$$P_k\xi = (k\xi_1 + (n-k)\xi_n)f_1 + (\xi_n - \xi_1)f_{k+1}$$

Since the value of $\|\xi\|$ is preserved under a permutation of components of ξ we can assume without loss of generality that $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$. We have then

$$\begin{aligned}\|\xi\| &= \|(\xi_2 - \xi_1)f_2 + \ldots + (\xi_n - \xi_{n-1})f_n\| \\ &\leq (\xi_2 - \xi_1)\|f_2\| + \ldots + (\xi_n - \xi_{n-1})\|f_n\|\end{aligned} \tag{4}$$

On the other hand

$$\|\xi\| \geq \|P_k\xi\| = (\xi_n - \xi_1)\|f_k\|. \tag{5}$$

Putting $\xi = f_m$ in (5) we get $\|f_m\| \geq \|f_k\|$ so that $\|f_2\| = \ldots = \|f_n\| = \alpha > 0$. Now (4) and (5) yield $\alpha(\xi_n - \xi_1) \leq \|\xi\| \leq \alpha(\xi_n - \xi_1)$. Finally we obtain that for arbitrary $\xi \in R^n$ $\|\xi\| = \alpha \max_{i,j} (\xi_i - \xi_j)$.

Putting $\alpha = 1$, we obtain for $\eta, \xi \in R^n$ the distance defined by the above pseudonorm to be $\|\eta - \xi\| = \max_{i,j}(\eta_i + \xi_j - \xi_i - \eta_j)$ Going back to O_n^+ we get for $v, u \in O_n^+$, $\eta = W(v)$, $\xi = W(u)$, $d(v,u) = \max_{i,j} (\ln v_i + \ln u_j - \ln u_i - \ln v_j)$ which is the Hilbert's pseudometric (1). □

Remark 1 The use of the pseudonorm $\|\xi\| = \max_{i,j} (\xi_i - \xi_j)$ in the study of stochastic matrices goes back to A.A. Markov. Moreover, for a stochastic $P = \{p_{ij}\}$

$$\begin{aligned}\tau_1(P) &\overset{\text{def}}{=} \sup_{\xi:\|\xi\|>0} \frac{\|P\xi\|}{\|\xi\|} = \frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|p_{is} - p_{js}| \\ &= 1 - \min_{i,j}\sum_{s=1}^{n}\min(p_{is}, p_{js})\end{aligned} \tag{6}$$

see [Sen], in particular Theorem 3.1. P being stochastic preserves the subspace $L \subset R^n$ and so can be factored to the quotient space R^n/L. $\tau_1(P)$ is the norm of the corresponding linear map of R^n/L into itself with respect to the norm on R^n/L induced by the pseudonorm $\|\cdot\|$.

Remark 2 Let for any non-negative allowable A (allowable = with no zero rows or columns)

$$\tau_B(A) = \sup_{v,u\in O_n^+} \frac{d(Av,Au)}{d(v,u)}$$

i.e. $\tau_B(A)$ is the coefficient of contraction of the Hilbert's metric under the action of A on PO_n^+. Let further for a given A and $v \in O_n^+$ $D_1 \in \mathfrak{D}$,

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