analogously, for  $t \in \mathbb{R}$  we consider *t-superharmonic* and *t-harmonic* functions:

$$\mathcal{S}_t = \mathcal{S}_t(X, P) = \{f \mid Pf \leq t \cdot f\} \quad \text{and} \quad \mathcal{H}_t = \mathcal{H}_t(X, P) = \{h \mid Ph = t \cdot h\}.$$

### 3. Spectral radius, amenability and law of large numbers

#### A. Spectral radius, isoperimetric inequalities and growth

The *Green kernel* is defined by

$$G(x, y \mid z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n, \quad x, y \in X, z \in \mathbb{C}.$$

In addition, define the stopping time

$$s^y = \inf \{n \geq 0 \mid Z_n = y\},$$

and set

$$F(x, y \mid z) = \sum_{n=0}^{\infty} \Pr_x [s^y = n] z^n.$$

Then, in the common domain of convergence of these power series,

$$G(x, y \mid z) = F(x, y \mid z) G(y, y \mid z).$$

If  $z = 1$ , then we omit the variable  $z$  and just write  $G(x, y)$  and  $F(x, y)$  for the resulting kernels on  $X \times X$ . Thus  $G(x, y)$  is the expected number (possibly infinite) of visits to  $y$  of  $Z_n$  starting from  $x$ , and  $F(x, y)$  is the probability of ever reaching  $y$  when starting in  $x$ .

The radius of convergence of the power series  $G(x, y \mid z)$  is  $1/\rho$ , where

$$\rho = \rho(P) = \limsup_{n \rightarrow \infty} p^{(n)}(x, y)^{1/n},$$

which is independent of  $x$  and  $y$  by irreducibility. Usually,  $\rho(P)$  is called the *spectral radius* of  $P$ . (However, this terminology may be misleading in the non-reversible case; Vere-Jones [232] uses *convergence norm* instead.) When it arises from a random walk with law  $\mu$  on a group, then we write  $\rho(\mu)$  instead of  $\rho(P)$ . The following useful characterization of  $\rho(P)$  is due to Pruitt [186].

**THEOREM 3.1.** *For irreducible  $(X, P)$ ,  $\rho(P)$  is the minimum among all  $t > 0$  for which  $\mathcal{S}_t^+$  is nonempty. If, in addition,  $(X, P)$  has finite range, then also  $\mathcal{H}_{\rho(P)}^+$  is nonempty.*

It is obvious that  $0 < \rho \leq 1$ . The property that  $\rho = 1$  (or  $\rho < 1$ , respectively) is related to the underlying structure. For groups, the following has been proved by Kesten [128, 129] and by Day [49].

**THEOREM 3.2.** *For an (irreducible) random walk on a group  $\Gamma$  with law  $\mu$ , one has:*

- (1) *if  $\rho(\mu) = 1$ , then  $\Gamma$  is amenable;*
- (2) *if  $\Gamma$  is amenable, then  $\rho(\mu) = 1$  for every symmetric  $\mu$ .*

For a subgraph  $U$  of  $X$ , let  $\partial U$  be the set of edges having precisely one endpoint in  $U$ , and  $\text{Vol}(U) = \sum_v \deg(x)$ .

(IS) We say that  $X$  satisfies a *strong isoperimetric inequality* if  $\iota(X) > 0$ , where

$$\iota(X) = \inf \left\{ \frac{|\partial U|}{\text{Vol}(U)} \mid U \text{ a finite subgraph of } X \right\}.$$

By Følner's criterion, a finitely generated group is nonamenable if and only if its Cayley graph(s) satisfy (IS). For all questions related with amenability, a good reference is, for example, Pier [184]. The above theorem generalizes to arbitrary graphs as follows; see Kaimanovich [112].

**THEOREM 3.3.** (1) *If  $(X, P)$  satisfies (SR) and (UI), then  $\rho(P) = 1$  implies  $\iota(X) = 0$ .*

(2) *If  $(X, P)$  satisfies (SR), (UI) and (TD), then  $\rho(P) = 1$  if and only if  $\iota(X) = 0$ .*

For a list of conditions which are equivalent with  $\rho(P) < 1$ , see the paper by Gerl [88] and its significant generalizations by Kaimanovich [112]. Let  $(X, P)$  be reversible and  $\mathcal{N}$  be the associated network. For  $U \subset X$ , define with respect to  $\mathcal{N}$

$$\text{Vol}(U) = \sum_{x \in U} c(x) \quad \text{and} \quad \text{Surface}(U) = \sum_{x \in U} \sum_{y \in X \setminus U} c(x, y).$$

(IS( $\mathcal{N}$ )) We say that  $\mathcal{N}$  satisfies a strong isoperimetric inequality if  $\iota(\mathcal{N}) > 0$ , where

$$\iota(\mathcal{N}) = \inf \left\{ \frac{\text{Surface}(U)}{\text{Vol}(U)} \mid U \subset X, \text{Vol}(U) \text{ finite} \right\}.$$

One of the generalizations due to Kaimanovich [112] is the following.

**THEOREM 3.4.** *Let  $(X, P)$  be reversible with associated network  $\mathcal{N}$ . Then  $\iota(\mathcal{N}) = 0$  if and only if  $\rho(P) = 1$ .*

In particular,  $\iota(X)$  is the isoperimetric number of the network associated with the SRW:  $\iota(X) = 0$  if and only if  $\rho(P) = 1$  for the SRW, and this holds for locally finite  $X$  even without assuming bounded vertex degrees (this has been proved before by Dodziuk and Kendall [58]). For further references, see Mohar and Woess [162].

We also mention the following extensions of Theorem 3.2 to group-invariant random walks on graphs; see Soardi and Woess [202] and Salvatori [190].

**THEOREM 3.5.** *Let  $X$  be quasi-transitive. Then the following statements are equivalent.*

- (1)  $\iota(X) = 0$ .
- (2) *Some ( $\Leftrightarrow$  every) quasi-transitive, closed subgroup of  $\text{Aut}(X)$  is amenable and unimodular.*
- (3)  $\rho(P) = 1$  for some ( $\Leftrightarrow$  every) reversible random walk on  $X$  whose conductances are invariant under a quasi-transitive subgroup of  $\text{Aut}(X)$ .

For related results, see also de la Peña and Takane [175] and Sy and Sunada [210].

In particular, one can use Theorem 3.5 to prove that every quasi-transitive graph with more than two ( $\Leftrightarrow$  infinitely many) ends satisfies (IS); see Soardi and Woess [202] and, for a different proof, Mohar [161]. (For the definition of ends, see Section 7 below.) For examples of graphs which are not necessarily quasi-transitive and satisfy (IS), see, for example, Dodziuk [57], Gerl [86], Ancona [4], Soardi [200] and Calogero [23].

Let  $X$  be a vertex-transitive graph with degree  $k \geq 3$  having a simple cycle (minimal closed path) of length  $\ell \geq 3$ . Paschke [174] shows that there is a universal one,  $\tilde{X} = \tilde{X}_{k, \ell}$ , among these graphs, which covers every such  $X$ . It is constructed explicitly as the Cayley graph (with respect to the ‘natural’ generators) of the free product of  $\mathbb{Z}_\ell$  with  $k-2$  copies of  $\mathbb{Z}_2$ . (See Section 6.D for a closer description of free products and their Cayley graphs.) In particular, for the SRW  $P$  on  $X$  and  $\tilde{P}$  on  $\tilde{X}$ , one obtains  $\rho(P) \geq \rho(\tilde{P})$ . The latter number is calculated explicitly by Paschke [174]; it is strictly larger than  $2\sqrt{(k-1)/k}$ .

**COROLLARY 3.6.** *For the SRW on a transitive graph  $X$  with degree  $k \geq 3$ , one has  $\rho(P) = 2\sqrt{(k-1)/k}$  if and only if  $X$  is a tree.*

This generalizes an analogous result of Kesten [128] for groups.

For use in Sections 3–6, we introduce the following refinement of condition (IS).

(IS<sub>q</sub>) We say that  $X$  satisfies a  $q$ -dimensional isoperimetric inequality ( $q \geq 1$ , real) if  $\iota_q(X) > 0$ , where

$$\iota_q(X) = \inf \left\{ \frac{|\partial U|^q}{\text{Vol}(U)^{q-1}} \mid U \text{ a finite subgraph of } X \right\}.$$

In many typical cases,  $q$  is an integer. For example, the integer lattice  $\mathbb{Z}^q$  (that is, its Cayley graphs when viewed as an Abelian group) satisfies (IS<sub>q</sub>) but does not satisfy (IS<sub>q+ε</sub>) for any  $\varepsilon > 0$ . Note that (IS)  $\Rightarrow$  (IS<sub>q</sub>)  $\Rightarrow$  (IS<sub>q'</sub>) if  $q \geq q' \geq 1$ . Finally, the growth sequence of  $X$  at vertex  $x$  is  $b(x, n) = |B(x, n)|$ , where  $B(x, n) = \{y \in X \mid d(y, x) \leq n\}$ .

(EG) We say that  $X$  has *exponential growth* if

$$\liminf_{n \rightarrow \infty} b(x, n)^{1/n} > 1.$$

Conversely, if

$$\limsup_{n \rightarrow \infty} b(x, n)^{1/n} = 1,$$

then  $X$  is said to grow *subexponentially*.

(PG<sub>q</sub>) We say that  $X$  has *polynomial growth with degree at most  $q$*  ( $q \geq 1$ , real), if

$$\limsup_{n \rightarrow \infty} b(x, n)/n^q < \infty.$$

If  $q$  cannot be chosen smaller in (PG<sub>q</sub>) then we omit the ‘at most’. Once more,  $\mathbb{Z}^q$  ( $q$  integer) satisfies (PG<sub>q</sub>). It is not hard to prove that (IS<sub>q</sub>) implies  $b(x, n) \geq Cn^q$  for a constant  $C > 0$  independent of  $x$ . Analogously, (IS) implies  $b(x, n) \geq C\lambda^n$  for constants  $C > 0$  and  $\lambda > 1$  independent of  $x$ . For more details and references, see, for example, Wagon [235] or Mohar and Woess [162].

## B. Law of large numbers

The classical law of large numbers applies, in particular, to random walks on  $\mathbb{Z}^q$  whose law has finite first moment. This generalizes to random walks on groups as follows.

**THEOREM 3.7.** *Let  $\mu$  be a probability measure on the finitely generated group  $\Gamma$ , having finite first moment with respect to the Cayley graph metric  $d = d_A$ .*

(1) *There is a constant  $\alpha = \alpha(\mu, A)$  such that*

$$\lim_{n \rightarrow \infty} \frac{d(Z_n, o)}{n} = \alpha \quad \text{Pr}_o\text{-almost surely.}$$

(2) *If  $\alpha = 0$ , then  $\rho(\mu) = 1$ , and  $\Gamma$  is amenable.*

(3) *If  $\mu$  is symmetric and finitely supported, then  $\alpha = 0$  if and only if all bounded harmonic functions of the random walk are constant.*

(1) is a consequence of Kingman’s [136] subadditive ergodic theorem; see Guivarc’h [100]. For (2), see Kaimanovich and Vershik [114, 115] and Guivarc’h

[100]. (3) relates the law of large numbers with the Poisson boundary (see Section 7.B below), and arises as a corollary of a ‘long range estimate’ of Varopoulos [224], simplified by Carne [24].

**THEOREM 3.8.** *Assume that  $(X, P)$  is reversible and  $(NN)$ . Then*

$$p^{(n)}(x, y) \leq 2 \sqrt{\left(\frac{c(x)}{c(y)}\right)} \exp\left(-\frac{d(x, y)^2}{2n}\right)$$

for all  $x, y \in X$ ,  $n \geq 1$ .

If  $\alpha = 0$  in Theorem 3.7, then one would like to have further information on the rate of escape of the random walk. In the classical Euclidean case, answers to this question are given by the law of the iterated logarithm on the one hand, and (for SRW on  $\mathbb{Z}^q$ ,  $q \geq 3$ ) by a result of Dvoretzky and Erdős [66] on the other. Both results have been generalized to groups with polynomial growth in a recent paper by Hebisch and Saloff-Coste [104].

**THEOREM 3.9.** *Let  $\Gamma$  be a finitely generated group with polynomial growth with degree  $q$ , equipped with a Cayley graph metric  $d$ . Let  $\mu$  be a finitely supported, symmetric probability measure on  $\Gamma$  satisfying  $\mu(o) > 0$ . Then for the associated random walk, one has the following.*

(1) *There is a constant  $C > 0$  such that*

$$\frac{1}{C} \leq \limsup_{n \rightarrow \infty} \frac{d(Z_n, o)}{\sqrt{(n \log \log n)}} \leq C \quad \text{Pr}_o\text{-almost surely.}$$

(2) *If  $q > 2$ , then for any decreasing function  $\psi$  on  $\mathbb{N}$  one has*

$$\liminf_{n \rightarrow \infty} \frac{d(Z_n, o)}{\psi(n) \sqrt{n}} = \begin{cases} \infty & \text{Pr}_o\text{-a.s. according to whether } \sum_n \psi(2^n)^{q-2} < \infty \\ 0 & = \infty \end{cases}$$

Note that a group  $\Gamma$  which has polynomial growth with degree  $q$  also satisfies  $b(x, n) \geq C'n^q$  (with  $C' > 0$ ), and  $q$  is integer; see Gromov [97].

At this point we also mention the work of Telcs [212, 213, 214], who relates the exit time from  $B(x, n)$  with the eigenvalues of two types of ‘Laplacians’ associated with the  $n$ -ball in an infinite graph.

Sawyer and Steger [193] study in detail the law of large numbers, the central limit theorem and the law of the iterated logarithm for random walks on homogeneous trees, viewed as Cayley graphs of free products of two-element groups (the same approach works for free groups). In the place of  $d_n(Z_n, o)$ , their results also apply to more general functions of  $(Z_n)$  such as, for example, the frequency of terminal strings in the word representation of elements of the tree.

On general graphs, results are not as sharp as in Theorem 3.7. The following is essentially contained in §4 of Kaimanovich and Woess [116].

**PROPOSITION 3.10.** (a) *If  $(X, P)$  satisfies  $(UM)$ , then*

$$\limsup_{n \rightarrow \infty} \frac{d(Z_n, x)}{n} \leq \bar{\sigma}(X, P) \quad \text{Pr}_x\text{-almost surely for every } x.$$

(b) If  $(X, P)$  satisfies (UI) and  $\rho(P) < 1$ , then there is a constant  $\underline{\sigma} > 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{d(Z_n, x)}{n} \geq \underline{\sigma} \quad \Pr_x\text{-almost surely for every } x.$$

(For proving (b), first observe that (UI) implies existence of  $A \geq 1$  such that  $p^{(n)}(x, y) \leq A^{d(x, y)} \rho(P)^n$ , then use the Borel–Cantelli Lemma.)

#### 4. The type problem

The random walk  $(X, P)$  is said to be *recurrent* if

$$\Pr_x[\exists n > 0: Z_n = x] = 1 \quad \text{for some } (\Leftrightarrow \text{all}) x \in X,$$

or, equivalently,

$$G(x, y) = \infty \quad \text{for some } (\Leftrightarrow \text{all}) x, y \in X.$$

If this does not hold, then  $(X, P)$  is said to be *transient*.

In the recurrent case, with probability one the random walk starting at  $x$  returns to  $x$  infinitely often; in the transient case, this event has probability zero. The type problem consists in deciding whether a given random walk is recurrent or transient. In particular, if the SRW is recurrent/transient, then one says that the graph  $X$  has the respective property.

Recall the following well-known recurrence criterion for an arbitrary irreducible Markov chain.

**THEOREM 4.1.**  *$(X, P)$  is recurrent if and only if every function in  $\mathcal{S}^+(X, P)$  is constant.*

##### A. The type problem for random walks on groups

The following is not difficult to prove; see Theorem 4.8 below. (Recall that all our random walks are irreducible).

**PROPOSITION 4.2.** *Let  $\Gamma$  be a finitely generated group. If some symmetric random walk on  $\Gamma$  is recurrent, then this is true for every symmetric random walk on  $\Gamma$  whose law has finite support.*

Indeed, for the conclusion it suffices to have a not necessarily symmetric random walk which is recurrent; see Baldi, Lohué and Peyrière [14] and Varopoulos [221], and compare with Theorem 4.16 below. Furthermore, for symmetric random walks, ‘finite support’ may be replaced by ‘finite second moment’ (with respect to the Cayley graph metric); see, for example, Mess [160, Theorem 10] or Ancona [5].

Thus it is justified to call a finitely generated group  $\Gamma$  *recurrent*, if this is true for some ( $\Leftrightarrow$  every) finite range symmetric random walk on  $\Gamma$ . Otherwise,  $\Gamma$  is called *transient*.

It seems likely that almost all of the work done so far concerning random walks on infinite graphs has its roots in some way or another in the influential paper by Pólya [185], who proved the following.

**THEOREM 4.3.** *The simple random walk on  $\Gamma = \mathbb{Z}^q$  with respect to the natural generators (that is, the unit vectors and their inverses) is recurrent for  $q = 1, 2$  and transient for  $q \geq 3$ .*

For a probability measure  $\mu$  on  $\mathbb{Z}^q$ , its (absolute) moment of order  $k$  is

$$|\mu|_k = \sum_x |x|^k \mu(x),$$

where  $|x|$  is the Euclidean length of  $x \in \mathbb{Z}^q$ . If  $|\mu|_1$  is finite, then the mean displacement (drift) of  $\mu$  is the vector

$$\bar{\mu} = \sum_x x \mu(x).$$

The following result gives a larger class of recurrent random walks on  $\mathbb{Z}^q$ ,  $q = 1, 2$ ; see Chung and Fuchs [43], Chung and Ornstein [44] and Spitzer [205, T8.1].

**THEOREM 4.4.** *Let  $\mu$  be a probability measure on  $\mathbb{Z}^q$ ,  $q \leq 2$ , such that  $|\mu|_q < \infty$  and  $\bar{\mu} = 0$ . Then the random walk with law  $\mu$  is recurrent.*

The following example, taken from Spitzer [205, p. 87], shows that a symmetric random walk on  $\mathbb{Z}$  may well be transient.

**EXAMPLE 4.5.** Let  $\mu$  be a symmetric probability measure on  $\mathbb{Z}$ , satisfying

$$0 < \lim_{n \rightarrow \infty} n^\alpha \mu(n) < \infty$$

for some  $\alpha > 1$ . Then the random walk with law  $\mu$  is recurrent if  $\alpha \geq 2$  and transient if  $\alpha < 2$ .

This, as well as the two-dimensional case in Theorem 4.4, is an application of the recurrence criterion of Kesten and Spitzer [134]: the random walk with law  $\mu$  on the Abelian group  $\Gamma$  is recurrent if and only if

$$\int_{\hat{\Gamma}} \operatorname{Re} \frac{1}{1 - \hat{\mu}(\xi)} d\xi = \infty,$$

where  $\hat{\Gamma}$  is the dual of  $\Gamma$ ,  $\hat{\mu}$  the Fourier transform of  $\mu$  and  $d\xi$  the Haar measure on  $\hat{\Gamma}$ .

Now let  $\Gamma$  be a group which contains  $\mathbb{Z}^q$  as a subgroup of finite index. Then it is not hard to deduce from Theorem 4.3 that  $\Gamma$  is recurrent if and only if  $q \leq 2$ ; see also Dudley [64]. The conjecture that a finitely generated group is recurrent if and only if it has polynomial growth with degree at most 2 is usually attributed to Kesten [131] (but not stated there explicitly). By Gromov's classification of groups with polynomial growth [97], this means that the group must have  $\mathbb{Z}$  or  $\mathbb{Z}^2$  as a subgroup of finite index (or be finite itself). It is worthwhile to remark that the analogue of Kesten's conjecture was first proved for random walks on connected Lie groups; see Baldi [13]. For finitely generated (discrete) groups, the problem has been settled in the 1980s in a series of remarkable contributions by Varopoulos, culminating in his note [226], which contains the positive answer to Kesten's conjecture.

**THEOREM 4.6.** *A finitely generated group is recurrent if and only if it grows polynomially with degree at most two.*

For a nice exposition of Varopoulos' proof, see Ancona [5]: if the given group  $\Gamma$  has growth function

$$b(n) = b(x, n) \geq C \cdot n^{2+\varepsilon} \quad (C, \varepsilon > 0),$$

then a symmetric probability measure  $\mu$  is constructed such that

$$\text{supp } \mu = \Gamma, \quad |\mu|_2 < \infty \quad \text{and} \quad \mu^{(n)}(o) = O\left(\frac{1}{n^{1+\varepsilon}}\right).$$

Now, for a group which does not have polynomial growth with degree one or two, from observations hidden in Gromov [97], one obtains that  $b(n)$  is bounded below by  $Cn^3$ . This implies the 'only if' part (using the fact that it is sufficient to find a transient symmetric random walk with finite second moment). The 'if' part is obvious, for example, from Nash-Williams' recurrence criterion (Corollary 4.14 below). We shall return to Varopoulos' results in more detail in Section 6.

Trofimov [219] shows that every transitive graph with polynomial growth is very close to a Cayley graph of some group with polynomial growth (with the same degree); see Woess [245] for a short proof based on a theorem of Losert [149]. A thorough examination of Losert's paper yields that  $b(n) \geq Cn^3$  for every transitive graph which does not have polynomial growth with degree  $\leq 2$ . One can then apply Theorem 2 of Varopoulos [228] or Theorem 4.1 of Hebisch and Saloff-Coste [104] to obtain that the SRW is transient. This carries over easily to quasi-transitive graphs.

**COROLLARY 4.7.** *The SRW on a quasi-transitive graph  $X$  is recurrent if and only if  $X$  has polynomial growth with degree at most two.*

### B. The type problem for reversible Markov chains

Let us now pass to recurrence criteria for general reversible random walks. To this end, it will be convenient to introduce a potential theoretic setup; see Yamasaki [249, 250], T. Lyons [152], Varopoulos [223, 224] and Kaimanovich [112].

Given  $(X, P)$  reversible, consider the associated network  $\mathcal{N} = (X, E_P, r)$  as described in Section 2. Consider the real Hilbert spaces  $\ell^2(X, c)$  with inner product

$$(f, g) = \sum_{x \in X} f(x)g(x)c(x)$$

and  $\ell^2(E_P, r)$  with inner product

$$\langle u, v \rangle = \sum_{e \in E_P} u(e)v(e)r(e).$$

The *gradient operator* is

$$\nabla: \ell^2(X, c) \rightarrow \ell^2(E_P, r), \quad \nabla f(e) = \frac{f(e^+) - f(e^-)}{r(e)}.$$

It has norm  $\leq 2$ . Its adjoint is given by

$$\nabla^* u(x) = \frac{1}{c(x)} \left( \sum_{e: e^+ = x} u(e) - \sum_{e: e^- = x} u(e) \right).$$

The Laplacian is

$$\Delta = -\nabla^* \nabla = P - I,$$

where  $I$  is the identity operator on functions on  $X$ , and  $P$  is the transition operator of our random walk, as given in Section 2.B. We remark here that for an arbitrary reversible Markov chain,  $\rho(P)$  is the norm of  $P$  as a transition operator on  $\ell^2(X, c(\cdot))$  (Kersting [127]).

Consider the space  $\mathcal{D}(\mathcal{N})$  of all functions  $f$  on  $X$  (not necessarily in  $\ell^2(X, c)$ ) such that  $\nabla f \in \ell^2(E_P, r)$ . If  $f$  is such a function, then its Dirichlet sum is

$$D(f) = \langle \nabla f, \nabla f \rangle = \sum_{e \in E_P} (f(e^+) - f(e^-))^2 / r(e).$$

This is a quasi-norm, its kernel consisting of the constants.  $\mathcal{D}(\mathcal{N})$  can be equipped with an inner product by choosing a reference vertex  $o \in X$ :

$$(f, g)_D = \langle \nabla f, \nabla g \rangle + f(o)g(o).$$

Thus  $\mathcal{D}(\mathcal{N})$  becomes a Hilbert space whose topology does not depend on the particular choice of  $o$ . By  $\ell_0(X)$  we denote the linear space of finitely supported functions on  $X$ , and by  $\mathcal{D}_0(\mathcal{N})$  its closure in  $\mathcal{D}(\mathcal{N})$ .

Given a vertex  $x_0 \in X$  and a real number  $i$ , a finite energy flow from  $x_0$  to  $\infty$  with value  $i_0$  on  $\mathcal{N}$  is a function  $u \in \ell^2(E_P, r)$  such that

$$\nabla^* u(x) = \begin{cases} -i_0/c(x), & \text{if } x = x_0, \\ 0, & \text{otherwise.} \end{cases}$$

Its energy is  $\langle u, u \rangle$ .

The capacity of a set  $A \subset X$  is

$$\text{cap}(A) = \inf \{ D(f) \mid f \in \ell_0(X), f \equiv 1 \text{ on } A \}.$$

Finally, we give the definition of the extremal length of a set of infinite paths in  $(X, E_P)$ . An infinite  $P$ -path is a sequence  $\pi = [x_0, x_1, x_2, \dots]$  of distinct vertices in  $X$  such that  $c(x_{n-1}, x_n) > 0$  for all  $n$ . We write  $x_n(\pi) = x_n$ , and  $e_n(\pi)$  for the (oriented) edge in  $E_P$  whose endpoints are  $x_{n-1}$  and  $x_n$ . If  $\Pi$  is a set of infinite  $P$ -paths, then its extremal length  $\lambda(\Pi)$  is given by

$$\lambda(\Pi)^{-1} = \inf \left\{ \langle u, u \rangle \mid u \in \ell_+^2(E_P, r), \sum_{n=1}^{\infty} u(e_n(\pi)) \geq 1 \text{ for all } \pi \in \Pi \right\}.$$

With these ingredients, we can formulate the following useful collection of necessary and sufficient transience criteria.

**THEOREM 4.8.** *For a reversible random walk  $(X, P)$  and the associated network  $\mathcal{N}$ , the following statements are equivalent.*

- (a) *The random walk is transient.*
- (b) *For some ( $\Leftrightarrow$  every)  $x \in X$ , there is a finite energy flow from  $x$  to  $\infty$  with nonzero value.*
- (c) *For some ( $\Leftrightarrow$  every) nonvoid finite  $A \subset X$ ,  $\text{cap}(A) > 0$ .*
- (d) *The constant function 1 does not belong to  $\mathcal{D}_0(\mathcal{N})$ .*
- (e)  *$\mathcal{D}_0(\mathcal{N}) \subset \mathcal{D}(\mathcal{N})$  properly.*
- (f) *For some ( $\Leftrightarrow$  every)  $x \in X$ ,  $\lambda(\Pi_x) < \infty$ , where  $\Pi_x$  is the set of infinite  $P$ -paths starting at  $x$ .*

Criterion (b) is the discrete analogue of the Kelvin–Nevalinna–Royden criterion for Riemannian manifolds; compare with Lyons and Sullivan [155].

For the case when  $(X, E_p)$  is a locally finite graph (that is,  $(X, P)$  satisfies (FR)), the equivalences between (c), (d), (e) and (f) are due to Yamasaki [249], and the equivalence of these conditions with (a) and (b) can be found in Yamasaki [250]; see also Nakamura and Yamasaki [166]. None of these papers uses probabilistic notions; transience is hidden in the existence of a finite Green kernel (compare with Theorem 4.1 above). The potential theoretic equivalents of ‘recurrent’ and ‘transient’ are ‘parabolic’ and ‘hyperbolic’, respectively. For the SRW, a proof of (c)  $\Rightarrow$  (b) is given by Markvorsen, McGuinness and Thomassen [157].

In the non-locally finite case, T. Lyons [152] independently proved (a)  $\Leftrightarrow$  (b); Schlesinger [195] has another proof. For (a)  $\Leftrightarrow$  (c), see Varopoulos [223, p. 227]; Gerl [85] gives a polished proof. See also the very general results of Kaimanovich [112]. For these as well as the remaining equivalences, the methods of Yamasaki generalize to the non-locally finite case (Soardi and Yamasaki [204]).

Theorem 4.8 applies, of course, to symmetric random walks on groups. An application is the following generalization of Proposition 4.2.

**COROLLARY 4.9.** *If some random walk on  $X$  satisfying (SR) and (UI) is recurrent, then this is true for every random walk satisfying (SR), (UI) and (BR).*

In order to prove this, one can work out that the corresponding Dirichlet norms are comparable and apply criterion (d); see also Varopoulos [222, p. 256]. More generally, one can compare graphs as follows. Let  $X$  and  $X'$  be two graphs with corresponding discrete metrics  $d$  and  $d'$ , respectively.

(RI) We say that  $X$  and  $X'$  are *roughly isometric* if there are a mapping  $\phi: X \rightarrow X'$  and constants  $A > 0$ ,  $B \geq 0$  such that

$$A^{-1}d(x, y) - B \leq d'(\phi x, \phi y) \leq Ad(x, y) + B$$

for all  $x, y \in X$ , and

$$d'(x', \phi X) \leq B$$

for every  $x' \in X'$ .

If  $B = 0$ , then we say that  $X$  and  $X'$  are *metrically equivalent*. We remark that rough isometries can be defined between arbitrary metric spaces. The terminology used here is the one introduced by Kanai [117, 118], while Gromov [98] uses ‘quasi-isometric’ instead. Using the axiom of choice, rough isometry is an equivalence relation (otherwise one also has to specify a ‘rough inverse’ of  $\phi$ ). In analogy with Corollary 4.9 one obtains the following.

**THEOREM 4.10.** *Let  $X$  and  $X'$  be roughly isometric graphs with bounded vertex degrees. Then  $X$  is recurrent if and only if  $X'$  is recurrent.*

For a proof using capacity, see Kanai [118, Corollary 7]. A proof using flows is given by Markvorsen, McGuinness and Thomassen [157].

We explain how criterion (f) of Theorem 4.8 can become useful. The following is due to Kayano and Yamasaki [122] when  $(X, E_p)$  is locally finite, and carries over to the non-locally finite case (Soardi and Yamasaki [204]).

**THEOREM 4.11.** *Let  $(X, P)$  be reversible and  $\mathcal{N}$  be the associated network. If  $f \in \mathcal{D}_0(\mathcal{N})$  and  $x \in X$ , then  $\lambda(\Pi_x \setminus \Pi'_x) = \infty$ , where*

$$\Pi'_x = \{\pi \in \Pi_x \mid \lim_{n \rightarrow \infty} f(x_n(\pi)) = 0\}.$$

**COROLLARY 4.12.** *If for  $(X, P)$  and  $\mathcal{N}$  as above one can find  $f \in \mathcal{D}_0(\mathcal{N})$  and  $x \in X$  such that  $f(x_n(\pi))$  does not converge to zero for any  $\pi \in \Pi_x$ , then the random walk is recurrent.*

This purely analytic criterion has been successfully applied to the edge graphs of ‘quasi-normal’ tilings of the plane in a beautiful paper by Soardi [200]. We remark that these graphs may have exponential growth.

Criterion (d) of Theorem 4.8 can also be applied to deduce Nash-Williams’ recurrence criterion. Let  $(X, P)$  be reversible with associated network  $\mathcal{N}$ . Suppose that we have a partition of  $X$ :

$$X = \bigcup_{i \in I} X_i, \quad \text{such that } \chi_{X_i} \in \mathcal{D}_0(\mathcal{N}), \forall i \in I.$$

(The last condition holds in particular if  $\sum_{x \in X_i} c(x) < \infty$ .) Then we can define the *shortened network*  $\mathcal{N}'$  with vertex set  $I$  and conductance function

$$c'(i, j) = \sum_{x \in X_i, y \in X_j} c(x, y), \quad \text{if } i \neq j.$$

Note that  $\sum_j c'(i, j) < \infty$  for every  $i$ , so that also the shortened network arises from a reversible Markov chain  $(I, P')$ . If  $f' \in \mathcal{D}(\mathcal{N}')$  and we set  $f(x) = f'(i)$  for  $x \in X_i$ , then  $D_{\mathcal{N}}(f) = D_{\mathcal{N}'}(f')$ . Thus, if  $1 \in \mathcal{D}_0(\mathcal{N}')$  then  $1 \in \mathcal{D}_0(\mathcal{N})$ .

**THEOREM 4.13.** *Suppose that  $(X, P)$  is reversible and that  $(I, P')$  arises from  $(X, P)$  by shortening. If  $(I, P')$  is recurrent, then so is  $(X, P)$ .*

A particular instance is the following.

**COROLLARY 4.14.** *Under the assumptions of Theorem 4.13, suppose that  $I = \mathbb{N}_0$  and  $c'(i, j) = 0$  if  $|i - j| \geq 2$ . If*

$$\sum_{i=1}^{\infty} \frac{1}{c'(i-1, i)} = \infty,$$

*then  $(X, P)$  is recurrent.*

This is due to Nash-Williams [165] in the case when all the  $X_i$  are finite. For the case when  $\sum_{x \in X_i} c(x) < \infty$  for all  $i$  (which implies  $\chi_{X_i} \in \mathcal{D}_0(\mathcal{N})$ ), a proof has been given by Griffeath and Liggett [92]. For a still more general version, see McGuinness [158]: the assumptions there can be related in a simple way with the condition  $\chi_{X_i} \in \mathcal{D}_0(\mathcal{N})$  (P. M. Soardi, oral communication). For a discussion and nice applications of ‘shortening’, see Doyle and Snell [63].

Finally, another obvious application of criterion (c) or (d) of Theorem 4.8 is the following, also contained in the papers by Griffeath and Liggett [92] and T. Lyons [152] and in the book by Doyle and Snell [63].

**COROLLARY 4.15.** *Let  $(X, P_1)$  and  $(X, P_2)$  be two reversible random walks such that for the corresponding conductances,  $c_1(x, y) \geq c_2(x, y)$  for all  $x, y \in X$ . If  $(X, P_1)$  is recurrent, then so is  $(X, P_2)$ .*

Under certain conditions, one can also compare non-reversible Markov chains with reversible ones. This goes back to a very useful lemma of Baldi, Lohué and Peyrière [14], first applied to random walks on groups. For extensions, see Varopoulos [221, 222], Durrett [65] and Chen [39]. We state the main result of Chen [39].

An *excessive measure* for  $(X, P)$  is a strictly positive function  $m(\cdot)$  on  $X$  such that  $m(x) \geq \sum_y m(y)p(y, x)$  for all  $x \in X$ . Two positive measures ( $\equiv$  functions)  $m_1(\cdot)$ ,  $m_2(\cdot)$  on  $X$  are *equivalent* if  $C^{-1} \leq m_1(x)/m_2(x) \leq C$  for all  $x \in X$ , where  $0 < C < \infty$ .

**THEOREM 4.16.** *Let  $P$  and  $Q$  be two irreducible transition matrices over  $X$ . Assume that  $P$  has excessive measure  $m(\cdot)$ , and that  $Q$  is reversible and such that the associated total conductance  $c(\cdot)$  is equivalent with  $m(\cdot)$ .*

*If  $P \geq \varepsilon_0 Q$  elementwise for some  $\varepsilon_0 > 0$ , then there is a constant  $K > 0$  such that*

$$G_P\left(x, x \left| \frac{z}{2-z} \right.\right) \leq K \cdot G_Q(x, x | z) \quad \text{for all } x \in X, 0 < z < 1.$$

*In particular, transience of  $(X, Q)$  implies transience of  $(X, P)$ .*

We remark that Chen [39] also has a result in the spirit of ‘shortening’ for non-reversible Markov chains.

### C. Nearest neighbour random walks on trees and recurrence criteria

For the simple random walk on a *tree*, the transience criterion given by Theorem 4.8(b) can be translated into finiteness of a (generalized) *volume* of the tree; see Gerl [84] and Woess [239]. For trees, the proof becomes much more elementary [239]. Given a locally finite tree without vertices of degree one, its (ordinary) volume is the energy of the flow with value one at  $o$  which divides equally at each branching of the tree when going out from  $o$ . Woess [239] conjectured that every transient tree has a subtree with finite (ordinary) volume, but Berman and Konsowa [19] have constructed an instructive counterexample.

Thomassen [216, 217, 218] has linked isoperimetric inequalities for a locally finite graph  $X$  with the existence of transient subtrees. Let  $f$  be a nondecreasing, positive real function defined on the positive integers.

(IS<sub>f</sub><sup>o</sup>)  $X$  satisfies a *connected  $f$ -isoperimetric inequality rooted at  $o \in X$*  if there is a constant  $\kappa > 0$  such that

$$|\partial U| \geq \kappa \cdot f(\text{Vol}(U))$$

for every finite, connected subgraph  $U$  of  $X$  containing  $o$ .

Thomassen [218] proves the following remarkable result.

**THEOREM 4.17.** *If a locally finite graph  $X$  satisfies (IS<sub>f</sub><sup>o</sup>) with*

$$\sum_{k=0}^{\infty} \frac{1}{f(k)^2} < \infty,$$

*then  $X$  contains a transient subtree with maximum degree three.*

We remark that this does *not* follow from the results of Varopoulos discussed below in Section 6 (in the first place because there the isoperimetric inequalities have to be global).

Note that if  $X = T$  is a tree, then any (NN) random walk is reversible. For these random walks, Benjamini and Peres [18] have another nice transience criterion which is worthwhile being mentioned here. The following observations concerning the geometry of a tree  $T$  will also be of use in Section 7.

Given  $x, y \in T$ , there is a unique shortest path in  $T$  connecting the two, called the *geodesic segment*  $\overline{xy}$ . Fix a reference vertex  $o \in T$ . For  $x, y \in T$ , their *confluent* with respect to  $o$  is the unique vertex

$$c = c(x, y) = \overline{ox} \cap \overline{oy} \cap \overline{xy}.$$

Consider a nearest neighbour (irreducible!) random walk on  $T$  with transition matrix  $P$  and associated resistances  $r(e)$  for the edges of  $T$ . We define the *meeting height* of  $x, y \in T$  with respect to  $o$  and  $P$ :

$$(x|y)_P = (x|y)_{o,P} = \sum \{r(e) \mid e \text{ an edge of } \overline{ox}\},$$

where  $c = c(x, y)$ . In particular, if  $P$  is the SRW, then we omit the subscript  $P$ . Thus

$$(x|y) = d(o, c(x, y)).$$

**THEOREM 4.18.** *The nearest neighbour random walk  $(T, P)$  is transient if and only if there is a finite constant  $C$  such that for every  $n \geq 1$  there are distinct vertices  $x_1, \dots, x_n$  with average meeting height*

$$\frac{2}{n(n-1)} \sum_{i < j \leq n} (x_i | x_j)_P \geq C.$$

Note that the theorem fails if  $T$  is not locally finite. The proof by Benjamini and Peres of Theorem 4.18 involves the capacity of the boundary at infinity, that is, the *space of ends* of  $T$  (see Section 7 for the definition). For further results with a potential theoretic spirit concerning random walks on trees (besides those described in Section 7), see also R. Lyons [150, 151].

#### D. $\rho$ -recurrence

The random walk  $(X, P)$  is called  $\rho$ -recurrent (where  $\rho = \rho(P)$ ) if

$$\sum_{n=0}^{\infty} p^{(n)}(x, y) \rho(P)^{-n} = \infty$$

for some ( $\Leftrightarrow$  every)  $x, y \in X$ . This property has been studied for general irreducible Markov chains by various authors; see, for example, Vere-Jones [230, 231, 232] or Pruitt [186]. For groups, the situation is as follows (see the indications of Guivarc'h [100, p. 85]).

**THEOREM 4.19.** *Let  $\mu$  be an irreducible probability measure on the group  $\Gamma$  giving rise to a  $\rho(\mu)$ -recurrent random walk. Then there is a unique  $\rho(\mu)$ -harmonic exponential  $\phi$  on  $\Gamma$ , and  $\mu_0(x) = \mu(x) \phi(x) / \rho(\mu)$  defines a probability measure which gives rise to a recurrent random walk on  $\Gamma$ .*

Recall that  $\rho(\mu)$ -harmonic means that  $\sum_y \mu(x^{-1}y) \phi(y) = \rho(\mu) \phi(x)$  for all  $x$ , that is,  $\phi$  being an exponential,

$$\sum_x \mu(x) \phi(x) = \rho(\mu).$$

Thus, among finitely generated, infinite groups, only finite extensions of  $\mathbb{Z}$  and  $\mathbb{Z}^2$  carry  $\rho$ -recurrent random walks. On the other hand, for SRWs on arbitrary graphs, no satisfactory criterion for  $\rho$ -recurrence in terms of structure properties is known. Gerl [83] and Gerl and Woess [89] exhibit a variety of examples of SRWs on trees.

### 5. Periodicity, ratio limit theorems

#### A. The period of a random walk

The *period* of  $(X, P)$  is the number

$$d = d(P) = \gcd \{n \mid p^{(n)}(x, x) > 0\}, \quad x \in X.$$

It is well known that for a general irreducible Markov chain,  $d$  does not depend on  $x$ . If  $d = 1$ , then the Markov chain is called *aperiodic*. If  $d > 1$ , then one can decompose  $X = X_1 \cup \dots \cup X_d$  such that the restriction of  $P^d$  to each of the  $X_i$  is stochastic, irreducible and aperiodic. In this sense, aperiodicity is no severe restriction, and the results described below under the assumption of aperiodicity extend to the case  $d > 1$  with the obvious modifications. The following is well known (Kingman [135]).

LEMMA 5.1. *If  $x, y \in X$  and  $k$  is such that  $p^{(k)}(x, y) > 0$ , then*

$$p^{(m)}(x, y) = 0 \quad \text{if } m \not\equiv k(d),$$

while

$$\lim_{n \rightarrow \infty} p^{(nd+k)}(x, y)^{1/(nd+k)} = \rho(P).$$

For an irreducible random walk with law  $\mu$  on a group  $\Gamma$ , it turns out that

$$\bigcup_{n=0}^{\infty} (\text{supp } \mu)^{-n} (\text{supp } \mu)^n$$

is a normal subgroup with index  $d = d(\mu)$  in  $\Gamma$ ; its cosets give the partition for which the restriction of  $\mu^{(d)}$  becomes aperiodic. (Here, for a set  $S \subset \Gamma$ ,  $S^n$  is the set of all products of  $n$  elements in  $S$ , and  $S^{-n}$  the set of all inverses of the latter.) The proof is an easy exercise. In particular, aperiodicity is equivalent to the property that no coset of a proper normal subgroup of  $\Gamma$  contains  $\text{supp } \mu$ .

(SP) We say that  $(X, P)$  is *strongly periodic*, if for some integer  $n_0$

$$\inf_{x \in X} p^{(nd)}(x, x) > 0 \quad \text{for all } n \geq n_0,$$

where  $d = d(P)$ .

(SA)  $(X, P)$  is *strongly aperiodic* if (SP) hold with  $d = 1$ .

Note that (SP) holds for all random walks which are invariant under a quasi-transitive group action, in particular for random walks on groups.

For a random walk on a graph satisfying (SR) and (NN),

$$d(P) = 1 \text{ or } 2,$$

and  $d(P) = 2$  exactly when  $X$  is bipartite ( $X$  can be partitioned in  $X_1 \cup X_2$ , such that every edge has one endpoint in  $X_1$  and the other in  $X_2$ ). Thus  $d(P) = 1$  if and only if  $X$  has a circuit of odd length (a circuit of length  $\ell$  is a sequence  $x_0, x_1, \dots, x_\ell = x_0$  of successively adjacent vertices, with repetitions allowed).

Suppose, in addition to (SR) and (NN), that the one-step transition probabilities along the edges are bounded below by some  $\varepsilon_0 > 0$ . Then it is easy to see that in the bipartite case, one has (SP) (with  $d = 2$ ). If  $X$  is not bipartite, then (SA) holds precisely when there is an odd number  $\ell$  such that every vertex is contained in a circuit of length  $\ell$ .

For (SA), compare, for example, Molchanov [163]. Condition (SP) was used by Cartwright [29] for proving the first part of the following result.

**THEOREM 5.2.** *Suppose that  $(X, P)$  has period  $d$  and satisfies (SP). Then the following hold.*

(a) *The singularities of  $G(x, y | z)$  on its circle of convergence  $\{z \in \mathbb{C} \mid |z| = 1/\rho(P)\}$  are precisely the numbers*

$$\frac{1}{\rho(P)} e^{2\pi i k/d}, \quad k = 0, \dots, d-1.$$

(b) *If  $x, y \in X$  and  $k$  is such that  $p^{(k)}(x, y) > 0$ , then*

$$\lim_{n \rightarrow \infty} \frac{p^{((n+1)d+k)}(x, y)}{p^{(nd+k)}(x, y)} = \rho(P).$$

Part (a) will be useful in Section 6 (local limit theorems). It also answers a question raised by de la Harpe, Robertson and Valette [52].

**COROLLARY 5.3.** *Suppose that  $(X, P)$  satisfies (NN), (SR) and (SP). Then  $-\rho(P) \in \text{spec}(P)$  if and only if  $X$  is bipartite.*

Here,  $\text{spec}(P)$  is the spectrum of  $P$  as an operator on  $\ell^2(X, c)$ . Recall that this operator has norm  $\rho(P)$ .

Part (b) of Theorem 5.2 was proved by Gerl [78, 81] for aperiodic random walks on discrete groups; the extension to the above situation is straightforward (and, to a large extent, already contained in Gerl's proofs). For a similar result, see also Guivarc'h [99]. Indeed, condition (II) used by Guivarc'h suggests that in Theorem 5.2 one can replace (SP) by the following.

(SP')  $\text{gcd}\{k \in \mathbb{N} \mid \text{there are } n \in \mathbb{N}, c > 0, \text{ s.t. } P^{n+k} \geq c \cdot P^n \text{ elementwise}\} = d(P)$ .

We remark that in a different context, condition (SP'), with  $d = 1$ , was also used by Molchanov [164].

## B. Ratio limit theorems

For most of the following (and also Section 6) it will be convenient to assume that  $(X, P)$ , besides being irreducible, is also aperiodic. A (strong) ratio limit theorem is a result which describes convergence of

$$\frac{p^{(n)}(x, y)}{p^{(n)}(o, o)}, \quad n \rightarrow \infty, \quad (\times)$$

for arbitrary  $x, y \in X$  ( $o$  a reference point).

We remark that there also are other types of ratio limits which have been studied, for example, by Kesten and Spitzer [133] and others. Whenever one has a local limit theorem (see Section 6), one obtains a ratio limit theorem as a corollary. Here, we describe those results which do not arise in this way.

On  $X = \mathbb{N}_0$ , consider an (NN) random walk:  $p(k, \ell) > 0$  if and only if  $|k - \ell| = 1$ . Thus  $(\mathbb{N}_0, P)$  has period two and is reversible with easy to determine invariant measure (total conductance)  $c(\cdot)$ . Theorem 4.8(b) yields a simple recurrence criterion, already given by Karlin and McGregor [121]; for a proof using methods of continued fractions, see Gerl [83]. For every  $t \geq \rho(P)$ , there is a unique positive function  $h_t$  in  $\mathcal{H}_t(P)$  with  $h_t(0) = 1$ ; it is determined by a second-order recursion. Karlin and McGregor [121] have proved the following.

**THEOREM 5.4.** *For an (NN) random walk on  $\mathbb{N}_0$  with spectral radius  $\rho$ ,*

$$\lim_{n \rightarrow \infty} \frac{p^{(2n+\kappa)}(i, j)}{p^{(2n+\kappa)}(k, \ell)} = \frac{c(j)h_\rho(i)h_\rho(j)}{c(\ell)h_\rho(k)h_\rho(\ell)},$$

*if  $j - i$  and  $\ell - k$  are both even and  $\kappa = 0$ , or both odd and  $\kappa = 1$ .*

For generalizations, see Kersting [127].

Most known ratio limit theorems of form  $(\times)$  apply to random walks on groups; first results are due to Chung and Erdős [42], Kemeny [124] and Kesten [130] for integer lattices. The results generalize to arbitrary Abelian groups (Stone [208]).

**THEOREM 5.5.** *Let  $\Gamma$  be a finitely generated Abelian group and  $\mu$  an irreducible, aperiodic probability measure on  $\Gamma$ . Then there is a unique  $\rho(\mu)$ -harmonic exponential  $\phi$  on  $\Gamma$ , and*

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\mu^{(n)}(o)} = \frac{1}{\phi(x)} \quad \forall x \in \Gamma.$$

Note that Theorem 5.5 requires no moment condition.

For symmetric  $\mu$ , the following generalization (with a completely different, elegant proof) is due to Avez [9].

**THEOREM 5.6.** *Let  $\Gamma$  be an amenable group and  $\mu$  a symmetric, irreducible, aperiodic probability measure on  $\Gamma$ . Then*

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\mu^{(n)}(o)} = 1 \quad \forall x \in \Gamma.$$

One may ask which conditions are necessary in general to have

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\mu^{(n)}(o)} = \psi(x) \quad \forall x \in \Gamma,$$

where  $\Gamma$  is a finitely generated group and  $\mu$  an irreducible, aperiodic probability measure on  $\Gamma$ ; see Le Page [145], Gerl [81] and Guivarc'h [99]. Suppose that  $\mu$  is finitely supported. From the existence of the limit in Theorem 5.2(b), in the aperiodic case one obtains that the sequence  $(\mu^{(n)}/\mu^{(n)}(o))$  is bounded in  $n$  for every  $x$ . Hence it is vaguely compact, and it is not hard to see that every function  $\psi$  which arises as the limit of a subsequence must satisfy the convolution equation

$$\mu * \psi = \psi * \mu = \rho(\mu) \cdot \psi.$$

Thus one obtains the following ratio limit theorem; see Gerl [81].

**PROPOSITION 5.7.** *Let  $\mu$  be a finitely supported, aperiodic, irreducible probability measure on  $\Gamma$ . For a property  $\mathbf{P}$  of positive functions on  $\Gamma$ , suppose that*

- (1) *whenever some subsequence  $\mu^{(n_k)}(x)/\mu^{(n_k)}(o)$  converges pointwise to a function  $\psi(x)$ , then  $\psi$  must have property  $\mathbf{P}$ , and*
- (2) *the convolution equation  $\mu * \psi = \psi * \mu = \rho(\mu) \cdot \psi$  has a unique solution satisfying  $\mathbf{P}$  and  $\psi(o) = 1$ .*

*Then*

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\mu^{(n)}(o)} = \psi(x) \quad \forall x \in \Gamma.$$

If one takes for property  $\mathbf{P}$  just ‘positive’, then uniqueness in (2) will hold when  $\Gamma$  is Abelian (Stone [208])—this yields Theorem 5.5 when  $\mu$  is finitely supported—or when  $\Gamma$  is nilpotent (Margulis [156]); for the following, compare with Theorem 7.3 below.

**COROLLARY 5.8.** *Let  $\Gamma$  be nilpotent and  $\mu$  finitely supported as above. Then*

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\mu^{(n)}(o)} = \frac{1}{\phi(x)} \quad \forall x \in \Gamma,$$

*where  $\phi$  is the unique  $\rho(\mu)$ -harmonic exponential on  $\Gamma$ .*

Proposition 5.7 also applies, for example, to isotropic random walks on free groups—with  $\mathbf{P}$  being ‘isotropic’—compare Gerl [81]. However, in this case also a local limit theorem is known; see Section 7.C. For generalizations of ratio limit theorems of the above type to homogeneous spaces (that is, random walks which are invariant under a transitive group action), see Shur [197, 198].

## 6. The asymptotic behaviour of transition probabilities

In this section we shall be interested in the asymptotic behaviour of  $p^{(n)}(x, y)$  as  $n \rightarrow \infty$ , where  $x, y \in X$ . In addition to irreducibility of  $(X, P)$ , it will be once more convenient to assume aperiodicity, which is not an essential loss of generality.

A *local limit theorem* is a result which gives an explicit sequence  $(a_n)$  of positive numbers, such that

$$\lim_{n \rightarrow \infty} \frac{p^{(n)}(x, y)}{\rho(P)^n a_n}$$

exists and is nonzero for all  $x, y \in X$ , or at least for  $x = y = o$ , where  $o$  is a fixed reference vertex. A weaker version is one asserting that  $(p^{(n)}(x, y)/\rho(P)^n a_n)$  be bounded and bounded away from zero.

Let us start with (irreducible) random walks on  $\Gamma = \mathbb{Z}^q$ ; see Rvačeva [189], Spitzer [205] and Stone [209].

**THEOREM 6.1.** *Let  $\mu$  be an aperiodic probability measure on  $\Gamma = \mathbb{Z}^q$  such that  $|\mu|_2 < \infty$ . Then*

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{n^{-q/2} \rho(\mu)^n} = C\psi(x) \quad \forall x \in \mathbb{Z}^q,$$

*where  $C > 0$  and  $\psi$  is an exponential on  $\mathbb{Z}^q$ . In addition,  $\rho(\mu) = 1$  if and only if  $\bar{\mu} = 0$ , and  $\psi \equiv 1$  in this case.*

If the second moment is not finite and  $\mu$  is in a domain of attraction other than that of normal distribution, then the asymptotic behaviour will, of course, be different; see Rvačeva [189] and Stone [209]. Ney and Spitzer [167, Theorem 2.1] have a refined version of Theorem 6.1 in the case when  $\mu$  has exponential moment and  $\bar{\mu} \neq 0$ . Krámli and Szász [139] and Guivarc'h [101] have local limit theorems which apply—in our terminology—to random walks on graphs which satisfy (GP) with respect to the quasi-transitive action of  $\Gamma \leq \text{Aut}(X)$ ,  $\Gamma \cong \mathbb{Z}^q$ .

In order to explain how Theorem 6.1 can be generalized, we first consider reversible Markov chains.

#### A. Reversible Markov chains and Dirichlet inequalities

Let  $(X, P)$  be a reversible Markov chain and  $\mathcal{N}$  be the associated network. For  $1 \leq p \leq \infty$ , we consider the norms of functions on  $X$

$$\|f\|_p = \left( \sum_{x \in X} |f(x)|^p c(x) \right)^{1/p}, \quad p < \infty; \quad \|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Furthermore, besides the Dirichlet sum  $D(f) = \langle \nabla f, \nabla f \rangle$ , we also consider the Sobolev sum

$$S(f) = \sum_{e \in E_p} |\nabla f(e)| = \sum_{e \in E_p} \frac{|f(e^+) - f(e^-)|}{r(e)}.$$

Consider the following conditions.

(D<sub>q</sub>) (Dirichlet inequality) For  $p = 2q/(q-2)$ ,  $q \geq 2$  and some constant  $C$ ,

$$\|f\|_p \leq C \sqrt{D(f)}, \quad \forall f \in \ell_0(X).$$

(S<sub>q</sub>) (Sobolev inequality) For  $p = q/(q-1)$ ,  $q \geq 1$  and some constant  $C$ ,

$$\|f\|_p \leq C \cdot S(f), \quad \forall f \in \ell_0(X).$$

Varopoulos [223] has proved the following significant results.

**THEOREM 6.2.** *Let  $(X, P)$  be reversible.*

(a) If  $q \geq 1$ , then (S<sub>q</sub>) implies

$$\sup_{x, y \in X} \frac{p^{(n)}(x, y)}{c(y)} = O(n^{-q/2}) \quad \text{as } n \rightarrow \infty. \quad (*)$$

(b) If  $q \geq 2$ , then (D<sub>q</sub>) implies (\*).

(c) If  $q > 2$ , then (\*) implies (D<sub>q</sub>).

(d) If  $q > 2$ , then (S<sub>q</sub>) implies (D<sub>q</sub>).

The last statement is just the combination of (a) and (c), but Varopoulos [223] also gives a direct proof. As a consequence of (b), if  $q > 2$  then (D<sub>q</sub>) implies transience. To prove this, one does not need the (highly nontrivial) asymptotic estimate (\*): it is enough to use Theorem 4.8, observing that (D<sub>q</sub>) for  $q > 2$  implies  $1 \notin \mathcal{D}_0(\mathcal{N})$  (as otherwise  $\|1\|_p \leq C \sqrt{D(1)} = 0$  for  $p = 2q/(q-2)$ ). For random walks which are very well adapted to the (original) graph structure of  $X$ , one has the following.

**THEOREM 6.3.** *Let  $(X, P)$  satisfy (SR), (UI) and (BR). If  $q \geq 1$ , then (IS<sub>q</sub>) holds if and only if the network associated with  $(X, P)$  satisfies (S<sub>q</sub>).*

The proof of Varopoulos [223] is given for the SRW on the Cayley graph of a group only, but carries over immediately to the SRW on a graph with bounded vertex degrees; see the remarks at the end of §4 of [223]. After that, the extension to random walks with (SR), (UI) and (BR) is a straightforward exercise. As a by-product of Theorem 6.3, one obtains that  $(IS_q)$  for  $q > 2$  implies transience: this is generalized by the result of Thomassen mentioned in Section 4 (Theorem 4.17).

In view of the above considerations, the following geometric result, due to Kanai [117], is very useful.

**PROPOSITION 6.4.** *If  $X_1$  and  $X_2$  are two roughly isometric graphs with bounded vertex degrees, then  $i_q(X_1) > 0$  if and only if  $i_q(X_2) > 0$ . The same holds for the strong isoperimetric inequality.*

### B. Local limit theorems and growth of groups

Varopoulos [225, 226] has applied his results (Theorem 6.2) to groups.

**THEOREM 6.5.** *For a finitely generated group  $\Gamma$ , the following statements are equivalent.*

- (a)  $\Gamma$  has polynomial growth with (precise) degree  $q$ .
- (b) Some ( $\Leftrightarrow$  every) Cayley graph of  $\Gamma$  satisfies  $(IS_q)$ .
- (c) For some ( $\Leftrightarrow$  every) symmetric, irreducible, aperiodic random walk on  $\Gamma$  with finitely supported law  $\mu$ , there is  $0 < C < \infty$  such that

$$C^{-1}n^{-q/2} \leq \mu^{(n)}(o) \leq Cn^{-q/2} \quad \text{for every } n,$$

where  $o$  is the identity element.

Recall Gromov's famous theorem that a finitely generated group has polynomial growth if and only if it is nilpotent-by-finite [97]. In this case, the growth exponent has to be integer. Also note that in (c), the identity element  $o$  can be replaced by an arbitrary  $x \in \Gamma$  (possibly with a different  $C$ ): this follows from an elementary Harnack type estimate bounding  $\mu^{(n)}(x)/\mu^{(n)}(o)$ . For all other groups, the  $n$ -step transition probabilities decay faster; see Varopoulos [226].

**THEOREM 6.6.** *If  $\Gamma$  is finitely generated and not nilpotent-by-finite ( $\Leftrightarrow$  does not grow polynomially), then for every (irreducible) random walk on  $\Gamma$  with finitely supported law  $\mu$ , one has*

$$\mu^{(n)}(o) = O(n^{-q}) \quad \text{as } n \rightarrow \infty \quad \text{for all } q > 0.$$

The last two theorems constitute the conclusion of Varopoulos' proof of Kesten's conjecture on recurrent groups; see Section 4 above. The cited work of Varopoulos is partially difficult to keep track of. A condensed and very readable exposition of the quoted results of Varopoulos has been written by Mess [160]. The main method of Varopoulos [223] has been simplified by Coulhon and Saloff-Coste [46], and a detailed book by Coulhon, Saloff-Coste and Varopoulos [48] has appeared recently. Finally, a different proof of Theorem 6.5, (a)  $\Leftrightarrow$  (c), is given by Hebisch and Saloff-Coste [104]; their estimates are based directly upon the growth function and do not take the 'detour' via isoperimetric inequalities. Their main result is an (upper and lower)

'Gaussian' estimate for the  $n$ -step probabilities which is uniform in time ( $n$ ) and space ( $x \in \Gamma$ ). Compare this with the results of Spitzer [205, §7] and of Ney and Spitzer [167, Theorem 2.1] for  $\mathbb{Z}^q$ . Coulhon and Saloff-Coste [47] have an extension to a large class of—not necessarily group-invariant—random walks on graphs which grow at most linearly.

In Theorems 6.6 and 6.7, one can replace the finite support assumption by finiteness of the second moment of  $\mu$ :

$$\sum_{x \in \Gamma} d_A(x, o)^2 \mu(x) < \infty,$$

where  $d_A$  is the Cayley graph metric with respect to the finite symmetric set of generators  $A$ . (Finiteness of the moment is independent of such  $A$ .)

Recently, Varopoulos [229] has refined Theorem 6.6; see also Hebisch and Saloff-Coste [104].

**THEOREM 6.7.** *If  $\Gamma$  is finitely generated and its growth function satisfies*

$$b(n) \geq \exp(c_1 \cdot n^q), \quad c_1 > 0, 0 < q \leq 1,$$

*then*

$$\mu^{(n)}(o) = O(\exp(-c_2 \cdot n^{q/(q+2)})), \quad c_2 > 0,$$

*for every finitely generated, symmetric, irreducible probability measure on  $\Gamma$ .*

Note that there are examples of groups where  $0 < q < 1$  (Grigorchuk [93]).

A group is called *polycyclic* if it is solvable and all its subgroups are finitely generated. For a list of equivalent conditions, see, for example, Wolf [248], who also proves that such a group grows either polynomially (is nilpotent-by-finite) or exponentially. Alexopoulos [2] has proved that

$$\mu^{(n)}(o) \geq c_3 \cdot \exp(-c_4 \cdot n^{1/3}), \quad c_3, c_4 > 0,$$

if  $\Gamma$  is polycyclic, non-nilpotent-by-finite, and  $\mu$  is symmetric, irreducible and aperiodic. In combination with Theorem 6.7, this shows that  $\exp(-c \cdot n^{1/3})$  gives the right order of decay in the case of these groups.

Let us now turn to groups which are the typical antagonists of those having polynomial growth.

### C. Random walks on free groups and trees

Consider the free group  $\mathbb{F}_s$  with free generators  $x_1, \dots, x_s$ ,  $s \geq 2$ . Its Cayley graph with respect to the set of generators

$$A = A_s = \{x_1^{\pm 1}, \dots, x_s^{\pm 1}\}$$

is a homogeneous tree  $T_{2s}$  with degree  $2s$ . The *length* of  $x \in \mathbb{F}_s$  is  $|x| = d(x, o)$ , the distance between  $x$  and the unit element  $o$  in the tree.

Local limit theorems for random walks on free groups have been studied by various authors. For  $\mathbb{F}_2$  and  $\mu =$  the equidistribution on  $A_2$ , see Gerl [77]; for  $\mathbb{F}_2$  and  $\text{supp } \mu = A_2$  but not necessarily equidistributed, see Gerl [79, 80, 82]; and for  $\mathbb{F}_s$  ( $s \geq 2$  arbitrary) and  $\text{supp } \mu = A_s \cup \{o\}$ , see Gerl and Woess [90]. Gerl and Woess [90] also prove that the limiting function (measure) in the local limit theorem can be described in terms of a subshift of finite type (without using this terminology). We

mention that Lalley's [140] version of the 'nearest neighbour' local limit theorem gives an asymptotic equivalent of  $\mu^{(n)}(x)$  which is uniform in  $x$ . We also mention that the implicit equation for the Green kernel in the nearest neighbour case is already present, in more or less implicit form, in the papers of Dynkin and Maljutov [68] and Levit and Molchanov [147], and, in the terminology of formal language theory, in Chomsky and Schützenberger [40, p. 145].

**THEOREM 6.8.** *Let  $\mu$  be an aperiodic probability measure on  $\mathbb{F}_s$ . If*

(a) *supp  $\mu$  is finite, or*

(b)  *$\mu$  has finite first moment ( $\sum_x |x| \mu(x) < \infty$ ),*

$$\text{supp } \mu \subset \{x_i^k \mid i = 1, \dots, s; k \in \mathbb{Z}\} \quad \text{and} \quad \sum_{i=1}^s \sum_{k \in \mathbb{Z}} k \mu(x_i^k) = 0,$$

then for every  $x \in \mathbb{F}_s$ ,

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\rho(\mu)^n n^{-3/2}} = \psi(x) > 0.$$

With assumption (a), this significant result is due to Lalley [141], who also undertakes a profound study of various other aspects of finitely supported random walks on  $\mathbb{F}_s$ . With assumption (b), the result is a particular case of what is proved by Woess [240] for more general free products; see Subsection D below. As in the other 'nonisotropic' results cited above, the proofs in both cases use a combinatorial approach to finding equations for the Green function which then are studied by means of complex analysis. In case (b), Woess [240] requires symmetry, which can be generalized to the above condition by use of the recent result of Cartwright [29] given in Theorem 5.2(a). In particular, the Green function of a finite range random walk on a free group is algebraic; see Aomoto [7], Steger [207] or Woess [242]. (The latter reference exhibits a close relation with the theory of context-free languages.)

An *isotropic* probability measure on  $\mathbb{F}_s$  is one such that  $\mu(x)$  depends only on  $|x|$ . More generally, an *isotropic random walk* on a homogeneous tree  $T_k$  of (not necessarily even) degree  $k$  is one where  $p(x, y)$  depends only on  $d(x, y)$ . Note that  $T_{2s+1}$  is the Cayley graph of the free product  $\mathbb{F}_s * \mathbb{Z}_2$  with respect to the natural generators: thus an isotropic random walk on  $T_k$  can always be interpreted as a random walk with law  $\mu$  on an appropriate group. Local limit theorems for isotropic random walks have been given by Sawyer [192] and Picardello [176].

**THEOREM 6.9.** *For an isotropic (irreducible, aperiodic) random walk on  $T_k$  ( $k \geq 3$ ) one has*

$$\lim_{n \rightarrow \infty} \frac{p^{(n)}(x, y)}{\rho(P)^n n^{-3/2}} = \psi(d(x, y)) > 0.$$

This is due to Sawyer [192]; for the periodic case, see Picardello [176], who also has good estimates for the error term

$$p^{(n)}(x, y) - \psi(d(x, y)) \rho(P)^n n^{-3/2}.$$

The proofs use the 'isotropic' harmonic analysis associated with the tree; see, for example, Cartier [26], Letac [146] and the book by Figà-Talamanca and Picardello

[71]. On the other hand, Cartwright [27] shows that if one does not assume some moment condition or isotropy, then for every  $\lambda > 2$  one may find a probability measure  $\mu$  on  $\mathbb{F}_s$  ( $s \geq 2$ ) such that

$$\mu^{(n)}(o) \sim C\rho(P)^n n^{-\lambda}.$$

We briefly mention a few results concerning random walks on non-homogeneous trees. Gerl [83] and Woess [238] use continued fraction methods to study SRWs on *radial* trees: a tree  $T$  is radial if there is a root vertex  $o$  such that the vertex degrees depend only on the distance from  $o$ , that is, there is a sequence of positive integers  $d_n$  such that

$$\deg(x) = d_n, \quad \text{if } d(x, o) = n.$$

For example, if  $d_0 = 1$ ,  $d_1 = 2$  and  $d_n = k$  for  $n \geq 2$ , then for the SRW one has

$$p^{(2n)}(0, 0) \sim C_3 \cdot \left(\frac{8}{9}\right)^n \cdot n^{-3/2}, \quad \text{if } k = 3,$$

$$p^{(2n)}(0, 0) \sim C_4 \cdot \left(\frac{3}{4}\right)^n \cdot n^{-1/2}, \quad \text{if } k = 4,$$

$$p^{(2n)}(0, 0) \sim C_k \cdot \left(\frac{k-1}{2k-4}\right)^n, \quad \text{if } k \geq 5,$$

as  $n \rightarrow \infty$  (Gerl [83]).

The *natural spanning tree* of  $\mathbb{Z}^q$  is the one where the geodesic from the origin to vertex  $(k_1, \dots, k_q)$  consists of straight lines between the points  $(0, \dots, 0)$ ,  $(k_1, 0, \dots, 0)$ ,  $(k_1, k_2, 0, \dots, 0)$ , ...,  $(k_1, \dots, k_q)$ . Gerl [87] shows that the SRW on this tree satisfies

$$p^{(2n)}(0, 0) \sim C_q \cdot n^{-1+2^{-q}}, \quad \text{as } n \rightarrow \infty.$$

#### D. Random walks on free products

We now discuss random walks on a more general class of nonamenable groups. Let  $\Gamma_i$ ,  $i = 1, \dots, s$ , be finite or finitely generated groups with at least two elements, with common notation  $o$  for the identity element. The *free product*

$$\Gamma = \Gamma_1 * \Gamma_2 * \dots * \Gamma_s$$

is the group consisting of all elements which can be written as

$$x = x_1 x_2 \dots x_k, \quad \text{where } k \geq 0, \quad x_j \in \Gamma_{i_j} \setminus \{o\} \text{ for } j = 1, \dots, k \\ \text{and } i_{j+1} \neq i_j \text{ for all } j$$

(the identity  $o$  is obtained when  $k = 0$ ). Besides the group relations inherited from the  $\Gamma_i$ , no new ones are introduced. If  $A_i$  is a finite symmetric set of generators of  $\Gamma_i$  and  $X_i$  is the corresponding Cayley graph,  $i = 1, \dots, s$ , then the Cayley graph  $X$  of  $\Gamma$  with respect to  $A = A_1 \cup A_2 \cup \dots \cup A_s$  can be described as follows. Take countably many copies of each of the  $X_i$  to build up  $X$ : each vertex of  $X$  is the intersection of precisely one of each of the  $X_i$ . The copies are glued together in a tree-like way such that besides the ones in the copies, no new simple circuits are induced. (Analogously, one defines the free product of a countable number of groups, but this will not be discussed here.) Each  $\Gamma_i$  becomes a subgroup of  $\Gamma$ . For example, one can write  $\mathbb{F}_s = \mathbb{Z} * \dots * \mathbb{Z}$  ( $s$  times).

(FP) A probability measure on  $\Gamma$  is *adapted to the free product structure* if one has

$$\text{supp } \mu \subset \Gamma_1 \cup \dots \cup \Gamma_s.$$

Thus one can write  $\mu$  as a convex combination

$$\mu = a_1 \mu_1 + \dots + a_s \mu_s, \quad a_i > 0, \quad \sum_{i=1}^s a_i = 1,$$

where  $\text{supp } \mu_i \subset \Gamma_i$ ,  $i = 1, \dots, s$ . Extending the case of free groups, one can use a ‘combinatorial’ approach in order to describe the Green kernel of  $\mu$  on  $\Gamma$ .

For an irreducible probability measure  $\mu$  on an arbitrary group  $\Gamma$ , let

$$G_\mu(z) = G(z) = G(o, o | z) = \sum_{n=0}^{\infty} \mu^{(n)}(o) z^n$$

and

$$\theta = \theta_\mu = G_\mu(\rho(\mu)^{-1}) / \rho(\mu) \in (0, \infty].$$

Then one can find a function  $\Phi(t) = \Phi_\mu(t)$ , analytic in an open set which contains  $[0, \theta)$ , such that

$$G(z) = \Phi(zG(z))$$

for all  $z$  in an open set containing  $[0, r)$ .

**PROPOSITION 6.10.** *Let  $\Gamma = \Gamma_1 * \dots * \Gamma_s$  and  $\mu = \sum_i a_i \mu_i$  as above. Then one has the following.*

- (1)  $\theta_\mu \leq \bar{\theta} = \min\{\theta_{\mu_i}/a_i \mid i = 1, \dots, s\}$ , and  $\Phi_\mu$  is analytic in an open set containing the interval  $[0, \bar{\theta})$ .
- (2)  $\Phi_\mu(t) - 1 = \sum_{i=1}^s (\Phi_{\mu_i}(a_i t) - 1)$ .

Thus the implicit equation which describes  $G_\mu(z)$  can be obtained from those regarding the  $G_{\mu_i}(z)$ ,  $i = 1, \dots, s$ . This useful formula was first found independently and simultaneously by Woess [240] and by Soardi [199] and Cartwright and Soardi [31]. The version stated here is that of Woess [240]. A polished proof which unites the advantages of each of the two approaches can be found in Cartwright [27]. Independently, this was also found by McLaughlin [159] (who was interested in spectral theory) on the one hand and—in a much more general and rather different setting of  $C^*$ -algebras—by Voiculescu [233, 234]. Proposition 6.10 can be used to obtain local limit theorems in a large class of cases by combining the results of Woess [240] and Cartwright [29].

**THEOREM 6.11.** *Under the assumptions of Proposition 6.10, if  $\mu$  is aperiodic and  $\theta_\mu < \bar{\theta}$ , then*

$$\lim_{n \rightarrow \infty} \frac{\mu^{(n)}(x)}{\rho(\mu)^n n^{-3/2}} = \psi(x) > 0$$

for every  $x \in \Gamma$ .

Indeed, the hypotheses and Proposition 6.10 yield that  $\rho(\mu)^{-1}$  is a simple branching point of  $G_\mu(z)$ , and the absence of further singularities on the circle of convergence (Theorem 5.2(a)) allows the application of Darboux’s method (see, for example, Bender [16]). For the following, recall that we always assume that  $s \geq 2$  and that no  $\Gamma_i$  is trivial.

**COROLLARY 6.12.** *The conclusion of Theorem 6.11 holds if for each  $i$ ,  $\mu_i$  and  $\Gamma_i$  satisfy one of the following assumptions.*

- (1)  $\Gamma_i$  is finite. (In the case when  $|\Gamma_i| = 2$  for all  $i$ , one needs  $s \geq 3$ .)
- (2)  $\Gamma_i = \mathbb{Z}$ , and  $\mu_i$  has finite first moment,  $\bar{\mu}_i = 0$ .
- (3)  $\Gamma_i = \mathbb{Z}^q$ ,  $q \leq 4$ , and  $\mu_i$  has finite second moment,  $\bar{\mu}_i = 0$ .
- (4)  $\Gamma_i = \mathbb{Z}^q$ ,  $q \leq 4$ , and  $\mu_i$  is finitely supported.
- (5)  $\Gamma_i$  has polynomial growth with degree at most four, and  $\mu_i$  is finitely supported and symmetric.
- (6) All  $\Gamma_i$  and  $\mu_i$  are copies of the same group  $\Gamma_0$  and measure  $\mu_0$  on  $\Gamma_0$ , and  $s$  is sufficiently large.

Based on his improvement of the proofs, Cartwright [27, 28] has given interesting local limit theorems for particular classes of random walks related to the last of the above conditions.

If  $\Gamma_0 = \mathbb{Z}^q$ , then its *natural generators* are the integer vectors with Euclidean length one. If  $\Gamma = \mathbb{Z}^q * \dots * \mathbb{Z}^q$  ( $s$  times), then we consider the simple random walk on  $\Gamma$  to be the one whose law  $\mu$  is the equidistribution on the natural generators of all the factors.

**THEOREM 6.13.** *For the SRW on  $\Gamma = \mathbb{Z}^q * \dots * \mathbb{Z}^q$  ( $s \geq 2$  times), one has as  $n \rightarrow \infty$*

$$\begin{aligned} \mu^{(2n)}(o) &\sim C\rho(\mu)^{2n} n^{-3/2}, & \text{if } s \geq s_0(q), \text{ and} \\ \mu^{(2n)}(o) &\sim C\rho(\mu)^{2n} n^{-q/2}, & \text{if } 2 \leq s < s_0(q), \end{aligned}$$

where  $C = C_{q,s}$  is a positive constant and  $s_0(q)$  is an increasing function of  $q$  whose first values are

$$s_0(1) = s_0(2) = s_0(3) = s_0(4) = 2, \quad s_0(5) = 4, \quad s_0(6) = 6, \quad s_0(7) = 8, \quad s_0(8) = 11.$$

The proof of Cartwright [27] is based on showing that in the first case  $\theta_\mu < \bar{\theta}$ , while  $\theta_\mu = \bar{\theta}$  in the second. One could interpret this by saying that for a small number  $s$  of factors and high dimension  $q$ , the Euclidean structure of the factors has a strong influence on the behaviour of the SRW, whence the  $-q/2$  in the exponent on the right. Of course, Theorem 6.13 can be extended to a local limit theorem regarding all  $x \in \Gamma$ , but as  $\mu$  has period 2, one has to take into account the parities of  $n$  and  $x$ .

On the other hand, Cartwright [28] shows the following.

**THEOREM 6.14.** *Let  $\Gamma = \Gamma_1 * \Gamma_2$ , where none of the  $\Gamma_i$  is trivial and  $|\Gamma_i| \geq 3$  for at least one  $i$ . Then there is a symmetric probability  $\sigma$  on  $\Gamma$  with  $\text{supp } \sigma \subset \Gamma_1 \cup \Gamma_2$  such that*

$$\sigma^{(2n)}(o) \sim C\rho(\sigma)^{2n} n^{-3/2}, \quad \text{as } n \rightarrow \infty,$$

where  $C > 0$ .

Indeed, if  $\Gamma_1 = \Gamma_2 = \mathbb{Z}^q$ , then  $\text{supp } \sigma$  can be chosen as the set of natural generators. Many people believed that the following conjecture is true:

If  $\Gamma$  is a finitely generated group and  $\mu$  is some irreducible, finitely supported, symmetric probability measure on  $\Gamma$  such that

$$\mu^{(2n)}(o) \sim C\rho(\mu)^{2n} n^{-\lambda} \quad \text{as } n \rightarrow \infty,$$

where  $\lambda > 0$ , then this holds—with the same  $\lambda$ —for every other probability on  $\Gamma$  satisfying the same assumptions.

Combining Theorems 6.13 and 6.14, one sees that this conjecture is *false*. In order to underline it, we restate this surprising result.

**COROLLARY 6.15.** *Let  $\Gamma = \mathbb{Z}^q * \mathbb{Z}^q$ , where  $q \geq 5$ . Then for the simple random walk on  $\Gamma$  one has*

$$\mu^{(2n)}(o) \sim C \cdot \rho(\mu)^{2n} n^{-q/2} \quad \text{as } n \rightarrow \infty \quad (C > 0),$$

*while there is another symmetric probability measure  $\sigma$  on  $\Gamma$  with the same support, such that*

$$\sigma^{(2n)}(o) \sim C' \cdot \rho(\sigma)^{2n} n^{-3/2} \quad \text{as } n \rightarrow \infty \quad (C' > 0).$$

### E. Cartesian products

Let  $(X, P_1)$  and  $(Y, P_2)$  be two (irreducible) Markov chains. The *Cartesian product* of two graphs  $X, Y$  has vertex set  $X \times Y$ , and two pairs  $xy, x'y'$  are adjacent if  $x \sim x'$  and  $y = y'$ , or  $x = x'$  and  $y \sim y'$ . In accordance with this definition, the Cartesian product of  $P_1$  and  $P_2$ , indexed with parameter  $a$ ,  $0 < a < 1$ , is given by

$$Q_a = a \cdot P_1 + (1-a) \cdot P_2,$$

where  $P_1$  and  $P_2$  are extended to  $X \times Y$  in the natural way:

$$p_1(xy, x'y') = \begin{cases} p_1(x, x'), & \text{if } y = y', \\ 0, & \text{otherwise,} \end{cases} \quad p_2(xy, x'y') = \begin{cases} p_2(y, y'), & \text{if } x = x', \\ 0, & \text{otherwise.} \end{cases}$$

Cartwright and Soardi [32] prove the following simple and useful result for random walks on groups; it carries over immediately to the general case.

**PROPOSITION 6.16.** *If*

$$p_1^{(n)}(x, x') \sim C_1 \rho(P_1)^n n^{\lambda_1} \quad \text{and} \quad p_2^{(n)}(y, y') \sim C_2 \rho(P_2)^n n^{\lambda_2}$$

*( $C_1, C_2 > 0$ ), then*

$$q_a^{(n)}(xy, x'y') \sim C \rho(Q_a)^n n^{\lambda_1 + \lambda_2},$$

*as  $n \rightarrow \infty$  ( $C > 0$ ).*

### 7. Behaviour at infinity and harmonic functions

In this section we always consider an irreducible random walk  $(X, P)$  which is *transient*. In this case,  $Z_n \rightarrow \infty$  almost surely in the sense that for every  $x \in X$  and finite  $U \subset X$ ,

$$\Pr_x[Z_n \notin U \text{ for all but finitely many } n] = 1.$$

We would like to know more about the way in which  $Z_n$  tends to infinity, that is, we want to distinguish different limit points of  $Z_n$  'at infinity'. To formulate this more precisely, let us suppose that we have a compactification  $\bar{X}$  which is natural in some sense with respect to the given graph structure. By a compactification we always mean a compact Hausdorff space which contains (the vertex set of)  $X$  as a dense, discrete, open subset. Let  $\partial X = \bar{X} \setminus X$  denote the associated boundary. One can ask the following questions.

(i) Is  $\partial X$  suitable as a model for the points attained at infinity by  $Z_n$ , that is, is there a  $\partial X$ -valued random variable  $Z_\infty$  such that

$$\Pr_x[\lim_{n \rightarrow \infty} Z_n = Z_\infty] = 1 \quad \text{for all } x \in X?$$

(Convergence in the topology of  $\bar{X}$ .)

In terms of the trajectory space, this means the following. Let  $\Theta_x$  be the set of trajectories  $\xi = (x_n)_{n \geq 0}$  which converge in  $\bar{X}$  to some point of  $\partial X$ , and denote this limit by  $Z_\infty(\xi)$ . Then  $\Pr_x(\Theta_x) = 1$  for every  $x \in X$ .

Suppose the answer in (i) is positive. Then we can define the limiting distributions  $\nu_x$ ,  $x \in X$ , on  $\partial X$ : if  $B \subset \partial X$  is a Borel set, then

$$\nu_x(B) = \Pr_x[Z_\infty \in B].$$

The probability measures  $\nu_x$ ,  $x \in X$ , are mutually absolutely continuous (by irreducibility of  $(X, P)$ ) and satisfy

$$\nu_x = \sum_y p(x, y) \nu_y.$$

The family  $\nu = (\nu_x)_{x \in X}$  is called the *harmonic measure class* of  $P$  with respect to  $\partial X$ .

(ii) Is the *Dirichlet problem* for  $P$  solvable with respect to  $\bar{X}$ ? In probabilistic terms, this means that

$$\lim_{\substack{x \rightarrow \zeta \\ x \in X}} \nu_x = \delta_\zeta \quad \text{weakly for all } \zeta \in \partial X.$$

To formulate this in terms of harmonic functions, observe that if  $\phi \in L^\infty(\partial X, \nu)$ , then

$$h(x) = \int_{\partial X} \phi d\nu_x \quad \text{defines } h \in \mathcal{H}^\infty. \quad (**)$$

Now the Dirichlet problem can be stated as follows: does every  $\phi \in C(\partial X)$  (that is, continuous function) have a continuous extension to  $\bar{X}$  which is harmonic on  $X$ ? (If yes, then it is unique by the maximum principle.)

How good is our compactification? The more points we can distinguish at infinity, the better it is. The question is not yet precise.

(iii) Can every  $h \in \mathcal{H}^\infty$  be obtained as in (\*\*) for some  $\phi \in L^\infty(\partial X, \nu)$ ? In this case we say that  $(\partial X, \nu)$  is the *Poisson boundary*.

Note that the Poisson boundary (see below for a precise definition) is unique only up to measure theoretical isomorphism; compare with Kaimanovich and Vershik [115]. (Indeed, modifying  $\partial X$  by a piece of measure zero does not change its essential properties.)

(iv) Is the boundary big enough for representing all positive harmonic functions, that is, is  $\partial X$  the *Martin boundary* of  $(X, P)$ ?

(Of course,  $\partial X$  might also be larger than the Martin boundary.) We recall the construction of the latter, as introduced by Doob [59] and Hunt [107]. Fix a reference vertex  $o$  and define the Martin kernel

$$K(x, y) = \frac{G(x, y)}{G(o, y)} = \frac{F(x, y)}{F(o, y)}.$$

Then the *Martin compactification*  $\hat{X} = \hat{X}(P)$  is the (up to homeomorphism) unique minimal compactification of  $X$  such that  $K(\cdot, \cdot)$  extends continuously to  $X \times \hat{X}$ . The Martin boundary is  $\mathcal{M} = \mathcal{M}(P) = \hat{X} \setminus X$ . The extended kernels are still denoted by  $K(\cdot, \cdot)$ . The points of  $\mathcal{M}$  can be considered as the directions of convergence of the

Martin kernels in the second variable. For  $\zeta \in \mathcal{M}$ ,  $K(\cdot, \zeta)$  is often also called the Poisson kernel or Poisson–Martin kernel. We recall the basic features of  $\mathcal{M}$ ; see Kemeny, Snell and Knapp [126] or Dynkin [67].

- There is an  $\mathcal{M}$ -valued random variable  $\hat{Z}_\infty$  such that in the Martin topology

$$\Pr_x[\lim_{n \rightarrow \infty} Z_n = \hat{Z}_\infty] = 1 \quad \forall x \in X.$$

(That is,  $\hat{X}(P)$  fulfils the requirements of question (i).)

- If  $\hat{\nu} = (\hat{\nu}_x)_{x \in X}$  denotes the associated measure class of limiting distributions, then  $(\mathcal{M}, \hat{\nu})$  is the Poisson boundary.
- If  $h \in \mathcal{H}^+$  is minimal (that is,  $h(o) = 1$ , and if  $h \geq h'$ , where  $h' \in \mathcal{H}^+$ , then  $h'/h$  is constant), then  $h = K(\cdot, \zeta)$  for some  $\zeta \in \mathcal{M}$ .
- The *minimal Martin boundary*  $\mathcal{E} = \{\zeta \in \mathcal{M} \mid K(\cdot, \zeta) \text{ is minimal}\}$  is a Borel set, and for every  $h \in \mathcal{H}^+$  there is a unique Borel measure  $\nu^h$  on  $\mathcal{M}$  such that

$$\nu^h(\mathcal{M} \setminus \mathcal{E}) = 0 \quad \text{and} \quad h(x) = \int_{\mathcal{M}} K(x, \cdot) d\nu^h \quad \forall x \in X$$

(Poisson–Martin representation theorem).

We remark that the Dirichlet problem is trivially solvable for the one point compactification, but is not always solvable with respect to  $\mathcal{M}$ . In this sense, solvability of the Dirichlet problem does not necessarily imply that the given boundary is ‘good’ (big). In particular, a necessary, but not sufficient condition for solvability of the Dirichlet problem with respect to a given compactification  $\bar{X}$  is that  $\text{supp } \nu = \partial X$  for the harmonic measure on  $\partial X$ , that is, the whole boundary is active.

In several papers one can find the statement that the Martin boundary is the closure of the minimal boundary. This is wrong; see Subsection G below.

We now review various results giving answers to the above questions.

### A. Integer lattices, Abelian and nilpotent groups

The results in the lattice case were obtained by Spitzer [205] and Ney and Spitzer [167]; see also the preceding work of Doob, Snell and Williamson [60], Choquet and Deny [41] and Hennequin [105]. We first consider the lattice case when the mean displacement is zero.

**THEOREM 7.1.** *Let  $\mu$  be an (irreducible) probability measure on  $\mathbb{Z}^q$ ,  $q \geq 3$ , such that  $\bar{\mu} = 0$  and  $|\mu|_s < \infty$ , where  $s = 2$  for  $q = 3$  and  $q = 4$ , and  $s = q - 2$  for  $q \geq 5$ . Then the Martin compactification of  $\mathbb{Z}^q$  with respect to  $\mu$  is the one point compactification. In particular, all positive harmonic functions are constant.*

(Recall that for  $r = 1, 2$ , random walks satisfying the above hypotheses are recurrent.) When  $\bar{\mu} \neq 0$ , the situation is different. We shall be interested in the function

$$\Psi(c) = \sum_{x \in \mathbb{Z}^q} \mu(x) e^{c \cdot x}, \quad c \in \mathbb{R}^q$$

(where  $c \cdot x$  denotes scalar product) for  $c$  in the set

$$C_\mu = \{c \in \mathbb{R}^q \mid \Psi(c) \leq 1\}.$$

The latter is convex and, by irreducibility of  $\mu$ , bounded. We shall need good convergence properties of the series defining  $\Psi(c)$ . Thus we assume that

$$\sum_{x \in \mathbb{Z}^q} \mu(x) e^{t|x|} < \infty$$

for some  $t > \sup\{|c| \mid c \in C_\mu\}$ , and briefly say that  $\mu$  has finite exponential moment of sufficiently large order. By Doob, Snell and Williamson [60] and Choquet and Deny [41], even without any moment condition one always has that the minimal harmonic functions are precisely the functions  $h(x) = e^{c \cdot x}$ , where  $\Psi(c) = 1$ . The main results of Ney and Spitzer [167] are as follows.

**THEOREM 7.2.** *Let  $\mu$  be an (irreducible) probability measure on  $\mathbb{Z}^q$ ,  $q \geq 1$ , such that  $\bar{\mu} \neq 0$  and  $\mu$  has finite exponential moment of sufficiently large order.*

- (a) *The Martin boundary is the unit sphere  $\mathbb{S}_{q-1}$  in  $\mathbb{R}^q$ . The Martin topology of  $\hat{X}(\mu)$  is obtained by embedding  $\mathbb{Z}^q$  into the unit ball  $\mathbb{B}_q$  via the map*

$$x \mapsto \frac{x}{1 + |x|}$$

*and taking the closure.*

- (b) *If  $(y_n)$  is a sequence in  $\mathbb{Z}^q$  such that  $y_n/(1 + |y_n|) \rightarrow u \in \mathbb{S}_{q-1}$ , then  $K(x, y_n) \rightarrow e^{c \cdot x}$ , where  $c$  is the unique vector in  $\mathbb{R}^q$  such that  $\Psi(c) = 1$  and  $\nabla \Psi(c)$  is collinear with  $u$ .*
- (c) *In particular,*

$$\Pr_x \left[ \frac{Z_n}{|Z_n|} \rightarrow \frac{\bar{\mu}}{|\bar{\mu}|} \right] = 1 \quad \text{for all } x \in \mathbb{Z}^q.$$

Statement (c) follows, of course, from the law of large numbers. Thus, for every  $x$ ,  $\hat{\nu}_x$  is the point mass at  $\bar{\mu}/|\bar{\mu}|$ , all bounded harmonic functions are constant, and even when  $\bar{\mu} \neq 0$ , the one point compactification is enough to describe the Poisson boundary. However, note that (c) contains more information about the way in which  $Z_n \rightarrow \infty$ . On the other hand, the Dirichlet problem is not solvable with respect to  $\hat{X}(\mu)$ . Constantness of all bounded harmonic functions is true even without any moment condition; see Doob, Snell and Williamson [60].

As the function  $\Psi(c)$  is convex, for random walks on  $\mathbb{Z}$  there are at most two minimal harmonic functions. The results of Cartwright and Sawyer [30, §7] on trees contain an example of a random walk on  $\mathbb{Z}$  which has two minimal harmonic functions, but the whole Martin boundary has more than two points.

The results of Theorems 7.1 and 7.2 will generalize to random walks on groups having  $\mathbb{Z}^q$  as a subgroup of finite index: apparently, this could be done along the lines of Babillot [12], who considers the case of  $\mathbb{R}^q$ , but to the author's knowledge, nobody has carried out the details in the discrete case.

Of course, for an arbitrary Abelian group  $\Gamma$ , the minimal positive harmonic functions still are the positive exponentials  $\psi$  with  $\sum_x \mu(x) \psi(x) = 1$ .

For random walks on (discrete) *nilpotent* groups, the author does not know of results concerning the whole Martin compactification. However, the positive harmonic functions are known in terms of those on the Abelian group  $\Gamma/\Gamma'$ ; the following result is due to Margulis [156].

**THEOREM 7.3.** *Let  $\mu$  be an (irreducible) probability measure on the nilpotent group  $\Gamma$ . Then every positive harmonic function with respect to  $\mu$  is constant on the cosets of the commutator subgroup  $\Gamma'$  in  $\Gamma$ .*

Thus the minimal positive harmonic functions with respect to  $\mu$  are the exponentials on  $\Gamma$  which have integral one with respect to  $\mu$ . In particular, all bounded harmonic functions with respect to  $\mu$  are constant, that is, the Poisson boundary is trivial (the harmonic measure class is supported by one point). This had been proved previously by Dynkin and Maljutov [68]. Furthermore, if  $\mu$  is symmetric and finitely supported, then all positive harmonic functions are constant; compare also with Lyons and Sullivan [155].

### B. Entropy and the Poisson boundary

The pair  $(\mathcal{M}, \hat{\nu})$  is not the only realization of the Poisson boundary. In particular, for random walks on groups there is a great variety of equivalent ways to introduce the Poisson boundary; see the long list given by Kaimanovich and Vershik [115]. An important approach is due to Furstenberg [76]. Still another, more recent approach is due to Willis [237], who describes the Poisson boundary via an ideal in the group algebra associated with the law of the random walk. A powerful tool for answering question (iii) is provided by entropy criteria. For a probability measure  $\mu$  on a group  $\Gamma$ , its entropy is

$$H(\mu) = - \sum_{x \in \text{supp } \mu} \mu(x) \log \mu(x).$$

If  $H(\mu)$  is finite, then

$$h(\Gamma, \mu) = \lim_{n \rightarrow \infty} \frac{H(\mu^{(n)})}{n}$$

exists and is called the *entropy of the random walk* with law  $\mu$ .

**THEOREM 7.4.** *Let  $\mu$  be an (irreducible) probability measure on the group  $\Gamma$  satisfying  $H(\mu) < \infty$ . Then the Poisson boundary of the random walk with law  $\mu$  is trivial (all bounded harmonic functions are constant) if and only if  $h(\Gamma, \mu) = 0$ .*

The definition of  $h(\Gamma, \mu)$  is due to Avez [8, 11], who proved the ‘if’ part of the theorem. The completion is due to Kaimanovich and Vershik [114, 115] and, independently, to Derriennic [54]. As a corollary, Theorem 7.4 implies the following result, originally also due to Avez [10].

**COROLLARY 7.5.** *Let  $\mu$  be a finitely supported (irreducible) probability measure on a group  $\Gamma$  which has subexponential growth. Then the Poisson boundary of the random walk with law  $\mu$  is trivial.*

We also mention that triviality of the Poisson boundary (for irreducible  $\mu$ ) implies that  $\Gamma$  is amenable; see Avez [11] and Rosenblatt [188]. As a partial converse, Northshield [169] shows that for a finitely generated amenable group, the Martin boundary of any symmetric, irreducible random walk has a fixed point under the group action. In other words, the constant function 1 arises as a Martin kernel  $K(\cdot, \zeta)$  for some (not necessarily minimal!)  $\zeta \in \mathcal{M}$ .

For further results, extensions and examples concerning entropy and Poisson boundary, see Kaimanovich and Vershik [115], Derriennic [55] and the references given in these two substantial papers. In particular, Kaimanovich [108, 111] provides examples of random walks on amenable groups with nontrivial Poisson boundary: let  $\mathcal{G}_q$  be the group of all finitely supported functions  $\mathbb{Z}^q \rightarrow \{0, 1\}$  with addition modulo 2. Then  $\mathbb{Z}^q$  acts on  $\mathcal{G}_q$  by translations. The resulting semidirect product  $\mathbb{Z}^q \ltimes \mathcal{G}_q$  is finitely generated. If  $\mu$  is a probability measure on this group, let  $\tilde{\mu}$  denote its projection onto  $\mathbb{Z}^q$  (the first factor).

**THEOREM 7.6.** *Let  $\mu$  be an irreducible probability measure on  $\mathbb{Z}^q \ltimes \mathcal{G}_q$  with finite first moment. Then the Poisson boundary of the resulting random walk is nontrivial if and only if the random walk on  $\mathbb{Z}^q$  with law  $\tilde{\mu}$  is transient.*

Thus for  $q \geq 3$ , one always obtains nontrivial Poisson boundary. For generalizations and a variety of further interesting results concerning Poisson boundaries of random walks on solvable groups, see Kaimanovich [108, 111].

Ledrappier [143, 144] and Kaimanovich [109] use extensions of Theorem 7.4 to show that a given boundary of a random walk (in the sense of Furstenberg [76]) coincides with the Poisson boundary. Ledrappier considers discrete groups of real, invertible square matrices: for any irreducible probability measure with logarithmic moment, the Poisson boundary is the ‘natural’ one. As a corollary of his generalized entropy criterion, Kaimanovich [109] also gives a useful geometric criterion for recognizing the Poisson boundary.

**THEOREM 7.7.** *Let  $\Gamma$  be a finitely generated group with Cayley graph metric  $d$ , and let  $\mu$  be a probability measure on  $\Gamma$  with finite first moment with respect to  $d$ . Suppose that  $\partial\Gamma$  is a boundary for the associated random walk, that is, a  $\Gamma$ -space such that*

$$Z_n \rightarrow Z_\infty \in \partial\Gamma \quad \text{almost surely.}$$

*If there is a sequence of measurable ‘projections’  $\Pi_n: \partial\Gamma \rightarrow \Gamma$  such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} d(Z_n, \Pi_n(Z_\infty)) = 0 \quad \text{almost surely,}$$

*then  $\partial\Gamma$ , together with the associated harmonic measure, coincides with the Poisson boundary of the random walk.*

Note that  $\nu_x = \delta_x * \nu_0$  for the distribution of  $Z_\infty$  with respect to the initial point  $x$ , so that one usually speaks of  $(\partial\Gamma, \nu_0)$  as the Poisson boundary.

Given a finitely supported probability measure  $\mu$  on a group  $\Gamma$ , Bisch [20] explains how to define a natural lifting of  $\mu$  to a probability measure  $\tilde{\mu}$  on a free group  $\mathbb{F}_g$ ;  $\tilde{\mu}$  is supported by elements of length  $\leq 1$ . Bisch then shows (among other results) that

$$h(\Gamma, \mu) \leq h(\mathbb{F}_g, \tilde{\mu}),$$

and that equality holds if and only if  $\Gamma = \mathbb{F}_g$ .

As the articles of Kaimanovich and Vershik [115] and Derriennic [55] contain excellent and very complete introductions to the theory of entropy and Poisson

boundary, we do not go into further detail here. Also, we point out that a very detailed study of measure theoretic boundaries (Poisson boundary and tail boundary) has been carried out recently in a long article by Kaimanovich [113] for general Markov chains, also giving various examples.

### C. Trees

We suppose that our graph  $X = T$  is an infinite tree. Recall the observations concerning the geometry of  $T$  made in Section 4.C. We give the definition of the appropriate boundary. An *infinite path* in  $T$  is a one-sided infinite sequence of successively adjacent vertices without repetitions. Two infinite paths are said to be *equivalent* if they differ only by finitely many vertices. An *end* of  $T$  is an equivalence class of infinite paths. The set of all ends is denoted by  $\Omega$ . If  $x \in T$  and  $\omega \in \Omega$ , then there is a unique infinite path which represents  $\omega$  and starts at  $x$ . This is the *geodesic ray*  $x\omega$ . The definition of the confluent with respect to reference vertex  $o \in T$  carries over to  $T \cup \Omega$ : if  $\eta, \zeta \in T \cup \Omega$ , then  $c(\eta, \zeta)$  is the last common vertex on  $\overline{o\eta}$  and  $\overline{o\zeta}$ , if  $\eta \neq \zeta$  (while  $c(\omega, \omega) = \omega$  for  $\omega \in \Omega$ ), and  $(\eta | \zeta) = d(o, c(\eta, \zeta))$ . The *end topology* is discrete on  $T$ , while a neighbourhood base at  $\omega \in \Omega$  is given by the sets

$$\{\zeta \in T \cup \Omega \mid (\zeta | \omega) \geq n\}, \quad n = 1, 2, \dots$$

This topology is independent of the choice of  $o$ ; it is induced by the metric

$$\theta(\eta, \zeta) = \begin{cases} 0, & \text{if } \zeta = \eta, \\ e^{-(\eta | \zeta)}, & \text{otherwise.} \end{cases}$$

Thus the end compactification  $T \cup \Omega$  becomes a totally disconnected Hausdorff space. We first state the main result of the influential article by Cartier [25].

**THEOREM 7.8.** *Let  $(Z_n)$  be a transient random walk on  $T$  satisfying (NN). Then the Martin compactification coincides with the end compactification. The extended Martin kernels are locally constant and given by*

$$K(x, \omega) = K(x, c), \quad \text{where } c = c(x, \omega).$$

*Furthermore, all functions  $K(\cdot, \omega)$  ( $\omega \in \Omega$ ) are minimal harmonic.*

A particular instance is that of the free group  $\mathbb{F}_s$ . As observed in Section 5, its Cayley graph with respect to the free generators and their inverses is a homogeneous tree. Its ends can be represented by *infinite reduced words* in the free generators and their inverses. In this context, the Martin boundary for nearest neighbour random walks on  $\mathbb{F}_s$  (which may be viewed as a special case of Theorem 7.8) had been determined previously by Dynkin and Malyutov [68]. Derriennic [53] has generalized the result of Dynkin and Malyutov to arbitrary (irreducible) random walks on  $\mathbb{F}_s$  with finitely supported law  $\mu$ . Extending Derriennic's method, Picardello and Woess [179] have proved the following generalization of Theorem 7.8 (which also applies to Derriennic's situation).

**THEOREM 7.9.** *Let  $(Z_n)$  be a transient random walk on a tree  $T$  satisfying (UI) and (BR). Then the Martin compactification coincides with the end compactification, and all kernels  $K(\cdot, \omega)$  ( $\omega \in \Omega$ ) are minimal harmonic functions.*

Note that Picardello and Woess [179] give a slightly more restrictive definition of uniform irreducibility, but nothing changes with the definition of (UI) adopted here. We remark that the assumptions of Theorem 7.9 require that  $T$  has bounded vertex degrees, which is not necessary in the nearest neighbour case of Theorem 7.8. Indeed, the results of Theorem 7.8 carry over even to trees which are not locally finite, once the end compactification has been redefined properly (see Cartwright, Soardi and Woess [34]), but here we do not go into details concerning non-locally finite graphs.

For random walks on locally finite trees which do not have bounded range, the situation is more complicated. Cartwright and Sawyer [30] have proved the following.

**THEOREM 7.10.** *Let  $T = T_q$  ( $q \geq 3$ ) be a homogeneous tree and let  $P$  be the transition matrix of an isotropic random walk having finite first moment, that is,*

$$\sum_{x \in T} d(x, o) p(o, x) < \infty.$$

*Then the Martin compactification coincides with the end compactification, and the Martin kernels  $K(\cdot, \omega)$  ( $\omega \in \Omega$ ) are precisely those of the simple random walk on  $T$ .*

The latter kernels are, of course, well known and easy to calculate; see, for example, Cartier [26]. Theorem 7.10 generalizes an earlier result of Sawyer [192], where finiteness of the second moment is assumed. Cartwright and Sawyer [30] also have other conditions which assure that the Martin boundary of an isotropic random walk on  $T_q$  is the space of ends. On the other hand, they also have examples where the Martin boundary factors nontrivially over  $\Omega$ . In any case, for an arbitrary isotropic random walk on  $T_q$ , the positive harmonic functions coincide with those of the simple random walk. As a consequence, the minimal Martin boundary is  $\Omega$ , the random walk converges in the end topology, and the Dirichlet problem is solvable with respect to  $\Omega$ .

This does not hold only in the isotropic case. Questions (i) and (ii) have positive answers on arbitrary graphs under rather general assumptions with respect to the end compactification; see Subsection D below. In particular, the positive answers to (i) and (ii) given by Derriennic [53] for finite range random walks on  $\mathbb{F}_s$  carry over to arbitrary (irreducible) random walks on free groups; see Cartwright and Soardi [33] and Woess [244].

By Derriennic [53], question (iii) has a positive answer for finite range random walks on  $\mathbb{F}_s$ , when considering the space of ends. For arbitrary (irreducible) isotropic random walks on  $T_q$  ( $q \geq 3$ ), it also follows from the above results of Cartwright and Sawyer [30] that the Poisson boundary coincides with the space of ends. For not necessarily isotropic random walks on free groups, there is an intermediate result due to Kaimanovich [109], using the criterion of Theorem 7.7.

**THEOREM 7.11.** *Suppose that  $\Gamma$  has  $\mathbb{F}_s$  ( $s \geq 2$ ) as a subgroup with finite index. If  $\mu$  is an (irreducible) probability measure on  $\Gamma$  which has finite first moment, then the Poisson boundary coincides with the space of ends ( $\equiv$  infinite words of the free subgroup).*

Note that Theorem 7 of Kaimanovich [109] appears to be formulated for arbitrary free products. However, without additional effort, it applies only to free products of

finite or cyclic groups. These have a free subgroup with finite index; see Karrass, Pietrowski and Solitar [120]. On the other hand, the generalization to arbitrary groups having a finite index free subgroup is immediate.

We finally mention a few other results concerning harmonic functions on trees.

The papers of Cartier [25], Derriennic [53], Korányi and Picardello [137] and Picardello and Woess [179] give—under the respective assumptions—a (deterministic) Fatou theorem concerning boundary convergence of positive harmonic functions.

For a nearest neighbour random walk on  $\mathbb{Z}$ , it is easy to see that in the transient case every (not necessarily positive) harmonic function is a linear combination of the two Martin kernels at  $+\infty$  and  $-\infty$ . (In the recurrent case, we obtain all functions  $k \mapsto ak + b$ ,  $a, b \in \mathbb{R}$ .) This generalizes to transient nearest neighbour random walks on trees; see Cartier [25]: recall that the Martin kernels  $K(x, \cdot)$  are locally finite in the topology of  $\Omega$ . It turns out that every harmonic function has a unique integral representation with respect to a distribution on  $\Omega$ , that is, a linear functional on the space of locally constant functions. Various other proofs are available; see, in particular, Korányi, Picardello and Taibleson [138] and its appendix. A generalization to certain tree-like graphs is due to Picardello, Taibleson and Woess [178].

#### D. Hyperbolic graphs

A graph is called *hyperbolic* if the metric space  $(X, d)$  is hyperbolic in the sense of Gromov [98]. We give a brief outline: with respect to the reference vertex  $o$ , define for  $x, y \in X$

$$(x|y) = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)).$$

(In the case of a tree, this is the same quantity as defined above.) We say that  $X$  is hyperbolic if there is a  $\delta \geq 0$  (possibly large) such that for every  $x, w, y \in X$ ,

$$(x|y) \geq \min\{(x|w), (w|y)\} - \delta.$$

An equivalent, more suggestive definition is as follows: if  $x, y, w \in X$ , then a geodesic triangle  $xyw$  consists of three ‘geodesics’ (shortest paths—in general, these are not unique)  $\overline{xy}$ ,  $\overline{yw}$ ,  $\overline{wx}$ . The graph  $X$  is hyperbolic if there is a  $\delta' \geq 0$  (possibly large) such that for every geodesic triangle  $xyw$ , the following holds: for every vertex  $a \in \overline{xy}$  there exists  $b \in \overline{yw} \cup \overline{wx}$  such that  $d(a, b) \leq \delta'$ . (‘Hyperbolic triangles are  $\delta'$ -thin.’)

There is an extensive literature on hyperbolic metric spaces, in particular hyperbolic groups; see, for example, Gromov [98] and Ghys and de la Harpe [91]. A hyperbolic graph has a natural compactification: if  $a > 0$  is sufficiently small, then

$$\rho_a(x, y) = e^{-a(x|y)}, \quad x, y \in X,$$

is ‘almost’ a metric, that is,

$$\theta_a(x, y) = \inf \left\{ \sum_{i=1}^n \rho_a(x_{i-1}, x_i) \mid n \geq 0, x = x_0, x_1, \dots, x_n = y \in X \right\}$$

is a metric, and  $C \cdot \rho_a \leq \theta_a \leq \rho_a$  for some constant  $C > 0$ . If  $X$  is locally finite, then the completion  $\bar{X}$  of  $X$  with respect to  $\theta_a$  is compact, and the topology is independent of the particular choice of  $a$  (sufficiently small) and the base point  $o$ . This is the *hyperbolic compactification*, and  $\partial X = \bar{X} \setminus X$  is the *hyperbolic boundary* of  $X$ .

Typical examples are trees (they are hyperbolic with  $\delta = 0$ , and  $\partial X = \Omega$ , the space of ends), the Cayley graphs of finitely generated Fuchsian groups (if the group has no

parabolic elements, then the hyperbolic boundary coincides with the limit set), the Cayley graphs of many small cancellation groups (see Gromov [98]), or graphs arising from hyperbolic tessellations of the Poincaré plane (see Ancona [4]). The following significant result is due to Ancona [4].

**THEOREM 7.12.** *Suppose that  $(X, P)$  is such that  $X$  is hyperbolic and  $P$  satisfies (UI), (BR) and  $\rho(P) < 1$ . Then one has the following.*

- (a) *The Martin compactification coincides with the hyperbolic compactification, and all the extended kernels  $K(\cdot, \zeta)$ ,  $\zeta \in \partial X$ , are minimal harmonic. In particular, the random walk converges to  $\partial X$  almost surely.*
- (b) *If, in addition, the Green kernel vanishes at infinity, then the Dirichlet problem is solvable with respect to  $\partial X$ .*

The last condition means that for every  $\varepsilon > 0$ , there is a finite  $U \subset X$  such that  $G(x, o) < \varepsilon$  for all  $x \in X \setminus U$ . This holds in particular when  $P$  satisfies (UI), (BR) and (SR) and  $X$  satisfies (IS) (see Gerl [88] or Ancona [4]), or even when  $X$  satisfies  $(IS_q)$  for some  $q > 2$  (compare with Varopoulos [223]). Theorem 7.12 applies in particular to finite range random walks on non-exceptional Fuchsian groups: this case had been studied previously by Series [196]. We remark here that Series [196] also proves that the Poisson boundary arises in a natural way from a subshift of finite type. Theorem 7.12 also generalizes partially the result of Theorem 7.9 concerning trees, where, however, the hypothesis  $\rho(P) < 1$  is not needed. When  $P$  does not have bounded range, then less is known. The following is due to Kaimanovich and Woess [116].

**THEOREM 7.13.** *Assume that  $(X, P)$  is such that  $X$  is hyperbolic and  $P$  satisfies (UI) and (UM). Then the following hold.*

- (a) *If  $\rho(P) < 1$ , then the random walk converges almost surely to  $\partial X$ .*
- (b) *If there are  $\lambda < 1$  and  $M > 0$  such that*

$$p^{(n)}(x, y) \leq M\lambda^n \quad \forall x, y \in X, n \in \mathbb{N},$$

*then the Dirichlet problem is solvable with respect to  $\partial X$ .*

In particular, the condition in (b) is satisfied when (IS) and (SR) hold (Theorem 3.3). We also remark that for a random walk with law  $\mu$  on a hyperbolic group  $\Gamma$  (that is, the Cayley graphs of  $\Gamma$  are hyperbolic in the above sense), the results of Theorem 7.13 remain true under the following weaker assumptions:  $|\partial\Gamma| > 2$  (this is equivalent with  $|\partial\Gamma| = \infty$  and with  $\Gamma$  being nonamenable; see Gromov [98] and Ghys and de la Harpe [91]), and  $\mu$  is arbitrary irreducible (no moment condition is needed). This is proved by the same method as used by Cartwright and Soardi [33] for groups acting on trees, based on the work of Furstenberg [76]. The method applies to a whole class of compactifications of a certain type; see Woess [247].

### E. Planar graphs

We mention two recent papers by Northshield [168, 170] concerning SRWs on planar graphs. The first concerns nontriviality of the Poisson boundary: if the Green function is rapidly decreasing, then there are nonconstant bounded harmonic functions.

The methods are then refined in the second paper. Let  $X$  be a connected, infinite planar graph with bounded vertex degrees. Suppose that (i) the graph satisfies (IS), and (ii) every circuit (finite closed path in  $X$ ) surrounds only finitely many vertices.

For such a graph, Northshield constructs a boundary  $\partial X$  by the use of geodesics: a *geodesic ray*  $\alpha = [x_0, x_1, \dots]$  is an infinite path in  $X$  such that  $d(x_i, x_j) = |i - j|$ . Let  $\mathcal{G}$  denote the set of all geodesic rays starting at  $x_0 = o$ , our reference vertex. Let  $\alpha, \beta \in \mathcal{G}$  have only finitely many vertices in common. Delete their initial pieces until the last common points. We are left with two rays,  $\alpha', \beta'$ , which meet only in the first point. Their union cuts  $X$  into two *sectors*,  $(\alpha, \beta)$  and  $(\beta, \alpha)$ . (Think of first following  $\alpha'$  from infinity in the inverse order and then  $\beta'$  out to infinity. This subdivides  $X$  into a right-hand side— $(\alpha, \beta)$ —and a left-hand side— $(\beta, \alpha)$ .) If  $\alpha$  and  $\beta$  meet infinitely often, then define  $(\alpha, \beta) = (\beta, \alpha) = X$ . Recall that a set  $A \subset X$  is called *transient* if

$$\Pr_x[Z_n \in A \text{ infinitely often}] = 0 \quad \text{for all } x.$$

Under the above assumptions, every finite union of rays is transient (Northshield [168]). Thus, given  $\alpha, \beta \in \mathcal{G}$ , with probability one the SRW trajectories have to be absorbed in one of the two sectors. Define

$$\alpha \sim \beta \Leftrightarrow \Pr_o[Z_n \text{ is absorbed in } (\alpha, \beta)] = 1 \text{ or } \Pr_o[Z_n \text{ is absorbed in } (\beta, \alpha)] = 1.$$

This is an equivalence relation. Set  $\partial X = \mathcal{G} / \sim$ , the *circle boundary*. Then  $\bar{X} = X \cup \partial X$  carries a natural topology, the *cone topology*; compare, for example, with Anderson and Schoen [6]. Thus, one obtains a compactification of  $X$ . Note that this compactification is not purely ‘geometric’, as it relies on certain absorbing sets for the SRW. The main results of Northshield [170] are as follows.

- THEOREM 7.14.** (1)  $\partial X$  is homeomorphic with a circle.  
 (2) The SRW converges almost surely to a  $\partial X$ -valued random variable.  
 (3) The corresponding harmonic measure class carries no positive point mass, and the Dirichlet problem is solvable with respect to  $\bar{X}$ .

For a homogeneous tree, the space of ends is a Cantor set. There is a natural continuous surjection of  $\Omega$  onto the circle boundary which is one-to-one with the exception of a countable set of points, where it is two-to-one. For planar hyperbolic graphs, the circle boundary with the corresponding harmonic measure class coincides with the Poisson boundary. For a triangulation of the plane satisfying (IS),  $\bar{X}$  is the Martin compactification. Compare this with the results of DeBaun [50] on Dirichlet finite harmonic functions.

## F. Ends of graphs and groups

The end compactification can be defined for an arbitrary graph  $X$ ; see Freudenthal [74] and Halin [103]. Once again, an end is an equivalence class of one-sided infinite paths: two such paths are equivalent if there is a third one which meets each of the two infinitely often. Again, the set of ends is denoted by  $\Omega$ . The end topology of  $X \cup \Omega$  arises as follows: if  $U \subset X$  is finite, then  $X \setminus U$  decomposes into finitely many

connected components. If  $\omega \in \Omega$ , then precisely one of these components is such that every path representing  $\omega$  has all but finitely many vertices in that component:  $\omega$  ends up in this component. Given  $U$  and a component of  $X \setminus U$ , augment the latter by all  $\omega$  which end up in it. Thus if  $\omega \in \Omega$ , then there is precisely one augmented component, denoted by  $C(U, \omega)$ , which contains  $\omega$ . The family of all  $C(U, \omega)$ ,  $U \subset X$  finite, is a neighbourhood base at  $\omega$ . On  $X$  itself, the topology is discrete. Thus  $X \cup \Omega$  becomes a totally disconnected Hausdorff space, and  $\Omega$  is compact. In the case of a tree, this coincides with the construction described in Subsection C above.

We say that  $\omega \in \Omega$  has *finite diameter* if there is a neighbourhood base of the form  $\{C(U_i, \omega) \mid i \in I\}$ , such that  $\text{diam}(U_i) \leq K < \infty$  for all  $i$ . The set of all ends with finite diameter is denoted by  $\Omega_0$ .

If  $X$  is a hyperbolic graph, then it is easy to see that  $\text{id}_X$  extends to a continuous surjection  $\bar{X} \rightarrow X \cup \Omega$ , which maps the hyperbolic boundary onto  $\Omega$ . However, in general even a group with infinitely many ends is far from being hyperbolic. We also remark here that two graphs which are metrically equivalent have the same ( $\equiv$  homeomorphic) end compactifications; see, for example, Freudenthal [74]. (Local finiteness is important here!) In particular, if  $X^k$  is the graph with the same vertex set as  $X$ , and  $x \sim y$  in  $X^k$  if and only if  $1 \leq d(x, y) \leq k$ , then  $X^k$  has the same end compactification as  $X$ . If  $(X, P)$  satisfies (BR), then for sufficiently large  $k$ ,  $(X^k, P)$  will satisfy (NN). Thus when studying the interplay between the end structure of  $X$  and the behaviour of a random walk with bounded range, it is no loss of generality to assume (NN).

**THEOREM 7.15.** *Suppose that  $(Z_n)$  is a transient random walk with (BR) on  $X$ .*

- (a)  $Z_n$  converges almost surely in the end topology to some random end.
- (b) If  $|\Omega| \geq 2$ , then the Dirichlet problem is solvable with respect to  $\Omega$  if and only if the Green kernel vanishes at infinity.
- (c)  $\text{id}_X$  extends to a continuous surjection  $\tau: \hat{X}(P) \rightarrow X \cup \Omega$  such that  $\tau(\mathcal{M}) = \Omega$ .

Statement (a) is elementary, and also (b) is rather easy to prove: for (NN) random walks on trees, see Benjamini and Peres [18]; independently, an analogous proof for the general case has been given by Cartwright, Soardi and Woess [34], which applies even to (NN) random walks on non-locally finite graphs. In particular, (b) applies to finite range random walks on  $\mathbb{F}_s$ ,  $s \geq 2$ : indeed, in this case, the Green kernel vanishes at infinity because of nonamenability. For other proofs, see Derriennic [53] or Woess [244].

Statement (c) is proved under slightly more restrictive assumptions by Picardello and Woess [180]; see also Cartwright, Soardi and Woess [34] and the results for harmonic sheaves of Taylor [211]. Thus the Martin boundary is always ‘larger’ than  $\Omega$ , and we would like to know for which  $\omega \in \Omega$  the preimage  $\tau^{-1}(\omega)$  consists of only one point in  $\mathcal{M}$ . The method of the proof of Theorem 7.9 yields the following.

**PROPOSITION 7.16.** *Let  $(X, P)$  be transient, and assume that (UI) and (BR) hold. If  $\omega \in \Omega_0$ , then  $|\tau^{-1}(\omega)| = 1$ , that is,  $\omega$  can be considered as a point of  $\mathcal{M}$ . In addition,  $K(\cdot, \omega)$  is minimal harmonic.*

For more general conditions on  $\omega$  and  $P$  under which Proposition 7.16 remains true, see Picardello and Woess [180]. Compare also with the method of ‘ $\phi$ -chains’ of Ancona [4], where  $\rho(P) < 1$  is needed in addition.

If  $(Z_n)$  does not have bounded range, then one needs other conditions relating graph structure and transition matrix. Good answers to questions (i)–(iii) are available in the group-invariant case, but before discussing these, we state the following analogue of Theorem 7.13, also due to Kaimanovich and Woess [116], which does not need any group-invariance.

**THEOREM 7.17.** *Assume that (UI) and (UM) hold for  $P$ .*

- (a) *If  $\rho(P) < 1$ , then the random walk converges almost surely to a random end.*
- (b) *If there are  $\lambda < 1$  and  $M > 0$  such that*

$$p^{(n)}(x, y) \leq M\lambda^n \quad \forall x, y \in X, n \in \mathbb{N},$$

*then the Dirichlet problem is solvable with respect to  $\Omega$ .*

Note that for nonhyperbolic graphs, Theorem 7.17 cannot be interpreted as a consequence of Theorem 7.13.

Now consider an infinite, finitely generated group  $\Gamma$ . As any two Cayley graphs of  $\Gamma$  (with respect to finite sets of generators) are metrically equivalent, the end compactification of  $\Gamma$  does not depend on the particular Cayley graph. For an algebraic approach to ends of groups, see, for example, Stallings [206]. By Freudenthal [74],  $\Gamma$  has one, two or infinitely many ends. The case of one end is not suitable for our considerations (the end compactification does not contain enough structural information).

If  $\Gamma$  has two ends, then it has  $\mathbb{Z}$  as a subgroup with finite index. In particular, its Cayley graphs are metrically equivalent with a two-ended tree, and one can use (for example) Theorem 7.9 to see that for a transient random walk on  $\Gamma$  whose law has finite support, the Martin boundary consists of two points (both are minimal).

If  $\Gamma$  has infinitely many ends, then it contains  $\mathbb{F}_2$  as a subgroup. (There are several ways to verify this, for example by the ‘Ping-pong-lemma’; see, for example, de la Harpe [51].) In particular,  $\Gamma$  is nonamenable, and  $\rho(\mu) < 1$  for every irreducible probability measure on  $\Gamma$ .

**THEOREM 7.18.** *Assume that  $\Gamma$  is a finitely generated group with infinitely many ends, and that  $\mu$  is an (irreducible) probability measure on  $\Gamma$ .*

- (a) *The random walk with law  $\mu$  converges almost surely to a random end with finite diameter.*
- (b) *The Dirichlet problem for  $\mu$  is solvable with respect to  $\Omega$ .*
- (c) *If  $\text{supp } \mu$  is finite, then  $\Omega$  together with the harmonic measure class  $\nu$  (whose support is the whole of  $\Omega$ ) is a realization of the Poisson boundary of the random walk.*

These results arise in Woess [244] as a special case of a detailed study of random walks on graphs with infinitely many ends, whose transition probabilities are invariant under a vertex-transitive group of automorphisms. Statements (a) and (b) are derived by combining structural features of these graphs with a fruitful method of Furstenberg [75, 76]. In a graph theoretical setting, this method has first been applied to trees by Cartwright and Soardi [33]. For vertex-transitive graphs, the important property which is sufficient for (a) and (b) to hold as above is that the group which leaves the transition probabilities invariant (and acts transitively) be nonamenable, or,

equivalently, does not fix an end (see Soardi and Woess [202] for the latter equivalence). Statement (c) holds for graphs even if this group does fix an end, but then it may (but does not have to) happen that the support of  $\nu$  is just this end.

In the situation of Theorem 7.18(c), it is in general not true that  $\Omega$  coincides with the Martin boundary. In view of Proposition 7.16, this will be true in the case when all ends have finite diameter. It turns out that the finitely generated groups enjoying this property (that is,  $\Omega_0 = \Omega$ ) are precisely the groups which have a free subgroup with finite index; see Woess [243]. In other cases, the Martin boundary of a finite range random walk on a group  $\Gamma$  with infinitely many ends may be much larger than  $\Omega$ . The particular case when  $\Gamma = \Gamma_1 * \Gamma_2$  and  $\mu = a_1\mu_1 + a_2\mu_2$ , a convex combination with  $\text{supp } \mu_i \subset \Gamma_i$ , has been studied in detail by Woess [241]. Using the result of Proposition 6.10, the Martin compactification of  $(\Gamma, \mu)$  can be described completely in terms of the boundaries associated with  $\mu_i$  on  $\Gamma_i$ . (Woess [241] assumes finite support, but this is not relevant.)

### G. Cartesian products

Picardello and Sjögren [177] describe the minimal Martin boundary of the simple random walk on the Cartesian product of two homogeneous trees and derive some information about the topology of the Martin compactification. This is used to prove a (deterministic) Fatou type convergence theorem for certain positive harmonic functions.

Picardello and Woess [181] give a general description of the minimal Martin boundary of the Cartesian product of two arbitrary irreducible Markov chains in terms of the factors: consider  $(X, P_1)$  and  $(Y, P_2)$  and  $Q_a = a \cdot P_1 + (1-a) \cdot P_2$  on  $X \times Y$  as in Section 6.E. Then a positive function on  $X \times Y$  is minimal  $t$ -harmonic for  $Q_a$  if and only if it can be written as  $h_1(x)h_2(y)$ , where  $h_1$  is minimal in  $\mathcal{H}_t^+(X, P_1)$  and  $h_2$  is minimal in  $\mathcal{H}_{t_2}^+(Y, P_2)$ , with  $a \cdot t_1 + (1-a) \cdot t_2 = t$ . Previously, Molchanov [163, 164] had proved the analogous ‘only if’ part for the tensor product (direct product) of  $P_1$  and  $P_2$ .

A general understanding of the topology of the complete Martin compactification of a Cartesian product just in terms of the factors seems to be rather difficult, but some evidence is obtained (in particular for random walks on nonamenable groups) by Picardello and Woess [181]. At least partial understanding is easier when the factors have *stable Martin boundaries*—a notion introduced by Picardello and Woess [182]. In particular, Cartesian products may serve to construct simple examples where the Martin boundary contains nonminimal elements [181]: this will happen for Cartesian products of finite range random walks on infinite groups, if one of the factors is nonamenable.

The SRW on the Cartesian product of two homogeneous trees with degree  $\geq 3$  is such an example: in this case, the full Martin compactification has been computed by Picardello and Woess [183]. The minimal part is closed, but not dense in the Martin boundary.

### H. The Martin boundary for recurrent random walks

For a recurrent random walk  $(X, P)$ , all positive superharmonic functions are constant and hence it makes no sense asking for a compactification which may serve for an integral representation of all positive harmonic functions. Also, with

probability one the random walk will *not* tend to infinity. However, it still is of interest to study all positive functions  $h: X \rightarrow \mathbb{R}$  satisfying

$$\sum_{y \neq o} p(x, y) h(y) = h(x) \quad \text{for all } x \neq o, \quad (+)$$

where  $o$  is a reference point. This amounts to studying the positive harmonic functions of the transient Markov chain whose (substochastic) transition matrix is the restriction of  $P$  to  $X \setminus \{o\}$ . Thus one defines for  $x, y \in X$

$${}^o p(x, y) = p(x, y), \quad \text{if } y \neq o, \quad {}^o p(x, o) = 0,$$

$${}^o G(x, y) = \sum_{n=0}^{\infty} {}^o p^{(n)}(x, y) \quad \text{and}$$

$$K(x, y) = {}^o G(x, y) / {}^o G(o, y).$$

(We remark that  ${}^o p^{(n)}$  means  $({}^o p)^{(n)}$ .) Note that

$${}^o G(x, y) = E_x[\text{number of visits of } Z_n \text{ to } y \text{ for } n = 0, \dots, s^o],$$

where  $s^o$  is the instant of first visit to  $o$ .

The Martin compactification of the recurrent Markov chain  $(X, P)$  is then defined via the kernel  $K(\cdot, \cdot)$  precisely as in the transient case (see above). The resulting boundary  $\mathcal{M}$  is the Martin ('exit') boundary of  $(X, P)$ . For an account of the theory, see Kemeny, Snell and Knapp [126] or the—easier to read—original papers of Kemeny and Snell [125] and Orey [171]. We remark that most authors considered the Martin *entrance* boundary (indeed, this has more probabilistic significance, since there is no 'exit' in the recurrent case): it coincides with the exit boundary of the reverse chain. The latter is given by

$$\hat{p}(x, y) = m(y) p(y, x) / m(x),$$

where  $m(\cdot)$  is the stationary measure of  $(X, P)$ , by recurrence unique up to multiplication with constants. The Martin compactification does not depend on the base point  $o$ , that is, different base points give rise to canonically homeomorphic compactifications (the Markov chain is irreducible!).

A recurrent Markov chain is called *normal* if the series

$$A(x, y) = \sum_{n=0}^{\infty} (p^{(n)}(y, y) - p^{(n)}(x, y)) \quad \text{and} \quad B(x, y) = \sum_{n=0}^{\infty} \left( \frac{m(y)}{m(x)} p^{(n)}(x, x) - p^{(n)}(x, y) \right)$$

converge for every  $x, y \in X$ . In this case,  $A$  and  $B$  may serve as potential kernels; see Kemeny and Snell [125], Orey [171] and the references given there.

Normality and recurrent boundary theory for random walks on groups were studied in the 1960s by Kesten and Spitzer [134] and Kesten [131]; see also Spitzer [205]. Kesten has settled the problem completely (up to the minor question of whether infinite boundary might occur—after Varopoulos' result only possible for non-finitely generated groups). After that, to the author's knowledge not much further work on recurrent boundary theory has been done. We state the main results, restricting attention to finitely generated groups. Note that for groups,  $A(x, y)$  and  $B(x, y)$  must coincide, as  $m(\cdot)$  is the counting measure. Writing  $a(x) = A(o, x)$ , we have  $A(x, y) = a(x^{-1}y)$ . Below,  $d(\cdot, \cdot)$  denotes the Cayley graph metric with respect to a finite set of generators.

**THEOREM 7.19.** *Let  $\mu$  be an irreducible probability measure on the infinite, finitely generated group  $\Gamma$ , giving rise to a recurrent random walk. Then the random walk is normal, and the potential kernel is nonnegative and satisfies the convolution equation*

$$\mu * a - a = a * \mu - a = \delta_o.$$

- (1) *If  $\Gamma$  is a finite extension of  $\mathbb{Z}$  and  $\mu$  has infinite second moment ( $\sum_x d(o, x)^2 \mu(x) = \infty$ ), or if  $\Gamma$  is a finite extension of  $\mathbb{Z}^2$ , then the Martin boundary has precisely one point, and the only solutions of (+) are the constant multiples of  $a(x)$ .*
- (2) *If  $\Gamma$  is a finite extension of  $\mathbb{Z}$  and  $\mu$  has finite second moment, then the Martin boundary has precisely two points, and the solutions of (+) are all functions of the form*

$$c_1 a(x) + c_2 h_0(x),$$

*where  $c_1 > 0$  and  $|c_2| < \alpha c_1$  ( $\alpha > 0$  a constant determined by  $\mu$ ), and  $h_0$  is the unique harmonic function on  $\Gamma$  satisfying  $h_0(x) = x$  for all  $x \in \mathbb{Z} \subset \Gamma$ .*

We remark that in the case when  $[\Gamma : \mathbb{Z}]$  is finite, Kesten [131] considers finiteness (or not) of the second moment of the induced random walk on  $\mathbb{Z} \subset \Gamma$ . Along the lines of Kesten and Spitzer [134, Theorem 4.5], it is not hard to prove equivalence with the above condition concerning the second moment in the Cayley graph metric. Case (1) for finite extensions of  $\mathbb{Z}^2$  can be seen from Kesten [131, Lemma 4 and Theorem 3]. Also, Kesten [131] does not give the precise description of the solutions of (+) as above. In case (1), this is obvious, while case (2) is an easy adaption of the results of Kesten and Spitzer [134], where only Abelian groups are treated. In these two references, also non-finitely generated groups are considered. For  $\Gamma = \mathbb{Z}$  or  $\mathbb{Z}^2$ , the results were found before by Spitzer; see his book [205, §31].

### 8. Electric networks and harmonic functions with finite Dirichlet sum

In this section we always assume that  $(X, P)$  is reversible and (NN). Thus for the associated network  $\mathcal{N}$ , the edges coincide with those of the locally finite graph  $X$ ; we write  $E_p = E$ . (Indeed, many of the subsequent results do not require local finiteness.) We use the notation introduced in Sections 2.B and 4.B.

A *cycle* is a function  $Z \in \ell^2(E, r)$  with  $\nabla^* Z = 0$ . A *finite cycle* is one with finite support. Given a function  $i: X \rightarrow \mathbb{R}$  (usually finitely supported), an *electric current generated by  $i$*  is a function  $u \in \ell^2(E, r)$  satisfying the following two laws.

(ND) Kirchhoff's node law:

$$\nabla^* u(x) = -\frac{i(x)}{c(x)} \quad \text{for all } x \in X.$$

(LP) Kirchhoff's loop law:

$$\langle u, Z \rangle = 0 \quad \text{for every finite cycle } Z.$$

The most typical cases are  $i(x) = i_0(\delta_b - \delta_a)$ , where  $i_0$  is a constant and  $a, b \in X$  (current from  $a$  to  $b$  with value  $i_0$ ), and  $i(x) = i_0 \delta_{x_0}$ , where  $x_0 \in X$  (compare with Section 4.B). The generator  $i$  is said to be *balanced* if  $\sum_x i(x) = 0$  (absolutely convergent).

The first author to study the mathematical theory of infinite electric networks was Nash-Williams [165]; see also Kemeny, Snell and Knapp [126, §9.10]. A systematic study was then initiated independently by Flanders [72, 73] and Zemanian [251, 252]. The book by Doyle and Snell [63] gives a beautiful introduction and explains the interplay with Markov chain theory; Thomassen [215] gives a more combinatorial treatment. A recent monograph by Zemanian [253] is dedicated exclusively to the theory of infinite electric networks. Here, we review those aspects which are in closest relation with the topic of this survey.

The first questions which one typically has to deal with are those of existence and uniqueness of current when given a generator  $i$ . The existence problem involves the search for a *potential*, that is, a function  $g$  on  $X$  such that  $u = \nabla g$  (whence  $g \in \mathcal{D}(\mathcal{N})$ ). If there are one or more currents generated by  $i$ , then the ‘natural’ one will minimize the energy  $\langle u, u \rangle$ . Its existence is assured already, if there is a finite energy solution of (ND) only: Hilbert space theory tells us that in this case there will be a unique solution of (ND) with minimal energy, and that it must lie in the orthogonal complement of  $\ker \nabla^*$ , that is,  $\langle u, Z \rangle = 0$  for every cycle in  $\ell^2(E, r)$ , so that (LP) holds for  $u$ . This is called the *minimum energy flow* generated by  $i$ . Doyle [62] calls this the *odd flow*.

#### A. Existence criteria

See Flanders [72] and Zemanian [251] for the existence question; a good account and the most general criteria are given by Schlesinger [195].

**THEOREM 8.1.** *Let  $\mathcal{N}$  be a transient network (that is, the associated random walk is transient).*

- (1) *A sufficient condition for the existence of a current generated by  $i$  is that the series*

$$S = \sum_{x, y \in X} \frac{G(x, y) i(x) i(y)}{c(y)}$$

*converges absolutely. This is also necessary when  $i \geq 0$ .*

- (2) *If this is satisfied, then the minimum energy flow is given by  $u = \nabla g$ , where*

$$g(x) = \sum_{y \in X} G(x, y) \frac{i(y)}{c(y)},$$

*and its energy is  $\langle u, u \rangle = S$ .*

If  $i(x) = i_0 \delta_{x_0}$ , then one obtains a current from  $x_0$  to infinity; compare with Section 4.B. As mentioned by Schlesinger [195], Theorem 8.1 can be obtained from Kayano and Yamasaki [122] in the locally finite case; it also holds without local finiteness.

Next, we consider the recurrent case. Given a reference vertex  $o$ , let  ${}^o p(x, y)$  and  ${}^o G(x, y)$  be as in Section 7.H above. ( ${}^o G(x, y)$  is the expected number of visits to  $y$  up to—and possibly including—the first visit to  $o$ , when the random walk starts at  $x$ .)

**THEOREM 8.2.** *Let  $\mathcal{N}$  be a recurrent network.*

- (1) *If there is a current generated by  $i$ , then it is unique.*
- (2) *If  $\sum_x |i(x)| < \infty$  and*

$$S = \sum_{x, y \in X} \frac{{}^oG(x, y) i(x) i(y)}{c(y)}$$

*converges absolutely, then the current  $u$  generated by  $i$  exists if and only if  $i$  is balanced (conservation of current).*

- (3) *If these conditions hold, then the current is given by  $u = \nabla g$ , where*

$$g(x) = - \sum_{y \in X} {}^oG(x, y) \frac{i(y)}{c(y)},$$

*and its energy is  $\langle u, u \rangle = S$ .*

Uniqueness in the recurrent case was proved by Yamasaki [250]; see also Kayano and Yamasaki [122] when  $X$  is locally finite.

## B. Uniqueness of current and harmonic functions

If  $i$  has finite support and is balanced, then there is another candidate for the ‘natural’ electric current: take an exhaustion of  $X$  by finite connected subgraphs  $X_n$ ,  $X_n \subset X_{n+1}$  and  $\text{supp } i \subset X_1$ . Consider the finite networks  $\mathcal{N}_n$  obtained by restricting  $\mathcal{N}$  to  $X_n$ . As  $\mathcal{N}_n$  is finite, the classical theory yields the existence of a unique electric current  $u_n$  in  $\mathcal{N}_n$  generated by  $i$ . By Flanders [72],  $u_n$  converges to a finite energy solution of (ND) and (LP) on  $\mathcal{N}$ , independent of the particular exhaustion chosen. Thomassen [216] calls this *the electric current generated by  $i$* , while in Doyle’s [62] terminology this is the *even flow*. As noted by Zemanian [251], it does not always coincide with the minimal energy flow. This leads to the question of uniqueness of finite energy solutions of (ND) and (LP). Given two solutions, their difference will satisfy (ND) and (LP) with  $i \equiv 0$ , that is, the *homogeneous Kirchhoff equations*. By a standard argument (see Flanders [73, p. 328]), any finite energy solution  $u$  of (LP) has a potential  $h \in \mathcal{D}(\mathcal{N})$  with  $\nabla h = u$ . If  $u$  satisfies (ND) with  $i = 0$ , then we obtain  $\nabla^* \nabla h = 0$ , that is,  $Ph = h$ . This leads to the definition of the space of *harmonic functions with finite Dirichlet sum*:

$$\mathcal{H}\mathcal{D}(\mathcal{N}) = \{h \in \mathcal{D}(\mathcal{N}) \mid Ph = h\};$$

we write  $\mathcal{H}\mathcal{D}(X)$  when  $\mathcal{N}$  is the simple network of  $X$ .

Thus, electric current is unique if and only if  $\mathcal{H}\mathcal{D}(\mathcal{N})$  consists only of constants. In analogy with the continuous case, Yamasaki [250] proves that every  $h$  in  $\mathcal{H}\mathcal{D}(\mathcal{N})$  can be approximated by *bounded* functions in  $\mathcal{H}\mathcal{D}(\mathcal{N})$ . Thus one obtains the following hierarchy.

**PROPOSITION 8.3.** *Let  $\mathcal{N}$  be a network. Each of the following statements implies the next one.*

- (1)  *$\mathcal{N}$  is recurrent.*
- (2) *All positive harmonic functions on  $\mathcal{N}$  are constant.*

- (3) All bounded harmonic functions on  $\mathcal{N}$  are constant.
- (4) All functions in  $\mathcal{H}\mathcal{D}(\mathcal{N})$  are constant, and for any function  $i$  on  $X$ , there is at most one electric current generated by  $i$ .

**COROLLARY 8.4.** *Suppose that  $\Gamma$  is a nilpotent-by-finite group and that  $\mu$  is a finitely supported, irreducible probability measure giving rise to a reversible random walk. Then for the associated network  $\mathcal{N}$ ,  $\mathcal{H}\mathcal{D}(\mathcal{N}) = \{\text{constants}\}$ .*

This follows from (3), as the Poisson boundary is trivial (Dynkin and Malyutov [68]); see Theorem 7.3. All reversible random walks on groups with trivial Poisson boundaries provide further examples; see Section 7.B.

A *bounded automorphism* of a graph  $X$  is an element  $\phi$  of  $\text{Aut}(X)$  such that  $\sup_x d(x, \phi x) < \infty$ .

**THEOREM 8.5.** *Suppose that  $\mathcal{N} = (X, E, r)$  has  $\Gamma$ -invariant conductances for a given subgroup  $\Gamma \leq \text{Aut}(X)$ . Then in each of the following cases,  $\mathcal{H}\mathcal{D}(\mathcal{N}) = \{\text{constants}\}$ .*

- (a)  $\Gamma$  acts quasi-transitively and  $X$  has polynomial growth.
- (b) There is a constant  $\varepsilon_0 > 0$  such that  $r(e) \geq \varepsilon_0$  for all edges, and  $\Gamma$  contains a bounded automorphism without fixed points.

This is due to Soardi and Woess [203]; (a) is an application of (b) via the results of Trofimov [219] and Gromov [97]. Doyle [62] states that uniqueness of current holds if and only if for any pair of vertices  $a, b$ , the odd and the even flow from  $a$  to  $b$  with value one have the same energy. He indicates a proof that this is verified for the simple network, when  $X$  is quasi-transitive and  $\iota(X) = 0$  (in particular, for Cayley graphs of amenable groups and for the graphs in (a) above). For the following useful result, due to Soardi [201], recall the definition of rough isometry (Section 4.A).

**THEOREM 8.6.** *Let  $X$  and  $Y$  be two roughly isometric graphs with bounded vertex degrees. Then  $\mathcal{H}\mathcal{D}(X) = \{\text{constants}\}$  if and only if  $\mathcal{H}\mathcal{D}(Y) = \{\text{constants}\}$ .*

This generalizes immediately to the (NN) case when  $C^{-1} \leq r(e) \leq C$  for all edges ( $C > 0$ ), and yields yet another way to prove Theorem 8.5(b): by Trofimov [219], a quasi-transitive graph with polynomial growth is roughly isometric with the Cayley graph of a nilpotent group. We remark that the situation is different for positive or bounded harmonic functions (Martin/Poisson boundary): the example of T. Lyons [153] (see Section 9 below) can be transformed into two roughly isometric graphs with bounded vertex degrees, such that with respect to the SRW, one graph has  $\mathcal{H}^+ = \{\text{constants}\}$ , while the other has nonconstant functions in  $\mathcal{H}^\infty$ . See also Benjamini [17] for another example.

Another useful uniqueness criterion, in a spirit similar to Nash-Williams' criterion (Corollary 4.14), is due to Thomassen [216].

**THEOREM 8.7.** *Suppose that  $X$  can be decomposed into disjoint, finite subgraphs  $X_0, X_1, \dots$ , such that their union contains all vertices, and in  $X$  there are no edges between  $X_i$  and  $X_j$  when  $|i - j| > 1$ . If there is  $C > 0$  such that every pair  $x, y \in X_k$  is connected by a path in  $X_k$  of length at most  $C \cdot k$  (for all  $k$ ), then  $\mathcal{H}\mathcal{D}(X) = \{\text{constants}\}$ .*

This has been applied by Thomassen [216] to Cartesian products of graphs.

**COROLLARY 8.8.** *Let  $X, Y$  be two infinite, connected, locally finite graphs. Then for their Cartesian product,  $\mathcal{H}\mathcal{D}(X \times Y) = \{\text{constants}\}$ .*

Next, we turn our attention to nonuniqueness.

**THEOREM 8.9.** *If  $X$  has bounded vertex degrees, at least two ends and satisfies (IS), and if the resistances satisfy  $C^{-1} \leq r(e) \leq C$  ( $C > 0$ ), then  $\mathcal{H}\mathcal{D}(\mathcal{N})$  contains nonconstant functions.*

This is due to Soardi and Woess [203]. It applies, for example, when the conductances are invariant under a quasi-transitive subgroup of  $\text{Aut}(X)$  and  $X$  has more than two ( $\Leftrightarrow$  infinitely many) ends. In particular, the Cayley graphs of finitely generated groups with infinitely many ends carry nonconstant Dirichlet finite harmonic functions. As pointed out by C. Thomassen (oral communication), (IS) in Theorem 8.9 may be replaced by  $(IS_q)$  for any  $q > 2$ . Indeed, it is enough that the Green kernel vanishes at infinity.

In view of the above results, one might be tempted to believe that for one-ended vertex-transitive graphs, electric current is always unique. This is not true.

DeBaun [50] shows that the graph of a triangulated Riemannian surface has nonconstant Dirichlet finite harmonic functions if and only if the SRW is transient. Cartwright and Woess [35] show that on a graph  $X$  which can be embedded nicely in the hyperbolic disc and satisfies (IS), one can construct functions in  $\mathcal{H}\mathcal{D}(X)$  with prescribed boundary values arising from Lipschitz functions. This applies, for example, to Cayley graphs of nonexceptional Fuchsian groups. Paschke [172, 173] studies flows associated with Cayley graphs of groups. In [173] he considers the space of all passive electric currents (that is, all  $u = \nabla h$ ,  $h \in \mathcal{H}\mathcal{D}(X)$ ) on the simple network of a quasi-transitive graph. Assuming that a finitely generated group  $\Gamma$  acts on  $X$  with finite vertex-stabilizers, he associates with this space a numerical invariant (von Neumann dimension)  $\rho(\Gamma, X)$ . It turns out to depend only on  $\Gamma$ ,  $\rho = \rho(\Gamma)$ . (However, it is not an invariant of  $X$ , although  $X$  is roughly isometric with any Cayley graph of  $\Gamma$ .) In our terminology,  $\mathcal{H}\mathcal{D}(X) = \{\text{constants}\}$  if and only if  $\rho(\Gamma) = 0$ . A formula for  $\rho$  which involves vertex- and edge-stabilizers allows explicit calculation when  $X$  is a tree. Among various other results, Paschke [173] then proves the following relations.

When  $\Gamma_1$  is a finite index subgroup of  $\Gamma$ ,

$$\rho(\Gamma_1) = [\Gamma : \Gamma_1] \rho(\Gamma).$$

For the free product of two infinite groups,

$$\rho(\Gamma_1 * \Gamma_2) = \rho(\Gamma_1) + \rho(\Gamma_2) + 1.$$

For the Cartesian product of two infinite groups,

$$\rho(\Gamma_1 \times \Gamma_2) = 0.$$

(This is Corollary 8.8 for groups.)

### 9. Random walks and the classification of Riemannian manifolds

In potential theory, Riemannian manifolds are classified according to certain properties such as the following.

- Parabolicity or hyperbolicity.

(*Parabolic* means that all positive superharmonic functions are constant, or, in other words, that Brownian motion is recurrent; *hyperbolic* means transience of Brownian motion.) Note that this notion of hyperbolicity is completely different from hyperbolicity of metric spaces in the sense of Gromov [98].

- $\lambda(M) = 0$  or  $\lambda(M) < 0$ ,

where  $\lambda(M)$  is the top of the spectrum of the Laplace–Beltrami operator of  $M$  (operating on  $L^2(M)$ ).

- Nonexistence or existence of nonconstant bounded harmonic functions.

(Nonexistence of such functions is called the *weak Liouville property*.)

- Nonexistence or existence of nonconstant positive harmonic functions.

(Nonexistence of such functions is called the *strong Liouville property*.)

For further topics in the classification of Riemannian manifolds, see, for example, Sario and Nakai [191].

In the last decade, various authors have used ‘discrete approximation’ to give answers to the above questions of classification. We present a few results in this direction.

We start with observing that the definition of rough isometry (Section 4) makes sense for arbitrary metric spaces. Following Kanai [117] and Markvorsen, McGuinness and Thomassen [157], a *net* in a metric space  $(M, d_M)$  is a graph obtained via two parameters  $r$  and  $R$ ,  $0 < 2r < R$ : let  $X$  be a maximal set in  $M$  such that  $d_M(x, y) > r$  for all  $x, y \in X$ . If  $x, y \in X$ , then  $x \sim y$  if  $0 < d_M(x, y) \leq R$ . More precisely, this graph is called an  $(r, R)$ -*net* of  $M$ . Under quite general conditions on  $(M, d_M)$ , any of its nets (with the associated graph metric) is roughly isometric with  $(M, d_M)$  itself; see Markvorsen *et al.* [157].

A Riemannian manifold is said to have *bounded geometry* if its Ricci curvature is bounded from below and its injectivity radius is positive. Kanai [118] has proved the following.

**PROPOSITION 9.1.** *Let  $M$  be a complete Riemannian manifold with bounded geometry. Then every  $(r, 3r)$ -net of  $M$  is a connected graph with bounded vertex degrees, roughly isometric with  $X$  via the inclusion map  $X \hookrightarrow M$ .*

(Combining this with Theorem 3.3 of Markvorsen *et al.* [157], one sees that  $3r$  can be replaced by any  $R > 2r$ .) It turns out that rough isometry preserves many of the interesting properties.

**THEOREM 9.2.** *Let  $M$  be a complete Riemannian manifold with bounded geometry, and let  $X$  be a net of  $M$ . Then  $M$  is parabolic if and only if the SRW on  $X$  is recurrent. In particular, parabolicity is invariant under rough isometry of Riemannian manifolds with bounded geometry.*

For (different) proofs, see Varopoulos [222] and Kanai [118]. Markvorsen *et al.* [157] have applied this criterion successfully in both directions (that is, deciding

hyperbolicity of Scherk's surface—previously unknown—via recurrence of one of its nets, and vice versa, deciding recurrence of a certain graph in  $\mathbb{Z}^3$  via parabolicity of the 'helicoid'). For similar discretization procedures, see also Furstenberg [76], Lyons and Sullivan [155], Doyle [61] and the references given there.

Cheeger's [38] inequality relates  $\lambda(M)$  with the isoperimetric constant of  $M$ . Kanai [117] has proved that isoperimetric properties are preserved by rough isometry. Thus one obtains the following (Kanai [119]).

**THEOREM 9.3.** *Let  $M$  be a complete Riemannian manifold with bounded geometry, and let  $X$  be a net of  $M$ . Then  $\lambda(M) < 0$  if and only if  $\iota(X) > 0$ . In particular, the property that  $\lambda(M) < 0$  is invariant under rough isometry of Riemannian manifolds with bounded geometry.*

Compare this with the situation when  $M$  is the universal cover of a compact manifold  $M_0$  and  $\Gamma$  is the fundamental group of  $M_0$ . (The graph to be considered is the Cayley graph of  $\Gamma$ .) In this case,  $\lambda(M) = 0$  if and only if  $\Gamma$  is amenable: this has been proved before by Brooks [22] and Varopoulos [221].

As far as Liouville properties are concerned, a counterexample shows that they are not preserved by rough isometry: T. Lyons [153] constructs an example of a locally finite graph  $X$  carrying two strongly reversible, nearest neighbour random walks with transition matrices  $P, P'$  and associated conductances  $c(x, y)$  and  $c'(x, y)$  ( $x, y \in X$ ), respectively, with the following properties.

- $A^{-1}c(x, y) \leq c'(x, y) \leq Ac(x, y)$  for all  $x, y \in X$ , where  $A > 0$ .
- There are nonconstant positive functions in  $\mathcal{H}^\infty(P)$ , that is,  $(X, P)$  has nontrivial Poisson boundary.
- $\mathcal{H}^+(P')$  contains only constant functions, that is, the minimal Martin boundary for  $P'$  is trivial.

T. Lyons then associates with  $(X, P)$  and  $(X, P')$  two manifolds which are quasi-isometric (and hence also roughly isometric), such that one has the strong Liouville property, while the other does not even satisfy the weak version. Another example is given by Benjamini [17].

On the other hand, Kanai [117] proves the following, once more using discrete approximations: if a complete Riemannian manifold with bounded geometry is roughly isometric with  $\mathbb{Z}^q$ , where  $q \geq \dim M$ , then  $M$  has the strong Liouville property.

For the weak Liouville property, that is, constantness of bounded harmonic functions, see Kaimanovich [110], who shows that for the discretization of Furstenberg [76] and Lyons and Sullivan [155] the Poisson boundary is isomorphic with that of the manifold itself.

Grigor'yan [94, 95] studies conditions for the existence of nonconstant harmonic functions with finite Dirichlet integral on Riemannian manifolds. In particular, the criterion of Grigor'yan [94] shows that constantness of all functions in  $\mathcal{H}\mathcal{D}$  is invariant under quasi-isometry between manifolds. (Quasi-isometry is slightly more restrictive than rough isometry; note that Gromov [98] uses the terminology 'quasi isometric' for what is called 'roughly isometric' here.) Compare this with the discrete analogue (Theorem 8.6). Finally, note that rough isometry also preserves the validity of  $(IS_q)$  and  $(D_q)$  for graphs and complete Riemannian manifolds with bounded geometry; see Kanai [117, 119].

**THEOREM 9.4.** *Given a complete Riemannian manifold  $M$  with bounded geometry, and any of its nets  $X$ , the heat kernel on  $M$  satisfies*

$$p_t(\xi, \eta) = O(t^{-q/2}) \quad \text{as } t \rightarrow \infty \quad (\xi, \eta \in M)$$

*if and only if*

$$p^{(n)}(x, y) = O(n^{-q/2}) \quad \text{as } n \rightarrow \infty \quad (x, y \in X)$$

*for the SRW on  $X$ .*

For a proof in the case of a cocompact covering, see Varopoulos [227]. For the general case, see Coulhon [45], who proves the equivalence directly, and Chavel and Feldman [37], who deduce the result on manifolds by use of *modified isoperimetric inequalities* (and also discretization). The modification is that in the ratios defining the  $q$ -dimensional isoperimetric constant of  $M$  (compare with the definition of  $\iota_q(X)$  for graphs), one considers only those relatively compact, open submanifolds which have inradius bounded below by  $\varepsilon$ , where  $\varepsilon > 0$ . It turns out that for the large-time behaviour of the heat kernel, this gives the right type of isoperimetric inequality. Indeed, the net  $X$  of  $M$  satisfies  $(IS_q)$ ,  $q > 1$ , if and only if  $M$  satisfies the modified  $q$ -dimensional isoperimetric inequality for some ( $\Leftrightarrow$  every)  $\varepsilon > 0$ ; see Chavel and Feldman [36].

#### *Index of abbreviations*

(BR)	bounded range	[§2.A]
( $D_q$ )	$q$ -dimensional Dirichlet inequality	[§6.A]
(EG)	exponential growth	[§3.A]
(FP)	adapted to free product structure	[§6.D]
(FR)	finite range	[§2.A]
(GC)	group-invariant conductances	[§2.D]
(GP)	group-invariant transition probabilities	[§2.D]
(IS)	strong isoperimetric inequality	[§3.A]
( $IS_f^o$ )	connected $f$ -isoperimetric inequality rooted at $o$	[§4.C]
( $IS(\mathcal{N})$ )	strong isoperimetric inequality for networks	[§3.A]
( $IS_q$ )	$q$ -dimensional isoperimetric inequality	[§3.A]
(LP)	Kirchhoff's loop law	[§8]
(ND)	Kirchhoff's node law	[§8]
(NN)	nearest neighbour	[§2.A]
( $PG_q$ )	polynomial growth	[§3.A]
(R)	reversible	[§2.B]
(RI)	roughly isometric	[§4.B]
( $S_q$ )	$q$ -dimensional Sobolev inequality	[§6.A]
(SA)	strongly aperiodic	[§5.A]
(SP)	strongly periodic	[§5.A]
(SR)	strongly reversible	[§2.B]
(SRW)	simple random walk	[§2.A]
( $SP'$ )	strongly periodic (generalized)	[§5.A]
(TD)	tight distance distributions	[§2.A]
(UI)	uniformly irreducible	[§2.A]
(UM)	uniform first moment condition	[§2.A]

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(The numbers in parentheses at the end of each reference indicate the section(s) where it is cited.)

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