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On Uniform Contraction Generated by Positive Matrices

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ABSTRACT. It is shown that the Hilbert's metric in the positive orthant is the only Finsler metric that makes the action of any positive matrix into a contraction. The volume element defined by the Hilbert's metric is introduced and the contraction of this volume element is used in estimating below the spectral radius of a product of matrices or the maximal Lyapunov exponent of a product of random matrices. The connection between the contraction of the volume and the factorization of a positive matrix into diagonal, doubly stochastic and diagonal matrices is established.

In the study of products of positive matrices Hilbert's projective pseudometric ([Birk]) is a useful device (see [Sen] and the bibliography therein). Its importance comes from the fact that positive matrices act as contractions in the space of rays in the positive orthant with the Hilbert's metric.

Hilbert's metric is a Finsler metric, which allows for infinitesimal methods in dealing with it. Let us recall that a Finsler metric on a manifold is by definition a continuous family of norms in the tangent spaces; it allows the introduction of the length of a smooth curve and the metric in the ordinary sense as the infimum of the lengths of piecewise smooth curves connecting two points (this construction is analogous to the one in the Riemannian geometry using a Riemann metric). Actually the space of rays in the positive orthant with the Hilbert's metric is isometric with a normed linear space so that this Finsler metric is "flat."

We will prove that the Hilbert's metric is the only, up to a scalar factor, Finsler metric which makes the action of any positive matrix on the space of rays in the positive orthant into a contraction. Hilbert's metric is not Riemann so in particular there is no Riemann metric with the above property.

Hilbert's metric can be defined in the context of a cone in a linear

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