view of (9), if $A_1 \in \mathcal{P}$

$$\mu_{max} > \frac{1}{2}(E(\ln|\det A_1|) - E(\ln\tau_B(A_1))) \tag{12}$$

The estimate (12) was used in [Wojt 2] in estimation from below of the metric entropy of the billiard system in the stadium and in the asymptotic case of the stadium close to a disk it is better than more conventional estimates.

Generally $\sigma(A)$ and $\rho(A)$ are not readily computable. But there is an interesting connection between $\rho(A)$ and the factorization of A into diagonal, doubly stochastic and diagonal matrices.

A is called fully indecomposable if there are no permutation matrices Π_1, Π_2 such that $\Pi_2 A \Pi_1 = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ where A_1 is a $k \times k$ matrix, $1 < k < n - 1$.

For a fully indecomposable non-negative matrix A it was proved by Djokovic ([Djok]) and by London ([Lond]) that $\frac{F(Au)}{F(u)}$ attains minimum at a unique ray passing through a vector $v \in O_n^+$ and moreover if $D_1 = \mathrm{diag}\,(v_1, \ldots, v_n)$ and

$$D_2 = \mathrm{diag}\left(\left(\sum_{j=1}^{n} a_{1j}v_j\right)^{-1}, \ldots, \left(\sum_{j=1}^{n} a_{nj}v_j\right)^{-1}\right)$$

then $D_2 A D_1 = P$ is doubly stochastic. Hence

$$\begin{aligned}\rho(A) &= \inf_{u \in O_n^+} \left(\frac{F(Au)}{F(u)}\right)^{1/n} = \left(\frac{F(Av)}{F(v)}\right)^{1/n} = \left(\frac{F(D_2^{-1}D_2AD_1\mathbf{1})}{F(D_1\mathbf{1})}\right)^{1/n} \\ &= \left(\frac{F(D_2^{-1}P\mathbf{1})}{F(D_1\mathbf{1})}\right)^{1/n} = \left(\frac{F(D_2^{-1}\mathbf{1})}{F(D_1\mathbf{1})}\right)^{1/n} = (\det D_1D_2)^{-1/n}\end{aligned} \tag{13}$$

In [Djok] and [Lond] the minimization of F(Au) under the constraint F(u)=1 was used as a technical device (without reference to the Hilbert volume) to prove that a fully indecomposable non-negative A can be factored into A=LPR where L, R $\in \mathcal{D}$ are diagonal ($L = D_2^{-1}$, $R = D_1^{-1}$) and P is doubly stochastic (these papers contain the bibliography of the original proofs of this fact). Moreover it was shown that L, P and R are unique up to a scalar factor in L and R. Using the Hilbert volume in PO_n^+ this factorization can be understood in the following way. For a fully indecomposable matrix $A \in \mathcal{P}$ the Hilbert volume in PO_n^+ is contracted least at a unique ray in PO_n^+ passing through a vector $v \in O_n^+$. For a fully indecomposable doubly stochastic matrix (or its multiple) $v = \mathbf{1}$, in particular the ray has the direction of an eigenvector. Conversely if for a

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