$$\sigma(A) = \sup_{u\in O_n^+} \sup_{v',\dots,v^{n-1}\in R^n} \frac{F(u)}{F(Au)}\left|\frac{\omega_0(Au,Av',\dots,Av^{n-1})}{\omega_0(u,v',\dots,v^{n-1})}\right| = |\det A| \sup_{u\in O_n^+} \frac{F(u)}{F(Au)}. \qquad \square$$

It follows from the above computation that the infinitesimal coefficient of contraction of ω at a ray passing through $u \in O_n^+$ is equal to $|\det A| \frac{F(u)}{F(Au)}$; $\sigma(A)$ is its supremum over all $u \in O_n^+$.
On the other hand if $A \in \mathcal{P}$ is irreducible then it has the positive eigenvector $u_A \in O_n^+$ with the eigenvalue $r(A)$. The infinitesimal coefficient of contraction of ω at u_A is clearly equal to

$$\frac{|\lambda_2\cdot\ldots\cdot\lambda_n|}{(r(A))^{n-1}} = \frac{|\det A|}{(r(A))^n}$$

and it does not exceed $\sigma(A)$ i.e.

$$\frac{|\det A|}{(r(A))^n} \leq |\det A|\ (\rho(A))^{-n}$$

As a consequence

$$r(A) \geq \rho(A) \tag{10}$$

for irreducible $A \in \mathcal{P}$.

[Wojt 1] contains a straightforward proof of (10) valid for an arbitrary non-negative matrix A without reference to the Hilbert metric. It is also proved there that for $A \in \mathcal{P}$ $\rho(A) > |\det A|^{1/n}$ if and only if there are no permutation matrices Π_1, Π_2 such that $\Pi_1 A \Pi_2$ is a triangular matrix.

Clearly for non-negative A,B $\rho(AB) \geq \rho(A)\rho(B)$ so that (10) can be applied in estimating from below the spectral radius of a product of non-negative matrices or the maximal Lyapunov exponent of a product of random non-negative matrices. Indeed, if $A_1,\dots,A_N$ are non-negative matrices then

$$r(A_1 \dots A_n) \geq \rho(A_1) \dots \rho(A_N)$$

Further if A_1, A_2, ... is a stationary ergodic process with values in non-negative matrices then the maximal Lyapunov exponent μ_{max} is equal to

$$\mu_{max} = \lim_{k\to+\infty} \frac{1}{k}\ln||A_k\dots A_1|| \geq \limsup_{k\to+\infty}\frac{1}{k}\ln r(A_k\dots A_1)$$

where the limit exists almost surely ([Osel]) at least if $E(\ln\|A_1\|) < +\infty$. So we get

$$\mu_{max} \geq \lim_{k\to+\infty} \frac{\ln \rho(A_1)+\dots+\ln \rho(A_k)}{k} = E(\ln \rho(A_1)) \tag{11}$$

by the ergodic theorem.

In case a of 2 x 2 matrix $A \in \mathcal{P}$ $\sigma(A) = \tau_B(A)$ and (11) yields, also in

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