space ([Birk]). In this context Kohlberg and Pratt ([Koh-Pratt]) proved a general theorem on the unique role played by Hilbert's metric from which our Theorem 1 follows easily. Nevertheless we feel that our proof gives valuable insights into the special case of the positive orthant; in particular the Finsler character of the Hilbert's metric is not mentioned in [Koh-Pratt] (actually it is not mentioned in any of the recent papers in which Hilbert's metric is used or in the monograph [Sen]).

Let us introduce some notation. By 0, $\mathbf{1}$ we will denote vectors in R^n or the $n \times n$ matrices with all entries equal to 0 and 1 respectively. For two $n \times n$ matrices (or vectors in R^n) A, B, $A > B$ $(A \geq B)$ means that every entry of A is bigger (bigger or equal) than the respective entry of B. Let $0_n^+ = \{v \in R^n | v > 0\}$ and let

$\mathcal{P} = \{A \in GL(n,R) | A \geq 0\}$, $\mathcal{P}^+ = \{A \in GL(n,R) | A > 0\}$

$\mathcal{D}$ = the group of diagonal matrices in $\mathcal{P}$ (i.e. with positive diagonals).

For $v = (v_1, \ldots, v_n) \in 0_n^+$, $u = (u_1, \ldots, u_n) \in 0_n^+$ the Hilbert's projective pseudometric is given by the following formula:

$$d(v,u) = \max_{i,j} \ln \frac{v_i u_j}{u_i v_j} \tag{1}$$

$d(v,u)$ depends only on the rays passing through v and u so that it defines a metric on $P0_n^+ \subset RP^{n-1}$ - the set of rays in the positive orthant 0_n^+ as a subset of the projective space RP^{n-1}.

<u>Theorem 1</u> Hilbert's metric on $P0_n^+$ is a Finsler metric and it is the only, up to a scalar factor, Finsler metric on $P0_n^+$ such that the action of any $A \in \mathcal{P}^+$ on $P0_n^+$ is a contraction.

<u>Proof</u> Let $d(\cdot,\cdot)$ be a Finsler metric on $P0_n^+$ such that any $A \in \mathcal{P}^+$ acts on $P0_n^+$ as a contraction. By continuity then the action on $P0_n^+$ of any $A \in \mathcal{P}$ is non-expanding. We have that the action of $D \in \mathcal{D}$ and $D^{-1} \in \mathcal{D}$ on $P0_n^+$ is non-expanding. Hence $D \in \mathcal{D}$ acts on $P0_n^+$ as an isometry.

The group $\mathcal{D}$ is isomorphic with the additive group of R^n and it acts freely and transitively on 0_n^+ and on $P0_n^+$.

Let us introduce coordinates $\eta = (\eta_1, \ldots, \eta_n)$ in 0_n^+ by the formula: if $v = (v_1, \ldots, v_n) \in 0_n^+$ then $\eta_i = \ln v_i$, $i = 1, \ldots, n$. These coordinates introduce identification of 0_n^+ with R^n and the action of $\mathcal{D}$ on 0_n^+ becomes under this identification the action of R^n on R^n by translations. $P0_n^+$ is naturally identified with the quotient linear space R^n/L where $L = \{a\mathbf{1} \in R^n | a \in R\}$. The action of $\mathcal{D}$ on $P0_n^+$ becomes under this identification the action of R^n on the quotient space R^n/L by translations. Hence our Finsler metric on $P0_n^+$ defines a translationally invariant Finsler metric on the linear space R^n/L. Translationally invariant

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