Lemma 3.8.2 *Let* $0 \le k \le m$ *and* $c = \binom{2m}{k} \Big/ \binom{2m}{m}$. *Then*

$$\binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{k-1} < \frac{c}{2} \cdot 2^{2m}. \tag{3.12}$$

To digest the meaning of this, choose $m = 500$, and let's try to see how many binomial coefficients in the 1000th row we have to add up (starting with $\binom{1000}{0}$) to reach 0.5% of the total. Lemma 3.8.2 tells us that if we choose $0 \le k \le 500$ so that $\binom{1000}{k} \Big/ \binom{1000}{500} < 1/100$, then adding up the first k binomial coefficients gives a sum less than 0.5% of the total. In turn, Lemma 3.8.1 tells us a k that will be certainly good: any $k \le 500 - \sqrt{500 \ln 100} - \ln 100 = 447.4$. So the first 447 entries in the 1000th row of Pascal's Triangle make up less than 0.5% of the total sum. By the symmetry of Pascal's Triangle, the last 447 add up to another less than 0.5%. The middle 107 terms account for 99% of the total.

Proof. To prove this lemma, let us write $k = m - t$, and compare the sum on the left-hand side of (3.12) with the sum

$$\binom{2m}{m-t} + \binom{2m}{m-t+1} + \cdots + \binom{2m}{m-1}. \tag{3.13}$$

Let us denote the sum $\binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{m-t-1}$ by A, and the sum $\binom{2m}{m-t} + \binom{2m}{m-t+1} + \cdots + \binom{2m}{m-1}$ by B.

We have

$$\binom{2m}{m-t} = c\binom{2m}{m}$$

by the definition of c. This implies that

$$\binom{2m}{m-t-1} < c\binom{2m}{m-1},$$

since we know that binomial coefficients drop by a larger factor from $\binom{2m}{m-t}$ to $\binom{2m}{m-t-1}$ than they do from $\binom{2m}{m}$ to $\binom{2m}{m-1}$. Repeating the same argument,[1] we get that

$$\binom{2m}{m-t-i} < c\binom{2m}{m-i}$$

for every $i \ge 0$.

Hence it follows that the sum of any t consecutive binomial coefficients is less than c times the sum of the next t (as long as these are all on the left hand side of Pascal's Triangle). Going back from $\binom{2m}{m-1}$, the first block of t binomial coefficients adds up to A (by the definition of A); the next block

[1] In other words, using induction.

of t adds up to less than cA, the next block to less than c^2A, etc. Adding up, we get that

$$B < cA + c^2A + c^3A \dots$$

On the right-hand side we only have to sum $\lceil (m-t)/t \rceil$ terms, but we are generous and let the summation run to infinity! The geometric series on the right-hand side adds up to $\frac{c}{1-c}A$, so we get that

$$B < \frac{c}{1-c}A.$$

We need another inequality involving A and B, but this is easy:

$$B + A < \frac{1}{2}2^{2m}$$

(since the sum on the left-hand side includes only the left-hand side of Pascal's Triangle, and the middle element is not even counted). From these two inequalities we get

$$B < \frac{c}{1-c}A < \frac{c}{1-c}\left(\frac{1}{2}2^{2m} - B\right),$$

and hence

$$\left(1 + \frac{c}{1-c}\right)B < \frac{c}{1-c}\frac{1}{2}2^{2m}.$$

Multiplying by $1-c$ gives that $B < c\frac{1}{2}2^{2m}$, which proves the lemma. □

3.8.1 (a) Check that the approximation in (3.8) is always between the lower and upper bounds given in (3.9).

(b) Let $2m = 100$ and $t = 10$. By what percentage is the upper bound in (3.9) larger than the lower bound?

3.8.2 Prove the upper bound in (3.9).

3.8.3 Complete the proof of Lemma 3.8.1.

Review Exercises

3.8.4 Find all values of n and k for which $\binom{n}{k+1} = 3\binom{n}{k}$.

3.8.5 Find the value of k for which $k\binom{99}{k}$ is largest.