Hilbert metric.

Let $\pi: O_n^+ \to PO_n^+$ be the natural projection and V be the vector field in O_n^+, $V(v) = (v_1, \ldots, v_n)$. Let further ω_0 be the standard volume in O_n^+ i.e. $\omega_0 = dv_1 \wedge \ldots \wedge dv_n$ and $F: O_n^+ \to R$, $F(v) = v_1 \cdot \ldots \cdot v_n$.

Proposition 1. If ω is the volume element in PO_n^+ induced by the Hilbert metric then $\pi^*\omega = \text{const} \frac{1}{F(v)} \sum_{k=1}^{n} (-1)^{k+1} v_k dv_1 \wedge \ldots \wedge \widehat{dv_k} \wedge \ldots \wedge dv_n$ $= \text{const} \frac{1}{F(v)} i_V \omega_0$ where $\widehat{\ }$ means that we omit corresponding 1- form and $i_V \omega_0$ is the interior derivative of ω_0 by V.

Proof. Let us transport ω to R^n/L (see the proof of Theorem 1) via the change of variables W. We get the volume element ν in R^n/L which is translationally invariant, hence it must be up to a scalar factor equal to the standard Lebesgue volume. Let ν_0 be the standard Lebesgue volume in R^n, $\nu_0 = d\eta_1 \wedge \ldots \wedge d\eta_n$ and $p: R^n \to R^n/L$ be the natural projection. We have

$$p^*\nu = \text{const } i_1 \nu_0 = \text{const} \sum_{k=1}^{n} (-1)^{k+1} d\eta_1 \wedge \ldots \wedge \widehat{d\eta_k} \wedge \ldots \wedge d\eta_n$$

Since W is defined by $\eta_i = \ln v_i$ then $d\eta_i = \frac{dv_i}{v_i}$, $i = 1, \ldots, n$ and we get

$$\pi^*\omega = \text{const} \frac{1}{F(v)} \sum_{k=1}^{n} (-1)^{k+1} v_k \, dv_1 \wedge \ldots \wedge \widehat{dv_k} \wedge \ldots \wedge dv_n .$$

By the definition of the interior derivative we have then

$$\pi^*\omega = \text{const} \frac{1}{F(v)} i_V \omega_0 . \qquad \square$$

Let us introduce the coefficient of contraction of the volume element ω under the action of $A \in \mathcal{P}$ on PO_n^+:

$$\sigma(A) = \sup_{\substack{X \subset PO_n^+ \\ X\text{-compact}}} \int_{AX} \omega / \int_X \omega$$

Let further for any nonnegative matrix A

$$\rho(A) = \inf_{u \in O_n^+} \left(\frac{F(Au)}{F(u)}\right)^{1/n} .$$

Proposition 2. $\sigma(A) = |\det A| \, (\rho(A))^{-n}$

Proof It is easy to see that

$$\sigma(A) = \sup_{u \in O_n^+} \sup_{v', \ldots, v^{n-1} \in R^n} \left| \frac{\pi^*\omega(Au)(Av', \ldots, Av^{n-1})}{\pi^*\omega(u)\,(v', \ldots, v^{n-1})} \right|$$

By Proposition 1 we get

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