Finsler metric on a linear space is nothing else but a norm. A norm on R^n/L defines a pseudo-norm on R^n which is equal to zero exactly on the subspace L. Let us denote such a pseudo-norm on R^n, defined by our Finsler metric, by $\|\cdot\|$.

Let for $A \in \mathcal{P}$, $\Phi_A : R^n \to R^n$ be the action of A on O_n^+ in η-coordinates. The fact that Φ_A is non-expanding with respect to the pseudo-metric defined in R^n by the pseudo-norm $\|\cdot\|$ can be expressed infinitesimally as: for every $\eta \in R^n$ and every $\xi \in R^n$

$$\|D_\eta \Phi_A \xi\| \leq \|\xi\| \tag{2}$$

where $D_\eta \Phi_A$ is the differential of Φ_A at η.

Φ_A is the composition $R^n \xrightarrow{W^{-1}} O_n^+ \xrightarrow{A} O_n^+ \xrightarrow{W} R^n$ where $W : O_n^+ \to R^n$ is the change of coordinates $W(v) = (\ln v_1, \ldots, \ln v_n)$. So it is easy to see that $D_\eta \Phi_A = D_2 A D_1$ where $D_1, D_2 \in \mathcal{D}$, $D_1 = \mathrm{diag}(e^{\eta_1}, \ldots, e^{\eta_n})$, $D_2 = \mathrm{diag}((\sum_{j=1}^n a_{1j} e^{\eta_j})^{-1}, \ldots, (\sum_{j=1}^n a_{nj} e^{\eta_j})^{-1})$. Clearly $D_2 A D_1$ is a stochastic matrix from $\mathcal{P}$ so that (2) for every $A \in \mathcal{P}$ is equivalent to: for every stochastic matrix $P \in \mathcal{P}$

$$\|P\xi\| \leq \|\xi\| \quad \text{for all } \xi \in R^n. \tag{3}$$

In particular we get that for any permutation matrix Π

$$\|\Pi\xi\| = \|\xi\| \quad \text{for all } \xi \in R^n.$$

By continuity (3) must be true for an arbitrary stochastic matrix (not necessarily nondegenerate).

Let us introduce a basis $\{f_1, \ldots, f_n\}$ in R^n by $f_k = (\underbrace{\alpha_k, \ldots, \alpha_k}_{k-1}, \underbrace{\beta_k, \ldots, \beta_k}_{n-k+1})$ where $\alpha_k = -\frac{n-k+1}{n}$ and $\beta_k = \frac{k-1}{n}$ for $k = 2, \ldots, n$ and $f_1 = 1$. In particular f_1 is orthogonal to $f_2, \ldots, f_n$. For any $\xi = (\xi_1, \ldots, \xi_n) \in R^n$ we have

$$\xi = (\sum_{i=1}^n \xi_i) f_1 + (\xi_2 - \xi_1) f_2 + (\xi_3 - \xi_2) f_3 + \ldots + (\xi_n - \xi_{n-1}) f_n.$$

Let P_k, $k = 1, 2, \ldots, n-1$ be the following special stochastic matrices.

$P_k = \{p_{ij}\}$ where $p_{i1} = 1$ for $1 \leq i \leq k$, $p_{in} = 1$ for $k+1 \leq i \leq n$ and all the other entries are zero. For any $\xi \in R^n$

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