Infinite Electrical Networks: A Reprise

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Abstract — This is a tutorial paper on resistive infinite electrical networks presented at an undergraduate level. Rather than being a comprehensive survey on the wide variety of results existing in this subject, it introduces the basic ideas with several examples and puzzles, examines how the theory branches into two separate avenues of investigation, points out how infinite networks differ in their behavior from finite networks, and ends with a brief survey of the literature. No apology is offered for several cheeky remarks.

I. A Tale and a Moral

ONCE UPON A TIME, an electrical engineer set out to discover the value of the driving-point resistance R between two adjacent nodes a and b in an infinite resistive square grid of 1-Ω resistors shown in Fig. 1(a). He fulfilled his quest as follows: He first connected one of the terminals of a 1-A current source to node a and the other terminal to infinity, as shown in Fig. 1(b). (He reached infinity through some magic he happened to have at hand —this is a fairy tale.) That source current was directed toward node a. The engineer found that the 1 A split up evenly into 1/4-A currents flowing away from node a through the four grid branches incident to node a. This was to be expected because the grid was symmetric around node a. Next, he removed that current source and connected another 1-A current source between node b and infinity, but this time he directed the source current away from node b, as shown in Fig. 1(c). This resulted in a 1/4-A current flowing toward node b in each of the grid branches incident to b. He knew from the superposition principle that, when both sources were connected simultaneously, the current in the branch between a and b would be the sum of the currents occurring when each source was connected all by itself, that is, 1/2 A would flow, as shown in Fig. 1(d). Moreover, the simultaneous connection would both inject and extract 1 A at infinity; therefore, he concluded, the connection at infinity could be removed and the two 1-A sources could be combined into the single 1-A source shown in Fig. 1(d). Thus with the latter connection, a potential of 1/2 V would appear across the branch between nodes a and b, by virtue of Ohms law. In this way the engineer determined that $R = 1/2\ \Omega$.

"Suddenly, a mathematician appeared. "Stop," said he, "You can't do that."

"What do you mean I can't do that," retorted the engineer, "I just did."

"Look," said the mathematician, "if you're going to determine node voltages in a network by connecting current sources out to infinity, you've first got to tell us what the potential at infinity is. Moreover, your 1-A current source from infinity to some node is a discrete version of a 1-A current injection into an infinite uniform conducting plate; for the latter the potential function is proportional to $\log r$, where r is the radial distance from the point of current injection. Now, I've got some bad news for you," continued the mathematician. "The logarithm function has an infinite range. So, if you take the potential at infinity to be zero or, for that matter, any finite value, then any node you might point to will have an infinite potential. What you've really done is the following: With the first current-source connection to infinity, you have subtracted an infinite potential at node b from an infinite potential at node a to get a finite voltage drop between those nodes; in short, you've written $\infty - \infty = 1/4$. Similarly, with the second current-source connection to infinity, you have again obtained tacitly $\infty - \infty = 1/4$. Finally, appealing to superposition, you have added those two equations to obtain $\infty - \infty = 1/2$. You can't do that."

At a loss for a better response, the engineer shifted the argument. "Look, if we engineers stopped whenever you mathematicians told us to stop, we'd never get anything done. After all, it was our rockets that reached the moon, and they would never have been launched if we had to "prove" everything we did."

"Oh really? Tell me about the space shuttle," the mathematician shot back and then disappeared in a puff before the rejoinder could be made that it was an engineer who warned of the shuttle's problem.

Now, it happens that $R = 1/2$ is the "right answer." This can be seen by setting up (or simulating on a computer) a large finite square grid of $1 - \Omega$ resistors, connecting a 1-A current source across a central branch, and then noting that the voltage drop across the branch gets closer and closer to 1/2 as the grid is made larger and larger. Engineers "knew" this fact through their symmetry argument, faulty though it may be for mathematicians, while mathematicians "proved" it through a formal justification, fussy though it may be for engineers.

The moral of all this—at least for engineers—is that it really does not matter how heuristic one's reasoning is so long as the right answer is obtained. This remark is not meant to be either facetious or flippant. It asserts an important principle of engineering practice. One does not

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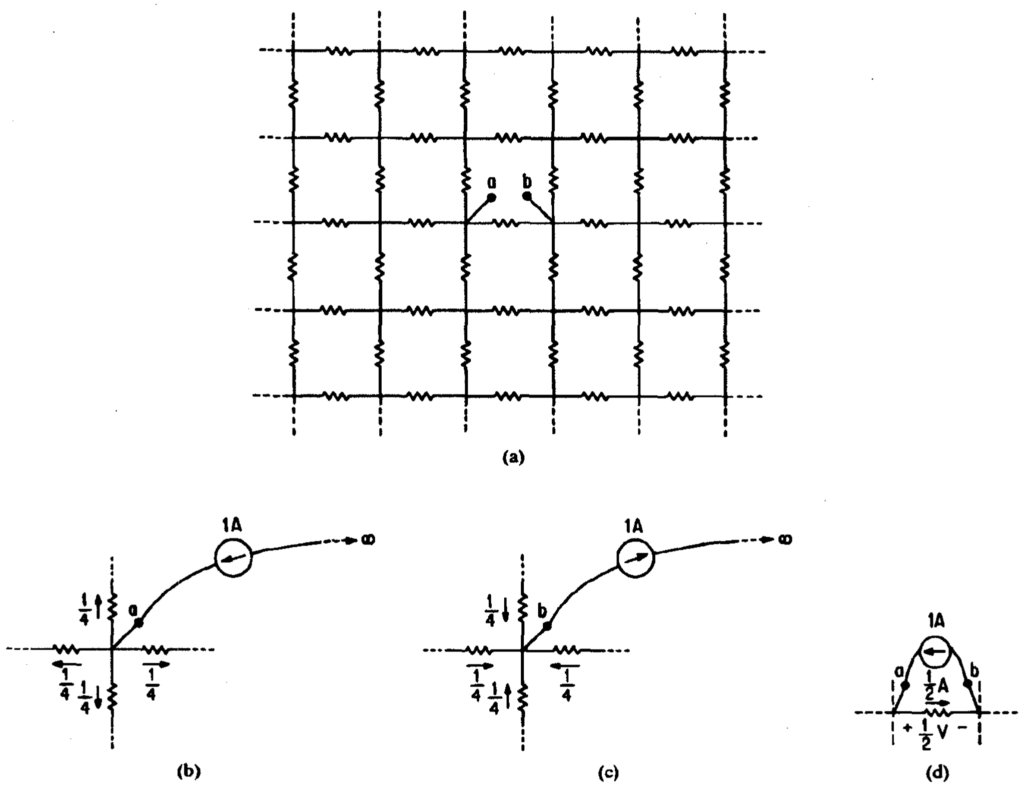

Fig. 1. (a) An infinite square grid, each branch of which is a 1-Ω resistor. The driving-point resistance R between the two adjacent nodes a and b is to be determined. (b) A current source injecting 1 A into node a from infinity. (c) A current source extracting 1 A from node b and sending it out to infinity. (d) The situation when both current sources are imposed simultaneously and then combined into a single source.

stop just because a theory fails us or a proof is unavailable. Any argument, any experiment, any hunch that leads to a worthwhile result is justified by that result. In engineering research and development, the ends do justify the means. (We are referring here to engineering methodology, not to unethical, immoral, or illegal means.) Of course, more successes and fewer failures are achieved when hunches are based upon valid reasoning. The point here is that engineering endeavor is generally quite different from mathematical endeavor.

II. Science, Technology, and Engineering

This paper is a tutorial exposition of a mathematical theory spawned by electrical engineering. Much new mathematics has been and continues to be generated by engineering, but this is incidental to engineering's primary function. So, let us digress still further to examine briefly how mathematics—and science as well—relate to engineering.

Engineering students, having struggled through physics, chemistry, and calculus during their freshman and sophomore years, are well aware of the fundamental roles that science and mathematics play in engineering. In fact, the college experience can lead to the view that engineering is simply applied science. In short, the scientist creates a theory or the mathematician proves a theorem, and the role of the engineer is to apply it somehow to benefit society; that is, scientists and mathematicians create, society consumes, and engineers are the middlemen. Hardly. Engineering is more than applied science. It also originates from another source: technology—useful products, structures, and techniques often empirically developed and not necessarily derived from scientific comprehension. (We shall henceforth use the word "technology" in this particular way.)

The flowering of science has occurred only during the last two or three hundred years, although its roots are considerably older—extending back about two or three millenia to its origins in the Middle East, India, and China, with incipient activities in other parts of the world such as Mesoamerica. Technology, on the other hand is older, indeed, very much older. A technological revolution occurred ten thousand or so years ago with the domestication of plants and animals, the development of agriculture, and the consequent rise of civilizations. However, technology is even older than that. Its roots are lost in prehistory, and archeologists tell us that technological artifacts date back to the evolutionary beginnings of humankind. Stone toolmaking and the controlled use of fire are just two of the attributes that set human beings apart from all other forms of life. In fact, some argue that there is a strong link between primitive technology and human evolution. (See

[23] or [30]). For instance, the opposable thumb enabled the handling of tools, and reciprocally the development of tools bestowed an evolutionary advantage upon the opposable thumb; so, tools and thumbs were symbiotic during the genesis of homo sapiens. More generally, a technological imperative, which is characteristic of human behavior, is an evolutionary heritage dating back perhaps many hundreds of thousands of years—or so the speculation goes.

Engineering derives not only from science but also from technology, and the impact of the latter should not be slighted. For example, the structural arch was used ubiquitously in Roman architecture without the Romans knowing the science of solid mechanics. More recent examples are the typewriter, the mechanical clock, and the bicycle. These were not designed from theoretical learning. Tinkerers, innovators, and inventors exploited practical workshop techniques to produce them as well as many other goods for which we now credit science.

The importance of empirical technological development is masked by the current prominence of high technology, which depends so heavily upon science. Electrical engineering students study Maxwell's equations but hear only anecdotes about Thomas Edison, who contributed so much to the initiation of the electrical industry. This is as it should be, for modern engineering practice requires scientific knowledge but hardly the history of inventions. Nonetheless, engineering students should be aware that many latter-day Edisons are contributing a steady stream of trial-and-error improvements without making sophisticated scientific analyses, that indeed inventiveness is crucial to engineering advancements but is not and probably cannot be taught in a formal way.

The purpose of these remarks is to emphasize the difference between engineering and science. Engineering has its own distinctive objective and should not primarily be judged as a science. The goal of science may be succinctly stated as the creation of new knowledge, whereas the aim of engineering is to utilize science and current technologies in order to design and improve new technologies. Those distinctive goals, the creation of knowledge and the development of technology, should not be confused. However, the interaction between science and mathematics on the one hand and engineering on the other has become so strong that these disciplines have become intertwined and in certain ways indistinguishable. For example, engineering is continually spawning new science and mathematics. Much of engineering research, especially in academia, is aimed at the creation of the science and mathematics that nourish engineering advancements. Indeed, mathematical and scientific endeavor is now an integral part of engineering. Unlike engineering in general, this activity should be held to the same standards that the sciences and mathematics impose upon themselves.

In particular, when an engineer publishes a "theorem" in an engineering journal, he purports to be doing mathematics and his theorem should be subjected to the criteria of rigor and proof that mathematics requires. The argument that $\infty - \infty = 1/2$ remains mathematically invalid, even though it may in some context be useful and warranted by its end result. In the prior section that end result was a fairly convincing argument that $R = 1/2$, so convincing in fact that one is surely justified in basing an engineering design upon it and relying upon a testing program to make sure that the design truly works.

The theory of infinite electrical networks is an example of mathematics spawned by engineering. It is not in the mainstream of either mathematics or engineering. The subject seems to be too much like electrical engineering to attract mathematicians and too much like mathematics to attract engineers. Nonetheless, it has been steadily developing, it is accessible to anyone with a knowledge of circuit theory and some elements of functional analysis, and, most importantly, it does have practical applications. The rest of this paper explains various parts of the subject at an undergraduate level of exposition. So, with no further apology, let us now plunge into the thicket of infinite electrical networks.

III. Them

They are all around us. We just haven't been paying attention. Mathematicians allude to them with a fancy name: "the exterior problem." Indeed, there are a variety of partial differential equations, such as Poisson's equation, the heat equation, the acoustic wave equation, and polarized forms of Maxwell's equations, whose finite-difference approximations are realized by electrical networks. So, if the domain at hand is infinite in extent, then a discretized analysis leads to an infinite electrical network.

For example, a current area of research is the computer-aided determination of the capacitance coefficients of VLSI interconnection wires. A typical model consists of several wires of various shapes above a grounded conducting plane, which represents the semiconducting chip. Laplace's equation is to be solved in the infinite region between the wires and above the conducting plane, when particular electrical potentials are assigned to the wires. Analytical solutions are available only for the simplest geometries (e.g., a single, infinitely long wire with a circular cross section). In fact, a numerical analysis must be used for virtually every practical model. The discretization required by the numerical analysis leads to an infinite, purely capacitive network [59]. The latter can be treated as a purely resistive network just by changing the meaning of the symbols. In that network, the ground plane is represented by an infinite node, that is, by a node having an infinity of incident branches. The conventional approach to this problem truncates the infinite network into a finite one by introducing an artificial grounded boundary surrounding all the wires but at some distance away from the wires. However, this introduces a truncation error. It would be better if the infinite network could be solved directly. This has recently been accomplished [59], and in fact the computations required by the infinite-network solution are considerably fewer than those needed by a reasonably accurate, finitely truncated network.

Another example is provided by the resistivity method of geophysical exploration, in which a large electrical current is injected into and extracted from the earth and the resulting potentials along the earth's surface are measured. The domain is now the semi-infinite region below a horizontal plane (the flat-earth model), and Laplace's equation again governs the phenomenon. Once again, a discretized analysis calls for a solution to an infinite resistive network [61], [62]. In the same way, the flow of petroleum from the earth into the bottom of an oil well can be represented by the flow of current in an infinite spherical grid of resistors [63]. As a final example, consider an electromagnetic method of geophysical exploration where a polarized electromagnetic wave is radiated into the earth. In this case, the appropriate model is an infinite *RLC* network [56].

There is a good reason to view infinite electrical networks as practical models of important problems and, therefore, as comprising a compelling research area. To be sure, the jump in complexity from finite networks to infinite ones is comparable to the jump in complexity from finite-dimensional spaces to infinite-dimensional spaces. On the other hand, the theory of infinite electrical networks is still in its puberty with many questions largely unexplored, especially with regard to computational problems. As compared to the networks research currently being applied to other areas, such as nonlinear, distributed, large-scale, active, or digital finite networks, the amount of effort being expended on infinite networks is meager. Infinite networks deserve better attention. Let's take a closer look at them.

IV. Some Precise Definitions

We start with a definition of a finite electrical network. With this in hand, we can examine the peculiar problems that can arise in formulating the idea of an infinite electrical network. The mathematical style we now adopt will last only through this section. The rest of the paper will be presented in a manner more common to engineering.

A finite electrical network is a finite graph upon which an analytical structure has been imposed. The graph is defined as the pair (N, B). N is any finite set, and its elements are called nodes. We think of the nodes as points in space, whereas in various applications they could represent other ideas, such as places where wires are connected together in electrical circuits, bus or train stations in transportation networks, atoms in crystalline structures, and so on. Actually, however, the definition of a graph does not require any interpretation to be assigned to a node; it can be simply an abstract entity. Furthermore, B is a family of unordered pairs of not necessarily distinct nodes; these pairs are called branches. Thus if n and m denote two distinct nodes, then the set $\{n, m\}$ and the family $\{n, n\}$ are both branches. Conventionally, a branch is visualized as a line connecting the nodal points n and m or connecting n to itself. Moreover, a "family" is different from a "set" in these definitions; whereas the elements are all different from one another in a set, this need not be so in a family, that is, an element may appear more than once in a family. We shall also denote a branch by the symbol b_j; no two branches will have the same index j, even among the repetitions of a node pair.

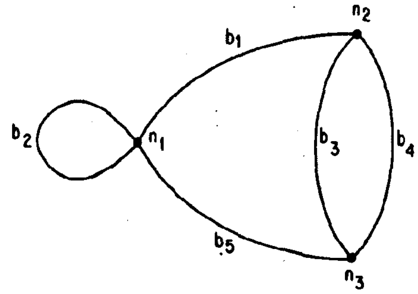

Fig. 2. A graph having a self-branch b_2 and two parallel branches b_3 and b_4.

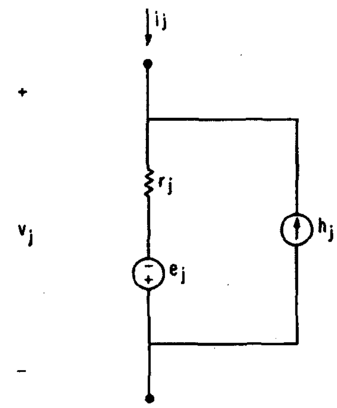

Fig. 3. The standard circuit symbolism for the five real numbers assigned to branch b_j. The branch's orientation is from top to bottom.

As an example, consider the graph in Fig. 2. The node set is $\{n_1, n_2, n_3\}$ and the branch family is $\{\{n_1, n_2\}, \{n_1, n_1\}, \{n_2, n_3\}, \{n_2, n_3\}, \{n_1, n_3\}\}$. We have designated those branches by b_1, b_2, b_3, b_4, and b_5, respectively. A branch with only one node, such as b_2, is called a self-branch, and branches with the same node pair, such as b_3 and b_4, are said to be in parallel.

In order to impose an analytical structure, we first specify an orientation to each branch. Then, each branch b_j is assigned four real numbers: v_j, i_j, e_j, and h_j, which we refer to, respectively, as the branch voltage, branch current, branch voltage source, and branch current source. Each of these quantities are said to be in the direction of the branch's orientation if it is positive, and opposite to that direction if it is negative. Furthermore, a fifth real number r_j, which is now required to be positive, is also assigned to b_j and is called its branch resistance. By definition, these quantities are related to each other by the following equation, namely, Ohm's law.

$$v_j + e_j = r_j(i_j + h_j). \tag{4.1}$$

Fig. 3 illustrates the analytical structure of a typical branch b_j using standard circuit symbolism.

To complete the definition of a finite network, two more laws are required. The first of these is Kirchhoff's current law, which is imposed at every node. Let n denote a particular node and let the branches incident to n be $b_{j_1}, b_{j_2}, \cdots, b_{j_K}$. Then, Kirchhoff's current law asserts that

$$\sum_{k=1}^{K} \pm i_{j_k} = 0 \tag{4.2}$$

where the plus sign is used if b_{j_k} is oriented toward n and the minus sign is used otherwise.

The second law is Kirchhoff's voltage law, which is required to hold around every loop of the network. A loop is a finite alternating sequence of nodes n_k and branches b_j: $n_{k_1}, b_{j_1}, n_{k_2}, b_{j_2}, \cdots, n_{k_M}, b_{j_M}, n_{k_1}$, where each branch is incident to the two nodes immediately preceding and succeeding it in the sequence and no node appears more than once in the sequence except for the starting and ending elements; those two elements are one and the same node, which appears no place else in the sequence. An orientation is assigned to every loop by choosing one of the two ways of tracing through the sequence (i.e., from left to right or from right to left). For a given loop Kirchhoff's voltage law asserts that

$$\sum_{m=1}^{M} \pm v_{j_m} = 0 \tag{4.3}$$

where the plus sign is used if the orientations of b_{j_m} and the loop agree and the minus sign is used otherwise. Since the network is finite, there are only a finite number of distinct loops.

Our complete definition of a finite network consists of the finite graph (N, B), the analytical structure illustrated in Fig. 3, and the three laws (4.1), (4.2), and (4.3). It can be proven [37] that every finite network has a unique voltage–current regime, that is, one and only one assignment of branch voltages and branch currents that satisfy (4.1) on every branch, (4.2) at every node, and (4.3) around every loop.

Actually, what has been defined here is a finite resistive network with independent voltage and current sources, but no dependent sources nor mutual coupling between branches. More complicated analytical structures can be defined for electrical networks by allowing v_j, i_j, e_j, and h_j to be functions of time or of a complex variable, by allowing r_j to be zero, negative, or complex, or by replacing r_j with a differential or integral operator or some still more general, linear or nonlinear, time invariant or time varying, mathematical operator. Furthermore, in our definition each branch is a two-terminal device, but n-terminal devices could be allowed; this would complicate both the constituent relationship (4.1) and the graph-theoretical basis of the network. We will not allow any of these generalizations because a discussion of the more complicated kinds of infinite networks would require a substantial amount of functional analysis.

Let us turn now to the idea of an infinite electrical network. A rigorous construction of such a concept leads to a series of perplexing choices, the first of them being just what kind of infinities should be allowed in the definition of the network's graph (N, B). To avoid the formidabilities of uncountably infinite sets, we shall restrict N to a finite or countably infinite set of nodes and B to a countably infinite (not finite) family of unordered pairs of not necessarily distinct nodes. As before, the members of N are all distinct. However, any member of B can appear in B finitely or infinitely many times. Thus an infinity of parallel branches may occur between a given pair of nodes.

As for the analytical structure, we begin exactly as before. Four real numbers v_j, i_j, e_j, and h_j and a positive number r_j are assigned to each branch b_j and interpreted in accordance with Fig. 3. The assumptions imposed up to this point can be illustrated with infinite-network diagrams, examples of which appear in most of the figures of this paper.

To complete our definition of an infinite electrical network, we need some laws to interrelate the five quantities of the analytical structure. Unfortunately, Ohm's law (4.1) and Kirchhoff's laws (4.2) and (4.3) are not enough to yield a unique voltage–current regime, except in certain trivial cases such as an infinite collection of finite networks. We now examine some particular infinite-network diagrams to illustrate various difficulties that can arise if insufficient care is taken in the definition of an infinite electrical network.

V. Hide and Go Seek

Let's play a game. I'll hide a mistake somewhere in the next paragraph, and your task is to find it. That a mistake is there will be obvious, for my argument will lead to a mathematical absurdity.

Consider the infinite ladder network indicated in Fig. 4(a). The resistance values are understood to extend infinitely to the right by continuing the indicated pattern of numbers. R denotes the driving-point resistance as measured from the two input terminals on the left. Upper and lower bounds on R can be obtained by using a basic principle of purely resistive networks, namely, any driving-point resistance is a monotone nondecreasing function of every branch resistance. As a result, R is no less than the driving-point resistance R_L obtained by letting every horizontal resistor in Fig. 4(a) tend to a short circuit. The limiting network is shown in Fig. 4(b). Specifically, the voltage drops become confined to the vertical resistors only, and a parallel connection of an infinity of resistors is obtained. By the formula for parallel resistances,

$$R \geq R_L = \frac{1}{\dfrac{1}{10} + \dfrac{1}{100} + \dfrac{1}{1000} + \cdots} = 9. \tag{5.1}$$

The aforementioned basic principle also implies that R is no larger than the driving-point resistance R_U obtained by letting every vertical resistor in Fig. 4(a) tend to an open circuit. This confines the current flow to the horizontal resistors, and we obtain the infinite series circuit of Fig.

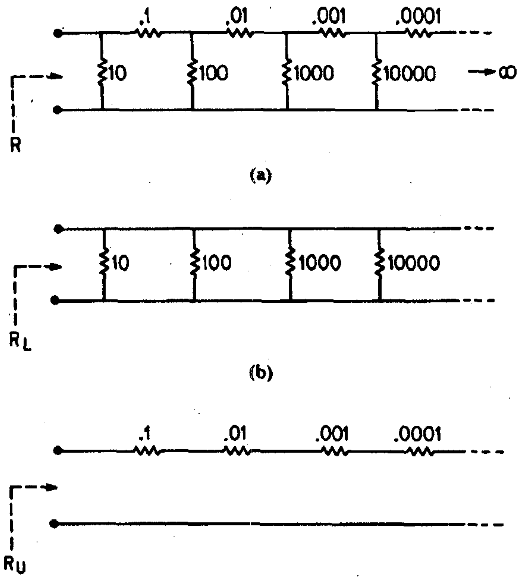

Fig. 4. (a) An infinite ladder network. All resistance values are given in ohms. R is the driving-point resistance as observed from the input terminals. (b) The infinite parallel circuit obtained by letting every horizontal resistor tend to a short circuit. (c) The infinite series circuit obtained by letting every vertical resistor tend to an open circuit.

4(c). Thus

$$R \leqslant R_U = 0.1 + 0.01 + 0.001 + \cdots = \frac{1}{9}. \tag{5.2}$$

Equations (5.1) and (5.2) taken together yield

$$9 \leqslant \frac{1}{9} \tag{5.3}$$

the promised absurdity. What went wrong? Review this paragraph to see if you can find the flaw, if you have not found it already. Hint: The given argument is perfectly correct for finite networks, but now we have an infinite one.

Here's the answer: We have tacitly used two different conditions at infinity. When computing R_L, we assumed in effect that there is no connection at infinity between the upper and lower horizontal portions of the network. This open circuit at infinity forces the currents to flow through the vertical resistors of Fig. 4(b). On the other hand, when computing R_U, we tacitly assumed that there was a short circuit at infinity, which completed the series connection of Fig. 4(c). All this shows that at least for some infinite networks it truly does matter what is occurring at infinity. In fact, for Fig. 4(a) we should deal with two driving-point resistances, R_{oc} for the case of an open circuit at infinity and R_{sc} for the case of a short circuit at infinity. Upon repeating the argument of the last paragraph for each case, we obtain $9 \leqslant R_{oc} \leqslant \infty$ and $0 \leqslant R_{sc} \leqslant 1/9$. We may say that "infinity is perceptible" to the observer at the input terminals of Fig. 4(a).

This matter can be explained still further by using the series and parallel rules for combining resistances to write the driving-point resistance—for *both* the open-circuit and short-circuit cases—as the infinite continued fraction

$$\cfrac{1}{0.1 + \cfrac{1}{0.1 + \cfrac{1}{0.01 + \cfrac{1}{0.01 + \cfrac{1}{0.001 + \cfrac{1}{0.001 + \cfrac{1}{0.0001 + \ddots}}}}}}} \tag{5.4}$$

It happens that this is a divergent continued fraction [39, p. 28]. In particular, its odd and even truncations converge to two different limits. Indeed, if we replace the nth horizontal resistor in Fig. 4(a) by an open circuit and then let $n \to \infty$, we obtain the odd truncations of (5.4), which converge to the value $R_{oc} = 9.001 \cdots > 9$, as can be seen with a hand calculator. Furthermore, if we replace the nth vertical resistor in Fig. 4(a) by a short circuit and then let $n \to \infty$, we get the even truncations of (5.4), which converge to $R_{sc} = 0.1098 \cdots < 1/9$.

We should mention that for some infinite ladder networks, it does not matter at all whether we have an open circuit or short circuit at infinity. For example, if we change every resistance value in Fig. 4(a) to 1 Ω, we obtain the infinite continued fraction

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

which converges [39, p. 120], that is, its odd and even truncations both converge to the same limit: $(\sqrt{5} - 1)/2$. In this case, infinity is not perceptible to an observer at the input terminals, in contrast to Fig. 4(a). Moreover, the argument of the second paragraph of this section now yields $R_L = 0$ and $R_U = \infty$, so no absurdity arises despite the two different tacit assumptions at infinity. In fact, we now have $R_{oc} = R_{sc} = (\sqrt{5} - 1)/2$. In this case, we might say that "infinity is imperceptible" to the observer at the input terminals.

One last matter before we leave this section. It seems intuitively clear that the short-circuit connection for R_{sc} can be made at the "ends at infinity" of the two horizontal portions of Fig. 4(a). However, how can we define the "ends at infinity" of more general kinds of infinite networks? We will need to do this if we are going to specify what occurs at infinity. This is discussed in Section IX and in much greater detail in [57].

VI. The Trouble with Kirchhoff

Mr. Kirchhoff is capable enough with finite networks, but he is not too reliable when it comes to infinite networks. See how he handles the infinite network of Fig. 5(a) wherein a 1-V voltage source in series with a 1-Ω resistor is connected to an infinite parallel circuit of 1-Ω resistors. The infinite parallel connection should be equivalent to a short circuit, according to the rule for combining parallel resistances, and so the voltage v between nodes a and b should be zero. Hence, the current i through the source ought to be 1 A, whereas the currents flowing through each of the purely resistive branches ought to be zero. However, calculus is unambiguous about the fact that an infinite series of zeros sums to zero. Therefore, we are led to conclude that 1 A flows toward node a while 0 A flows away from it.

Perhaps our supposition that $v=0$ is wrong; perhaps $v \neq 0$. If so, then $i=(1-v)$ A and the current flowing downward through every purely resistive branch is v A. In this case, calculus dictates that an infinite series of nonzero constants v, all identical, is infinite. So, now we have a finite current flowing toward node a, and an infinite current flowing away from it.

We have to conclude that Kirchhoff's current law fails at node a—and at node b too. Actually, we might blame the mathematicians who spoiled the calculus with their ϵ's and δ's. In the early days of calculus one could circumvent the above discrepancy in current flows by working with quantities that are extremely small (smaller in absolute value then any positive real number) but are nonetheless not zero: namely, the infinitesimals.

For instance, we could approximate our infinite network by using a large but finite number n of parallel 1-Ω resistors, as indicated in Fig. 5(b). Let us now denote the currents flowing in this finite network by $i_{n,0}, i_{n,1}, \cdots$, as shown. The first subscript n is simply a parameter indicating the number of purely resistive branches. By Ohm's law and Kirchhoff's laws, $v=1/(n+1)$, $i_{n,0}=n/(n+1)$, and $i_{n,j}=1/(n+1)$ for all $j=1,\cdots,n$. Moreover, Kirchhoff's current law at node a is

$$i_{n,0}=\sum_{j=1}^{n} i_{n,j}.$$

There is no problem in passing to the limit as $n \to \infty$ on both sides of this equation; we obtain $1=1$ so long as we sum first and then take the limit on the right-hand side. However, in passing to the limit for the current $1/(n+1)$ in each resistive branch, we have in effect interchanged these two processes: We have taken the limit and then summed. This is invalid, for we get $1=0$. (It would have been valid had $\sum_{j=1}^{n} i_{n,j}$ converged uniformly with respect to all n, but this unfortunately is not the case).

But wait; let's relax a bit. Let's say that the 1-A current i flowing toward node a in Fig. 5(a) splits up into an infinity of infinitesimals and that the sum of those infinitesimals is 1. That is, the current in each purely resistive branch of Fig. 5(a) is infinitesimally small and yet

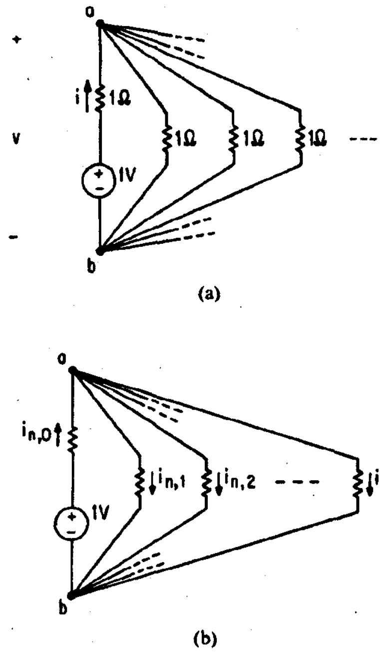

Fig. 5. (a) An infinity of 1-Ω resistors all connected in parallel with a branch having a 1-Ω resistor and a 1-V source in series. (b) A finite network that approximates the infinite network. All resistors are again 1 Ω.

Kirchhoff's current law continues to hold. Standard calculus does not allow this, but there is a nonstandard calculus that may [32], [35], that in fact rigorously resurrects the infinitesimals. I have felt for some time that a more satisfactory theory of infinite electrical networks, which eliminates the present discrepancy and others of a similar nature, might be built upon nonstandard analysis. However, the months—probably years—of effort it would take me to penetrate the mysteries of nonstandard analysis would go unrewarded if appreciation by engineers is the goal, for who would understand a nonstandard network theory? It is expedient to stay with standard calculus and to build a theory that at times allows the nonsatisfaction of Kirchhoff's current law at an infinite node and (i.e., at a node with an infinity of incident branches).

Standard calculus also forces at times the nonsatisfaction of Kirchhoff's voltage law. That law may fail if we try to apply it to an extended loop, that is, an infinite path that extends out to infinity in two directions and whose "ends at infinity" are shorted together. Such a network can be constructed by taking the electrical dual of Fig. 5(a). This is shown in Fig. 6. By interchanging voltages and currents in the arguments given above for Fig. 5(a), one can show that Kirchhoff's voltage law cannot be satisfied around that extended loop if all resistance values are 1 Ω.

However, it is not true that Kirchhoff's current law is never satisfied at an infinite node and that Kirchhoff's voltage law is never satisfied around an extended loop. A theory in which these laws do hold at certain infinite nodes and around certain extended loops exists [41], [44], [57] and will be described briefly in Section IX.

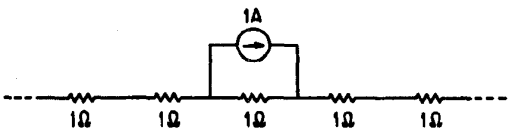

Fig. 6. An extended loop consisting of an infinity of 1-Ω resistors all connected in series with a branch having a 1-Ω resistor and a 1-A source in parallel. The network extends infinitely to the left and to the right and its "ends at infinity" are shorted together. Kirchhoff's voltage law cannot be satisfied around this extended loop.

VII. Two Roads Diverged

As yet, we have not indicated any method for determining a branch current in an infinite network. Here's one method that almost always works: Just guess!

For example, consider the infinite network of Fig. 7(a); e and all the r_j are given and it is desired to find the current i through the voltage source when Ohm's law and Kirchhoff's two laws are satisfied throughout the network. Any arbitrary choice of i is correct in the sense that there is a corresponding voltage–current regime that satisfies Ohm's law and Kirchhoff's two laws everywhere and for which i is the chosen value. Indeed, since i also flows through r_1 and r_2, the voltage drop upward across r_3 is $v_3 = i(r_1 + r_2) - e$, and so the current flowing upward through r_3 is $i_3 = v_3/r_3$. Then, Kirchhoff's current law applied to the two nodes of r_3 dictates that the currents flowing to the right in r_4 and to the left in r_5 are both $i + i_3$. Then, Kirchhoff's voltage law applied to the second loop shows that the voltage drop upward across r_6 is $v_6 = v_3 + (i + i_3)(r_4 + r_5)$. This determines the current in i_6, which by Kirchhoff's current law determines the currents in r_7 and r_8. These manipulations can be continued indefinitely, and thus there truly is a voltage–current regime of the asserted sort. A numerical example is shown in Fig. 7(b), where it has been assumed that e is 1 V and all the r_j are 1 Ω; upon choosing $i = 1$ and repeating the above manipulations, we obtain the indicated branch currents.

Had the network been finite, in particular, had r_6 been the last resistor toward the right, only one choice of i would be correct, namely, e/R where

$$R = r_1 + r_2 + \cfrac{1}{\cfrac{1}{r_3} + \cfrac{1}{r_4 + r_5 + r_6}}.$$

For any other choice of i, the above construction of a voltage–current regime would violate Kirchhoff's current law at the nodes of r_6 because of the absence of r_7 and r_8.

Let us return to Fig. 7(a). The power from the voltage source is ei watts, a finite amount. On the other hand, the power dissipated in all the resistors might well be—and generally is—infinite. This indeed is so for Fig. 7(b). What is happening is that there is an implicit power source out at infinity pumping energy into the network. By specifying i, we *indirectly* specify that power source at infinity. Upon choosing a different i, we specify indirectly a different (probably infinite) power source at infinity. Moreover, there is no way of directly specifying an infinite power

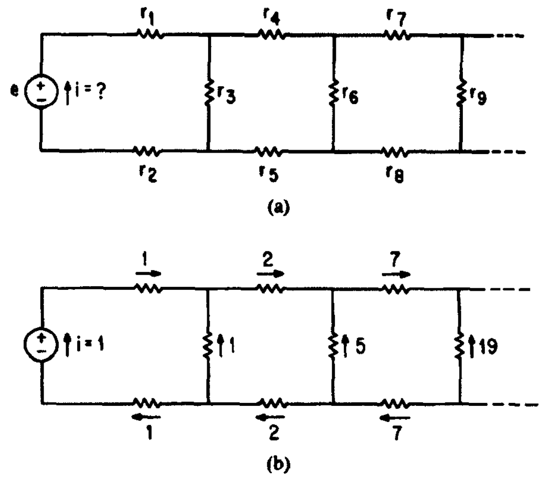

Fig. 7. (a) An infinite network in which e and all the r_j are given. It is desired to determine i. It turns out that i can be any number at all. (b) The branch currents when $e = r_1 = r_2 = \cdots = 1$ and i is chosen to be 1 as well.

source out at infinity; standard calculus does not allow us to distinguish one infinite source from another infinite source.

Special cases do arise where the power source at infinity is finite and perhaps zero. The zero source occurs when i is chosen to be e/R_{oc} or e/R_{sc}, where R_{oc} and R_{sc} are the driving-point resistances as observed from the voltage source e for, respectively, an open circuit and a short circuit at infinity. Furthermore, if the r_j decrease in value quickly enough as $j \to \infty$, it is possible to have a finite power source at infinity whose value can be assigned arbitrarily through an appropriate choice of i. This last statement is not obvious; it is a consequence of the results of [57].

We can state all this in a different way: Ohm's law and Kirchhoff's two laws do not by themselves determine a unique voltage–current regime (except for certain special infinite networks such as an infinite block-star network whose blocks are finite). This leads to two divergent ways of studying infinite electrical networks. One way is to impose only Ohm's law and Kirchhoff's laws and to examine the whole class of different voltage–current regimes that the network can have. Typical questions in this approach are the following. Can one always specify any permitted voltage–current regime by assigning currents to certain branches? Can arbitrary currents be assigned to more than one branch? If so, for how many branches can this be done, and how are those branches chosen? Does the above procedure for constructing a voltage–current regime always work, and, if not, can it be generalized to make it work? This is the less traveled road in the investigation of infinite electrical networks. A smaller body of results have been obtained along it. Nonetheless, the posed questions have all been answered. This is discussed in the next section.

The other road through infinite network theory starts with the following question. What additional requirements

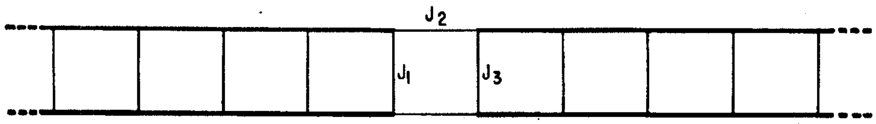

Fig. 8. An infinite network in which the maximum number of non-touching one-ended paths is four. The thick lines represent a set of four such paths. J_1, J_2, and J_3 are three possible joints.

must be added to Ohm's law and Kirchhoff's voltage and current laws in order to ensure the existence of a *unique* voltage–current regime? This has been answered in different ways and has led to various results that extend properties of finite networks. Much more has been achieved in this direction. It is in fact the main road in infinite network theory and leads to some practical applications. We will describe one such result in Section IX.

VIII. The One Less Traveled By

Let us now use only Ohm's law and Kirchhoff's laws as the governing equations. In this section we shall describe the key idea underlying an analysis of the many different voltage–current regimes that can appear in a given infinite network.

We are trying to determine how many degrees of freedom a network has with regard to arbitrary choices of certain branch currents. Those degrees of freedom arise from the fact that unspecified sources may be connected to the network out at infinity. This leads to the question of how many different ways one can reach out to infinity through the network using one-ended paths that do not touch. A one-ended path is simply a tracing along branches that starts at some node and continues on indefinitely without ever looping back on itself. It turns out that the more non-touching one-ended paths one can squeeze into the network, the more independent connections at infinity the network can sustain. So finally, the basic task is to find a maximal set of disjoint one-ended paths in the network —maximal with regard to the number of paths.

When this has been done, the next step is to extend the paths so that every node is in a path and the resulting paths remain non-touching. (Actually, this may require that the paths be extended into more complicated subnetworks, but in our examples they will remain one-ended paths.) Finally, we choose branches, called *joints*, such that the following conditions are satisfied: (i) Each joint connects together two of the extended paths. (ii) The subnetwork consisting of all the joints and extended paths has no loops, that is, it is a tree. The joints are the branches in which we may choose currents arbitrarily. This construction can be performed in many different ways. Therefore, an infinite network has in general many different sets of joints, but every set of joints will have the same number (i.e., cardinality) of joints. It is worth mentioning at this point that a method of analyzing an infinite network based upon the selected paths and joints has been devised for determining all the other currents in the network once the joint currents have been assumed [48].

These ideas may (indeed, should) strike you as being nebulous. Nonetheless, they do have precise versions and

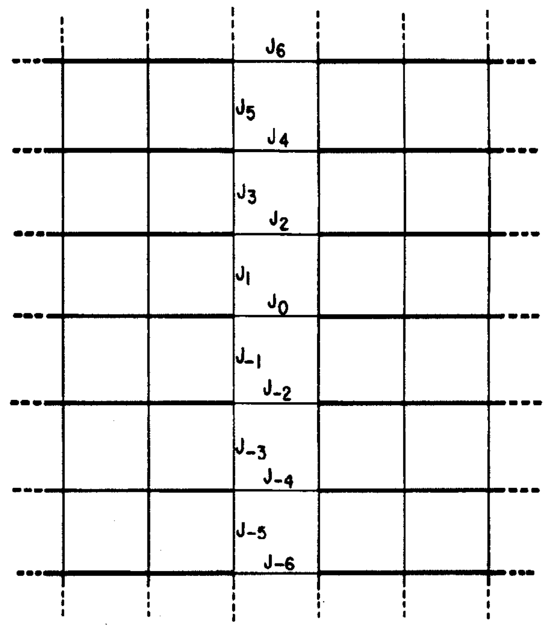

Fig. 9. A maximal set of non-touching one-ended paths is indicated for this infinite square grid. The branches J_k comprise an appropriate set of joints.

rigorous proofs, which however are beyond the scope of this paper. Our brief description along with the following examples should give you an intuitive appreciation of the key concept.

For instance, what is the maximum number of non-touching one-ended paths that can be drawn in the ladder network of Fig. 8? The answer is four. One possible choice of the four paths, which together contain all the nodes, is indicated by the thick lines. Then, J_1, J_2, and J_3 comprise one of the many possible sets of joints satisfying conditions (i) and (ii). On the other hand, the three branches incident at any one node (that is, three branches that form a "T") would fail to satisfy those conditions, for there is no way they could connect together four nontouching one-ended paths—nor would Kirchhoff's current law be satisfied in general at their central node if their currents were chosen arbitrarily. Note also that the number of joints for this network must be exactly three under conditions (i) and (ii).

As another example, consider the infinite square grid of Fig. 9. How many disjoint one-ended paths can be traced in it? An infinity of them is now possible, as is shown by the thick lines. Moreover, a complicated argument, which we skip, shows that this set of paths is maximal. Finally, the branches labeled J_k comprise a possible set of joints.

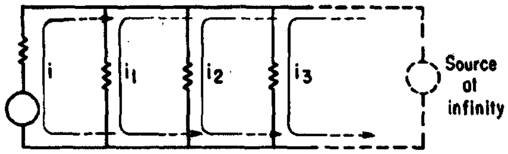

Fig. 10. The circuit of Fig. 5 redrawn with nodes a and b replaced by one-ended paths of short circuits. Upon choosing i arbitrarily, we can determine $i_1, i_2, i_3, \cdots$. Kirchhoff's current law is now satisfied at every node. The choice of i indirectly specifies a source of infinity, even though that source may be infinitely large.

Our method works even when infinite nodes are present. Those nodes behave as infinite paths of short circuits to which sources at infinity can be connected. A simple example of this is provided by Fig. 5(a), which can be redrawn as in Fig. 10 by replacing nodes a and b by one-ended paths of short circuits represented by the horizontal branches of Fig. 10. Upon choosing i arbitrarily as the single joint current, we can then determine the currents $i_1, i_2, i_3 \cdots$. In this case, Kirchhoff's current law is satisfied at every (now finite) node. Moreover, the choice of i indirectly specifies a source at infinity. This is one way of resolving the paradox regarding Fig. 5(a).

Have I piqued your curiosity? If someday you wish to see where these ideas lead to in mathematical circuit theory, read [46], [48], and [49].

IX. The Road not Taken

Unlike Robert Frost's traveler, we can easily return to where the two roads diverged and set out upon the one not as yet taken. So, let's do so, for that will prove to be the more rewarding journey.

Our aim now is to seek other requirements upon the infinite electrical network in addition to Ohm's law and Kirchhoff's laws that will eliminate all but one of the possible voltage–current regimes. A conspicuous requirement is finiteness for the total dissipated power. This works for some networks but not for all. The network in Fig. 4(a) with a 1-A current source connected to its input terminals has at least two finite-power regimes, one for an open circuit at infinity and another for a short circuit at infinity. Actually, it has an infinity of finite-power regimes, for any finite-power source can be connected out at infinity.

Thus it appears that a unique voltage–current regime might be established if two additional conditions are imposed, namely, finite total power dissipation and open circuits everywhere at infinity. Flanders [2] showed that this is truly so. He imposed the finite-power condition by allowing only a finite number of finite sources within the network, and he guaranteed the open-circuits-at-infinity condition by having the allowed current flows be the limits of current flows in an expanding sequence of finite subnetworks. He also assumed that there are no infinite nodes. More general theories, which relax these conditions, now exist [41], [44], [57]. Our aim in this section is to describe some of the key ideas in an intuitive fashion once again.

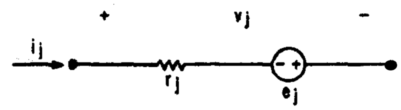

Fig. 11. The assumed Thevenin's equivalent form of every branch for the theorem in Section IX.

Given an infinite electrical network, number its branches with the index $j = 1, 2, 3, \cdots$. Assume that every branch appears in its Thevenin's equivalent form shown in Fig. 11. To insure that no more than a finite amount of power is dissipated in all the resistors, r_j, we require that the branch currents i_j satisfy

$$\sum i_j^2 r_j < \infty. \tag{9.1}$$

Throughout this section, Σ will denote a summation over all the indices j. With regard to the voltage sources, we require that

$$\sum e_j^2 / r_j < \infty. \tag{9.2}$$

It can be shown that the total power delivered by all the sources is never greater than $\Sigma e_j^2/r_j$, and so (9.2) insures that that power is finite as well.

Finally, we wish to allow current regimes that flow out to infinity, pass through connections there, and then flow back into the network. However, we do not want the connections at infinity to be infinite-valued sources; this would permit the many degrees of freedom discussed in the preceding section and would in fact violate (9.1).

The idea is to construct the sought-for current regime out of certain basic current flows; this is analogous to constructing a current flow in a finite network as a sum of fundamental mesh currents. One kind of basic current is a single flow around a finite loop; we will call it a *loop current*. Another kind is one that flows around an extended loop, that is, out along a one-ended path to infinity, through a short circuit at infinity, and then back along another one-ended path to the starting point; this will be called an *extended-loop current*. For example, a flow through J_3 and the two one-ended paths on the right in Fig. 8 would be an extended-loop current. Whenever an extended-loop current is allowed in the construction of a solution, it is tacitly being assumed that a short circuit at infinity connects the two one-ended paths in the extended loop. (Actually, finite sources at infinity could be allowed but this requires a more complicated construction [57].) Furthermore, one more condition must be imposed in order to satisfy (9.1): The sum of the resistances in the extended loop must be finite; if this is not so, the corresponding extended-loop current will not be allowed as a basic current.

In short, the basic currents are the loop currents for all the finite loops and the extended-loop currents for some selected extended loops. Upon adding a finite number of such basic currents, we obtain a current regime in the infinite network. The set of all current regimes that are constructed in this way as finite superpositions of basic currents will be denoted by K^0.

Finally, we can think of a current regime that is not in K^0, but whose power dissipations in the resistors r_j are arbitrarily close to those of some member of K^0. By "arbitrarily close" we mean that, given any $\epsilon > 0$ and the current regime $(i_1, i_2, i_3, \cdots)$ not in K^0, we can always find a current regime $(i_1', i_2', i_3', \cdots)$ in K^0 such that $\Sigma(i_j - i_j')^2 r_j < \epsilon$. Upon appending to K^0 all such current regimes, we obtain a larger set K, called the completion of K^0. All this is analogous to thinking of an irrational number as being arbitrarily close to a rational number and to the completion of the set of rational numbers to obtain the real line.

Here is what we have been preparing for.

Theorem: Under the condition (9.2), there is one and only one member $(i_1, i_2, i_3, \cdots)$ of K such that

$$\sum (e_j - r_j i_j) x_j = 0 \tag{9.3}$$

for every member $(x_1, x_2, x_3, \cdots)$ of K.

Although this theorem is brief, quite a lot is being said. For one thing, (9.3) is a generalization of Tellegen's theorem for finite networks. From it we get Kirchhoff's voltage law around every finite loop and the selected extended loops. Moreover, since the current regime $(i_1, i_2, i_3, \cdots)$ is a member of K, it follows from the way K was constructed that Kirchhoff's current law is satisfied not only at all the finite nodes but also at those infinite nodes whose incident conductances have a finite sum. Another consequence of the theorem is that $\Sigma i_j^2 r_j \leqslant \Sigma e_j^2 r_j^{-1}$, a bound alluded to previously. Still another one is the reciprocity theorem for infinite networks. All these results are easily derived from the theorem.

We could go on describing other results concerning finite-power regimes of infinite networks, but we have at this point introduced the general approach and the flavor of the subject. Let us therefore end this tutorial with a brief literature survey.

X. A Parade of Papers

This short survey is restricted to investigations of finite-power regimes in infinite resistive networks, for they comprise the bulk of the research on infinite electrical networks (if we choose to ignore the massive literature on waves in infinite electromagnetic lumped transmission lines and lattices). Most of the works in this area tacitly assume a unique voltage-current regime and do not explicitly state what assumptions are being imposed to achieve it. Finite power dissipation and open circuits at infinity are the usual unstated conditions.

Our literature citations are not all-inclusive, nor, on the other hand, are they just a cursory selection of some typical papers. They may however serve as a guide to various aspects of infinite-electrical-network theory. Moreover, the bibliographies of the works cited herein provide references to prior papers. Also, the reader can find papers appearing later on by searching the Science Citation Index under our references. Probably all significant works on infinite resistive networks can be traced in this fashion. The survey paper [45], which was written over a dozen years ago, contains 41 references.

Infinite resistive ladder networks can be viewed as discretizations of Laplace's equation in all of one-dimensional space. The same discretization in all of n-dimensional space leads to infinite n-dimensional resistive grids. Quite a lot has been determined about their electrical behavior; see [10], [12], [16]–[19], [22], [24], [25], [28], [34], [38], [40], [52], [62]. Infinite cylindrical and spherical grids are discussed in [58], [60], [63]. A few results on infinite hexagonal grids appear in [24], and [40]. If a ground node is introduced and each node of the grid is connected to it through a resistive branch, a grounded grid is the result; some recent works on infinite grounded grids appear in [50], [51], [56].

In 1971 Flanders [21] produced the first rigorous analysis of infinite resistive networks whose graphs need not have regular patterns. However, he did assume only a finite number of sources and only finite nodes. These restrictions were removed in [41]. Other extensions of the theory soon followed. Infinite networks whose branch resistances are positive operators, rather than scalars, appeared first for infinite ladder networks [42] and then for arbitrary networks [43]. The idea of short circuits at infinity was introduced in [44], and a theory for finite sources at infinity has been established quite recently [57].

Actually, not just infinite but also finite networks whose elements are operators on infinite-dimensional spaces have significance for infinite network theory because the operators may represent infinite subnetworks of scalar elements. There is a substantial literature on operator networks, a sampling of which is [1]–[9], [11], [20], [26], [27], [29], [31], [42], [43], [47], [50], [51], [56], [62].

A powerful alternative approach to infinite electrical networks was devised by Dolezal. Most of the results of his many papers on this subject appear in his two books [13], [14]. In Dolezal's approach the graph of the infinite network is represented by an incidence matrix, which, since the network is infinite, is in fact a Hilbert-space operator. This allows him to introduce operator–theoretic techniques. In contrast to all of the above papers, which discuss linear networks, his theory encompasses networks whose elements are nonlinear and even multivalued operators. Since graph-theoretic techniques are abandoned in favor of operator methods, his results are often difficult to visualize. Moreover, to understand them, one must climb steeply through functional analysis, but, be assured, there's gold in the Dolezal Hills.

Nonlinearities are also discussed in the more restricted context of infinite cascades of three-terminal and two-port networks [53]–[55]. Conditions are established under which those cascades have unique driving-point immittances, and the constructive proofs employed provide a means of computing those immittances. (The paper [49], which also encompasses nonlinearities, deals with infinite-power regimes in general.)

As was indicated in Section III, the theory of infinite grids has an important practical application to the exterior problem for various partial differential equations. When the domain of the equation is of infinite extent, a finite-difference approximation often leads to an infinite-electrical-network representation. The customary procedure is to truncate the infinite domain and to introduce artificial boundary conditions along a finite boundary. This in effect replaces the infinite network by a finite one and produces an error of truncation. However, if a solution to the infinite network can be computed, there is no need to truncate. Such is the case for certain infinite grids. As a result, new computational methods have been devised for certain exterior problems, which require less computation times and smaller memory storages and are of value in computer-aided simulation and design [36], [56], [59]–[63].

Finally, we should point out that infinite electrical networks arise surprisingly enough in probability theory, more specifically, in the theory of random walks on infinite graphs. A tutorial exposition of the subject is given in [15], and a more advanced treatment is provided in [33].

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