$D_1 = \mathrm{diag}(v_1,\ldots,v_n)$, $D_2 \in \mathcal{D}$, $D_2 = \mathrm{diag}((\sum_{j=1}^{n} a_{1j}v_j)^{-1},\ldots,(\sum_{j=1}^{n} a_{nj}v_j)^{-1})$.

From the proof of Theorem 1 it follows that $\tau_1(D_2AD_1)$ is the infinitesimal coefficient of contraction of the Hilbert's metric at $v \in O_n^+$. Hence

$$\tau_B(A) = \sup_{v\in O_n^+} \tau_1(D_2AD_1). \tag{7}$$

Explicit formula for $\tau_B(A)$ in terms of entries of A is well known ([Birk], [Sen]): $\tau_B(A) = (1 - \sqrt{\phi(A)})/(1 + \sqrt{\phi(A)})$ where $\phi(A) = \min_{i,j,k,\ell} \frac{a_{ik}a_{j\ell}}{a_{i\ell}a_{jk}}$ if $A > 0$ and $\phi(A) = 0$ otherwise. It follows from (6) and (7) that the formula for τ_B is equivalent to the following formula: for any $a, b \in O_n$, $a \neq 0$, $b \neq 0$

$$\inf_{v\in O_n^+} \sum_{s=1}^{n} \min\left(\frac{a_s v_s}{\sum_{i=1}^{n} a_i v_i}, \frac{b_s v_s}{\sum_{i=1}^{n} b_i v_i}\right) = \begin{cases} \frac{2\sqrt{\phi}}{1+\sqrt{\phi}} & \text{if } a>0 \text{ and } b>0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where $\phi = \min_{k,\ell} \frac{a_k b_\ell}{a_\ell b_k}$.

It would be interesting to have an independent proof of (8); it would provide an alternative proof of the formula for τ_B.

Let $r(A)$ denote the spectral radius of A. For nonnegative A $r(A)$ is by Perron-Frobenius theory equal to the maximal positive eigenvalue of A. The coefficient of contraction of the Hilbert metric $\tau_B(A)$ can be used in estimating the gap between $r(A)$ and the absolute value of any other eigenvalue ([Birk], [Sen] which contains the extensive bibliography). More precisely if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of a nonnegative allowable A, $r(A) = \lambda_1 > |\lambda_2| > \ldots > |\lambda_n|$, then $r(A)\tau_B(A) > |\lambda_k|$ $k = 2, \ldots, n$. As a consequence $(r(A)\tau_B(A))^{n-1} > |\lambda_2\ldots\lambda_n| = \frac{|\det A|}{r(A)}$ so that

$$r(A) > (\tau_B(A)))^{-n+1/n}|\det A|^{1/n} \tag{9}$$

It is clear that we will obtain a more precise estimate of $r(A)$ if we use the contraction of the volume element in PO_n^+ induced by the Hilbert metric instead of the contraction of the Hilbert metric itself.

Any Finsler metric on a manifold defines a volume element by the condition that unit balls in every tangent space have unit volume. If the Finsler metric is contracted under some diffeomorphism then the volume is also contracted. Our next goal is to express analytically in homogeneous coordinates $v_1:v_2: \ldots: v_n$ the volume element ω in PO_n^+ induced by the

Copyrighted material