Section 3.2: Spectra of Graphs Derived by Operations and Transformations

In this section we extend the results given in Chapter 2 of [CvDSA1] where some procedures for determining the spectra and/or characteristic polynomials of graphs derived from some simpler graphs by graph operations or transformations are described.

We start by describing several results which have been united by the concept of a *rooted product* given by C. D. Godsil and B. D. McKay [GoMK2] (see Theorem 3.8). Further results include formulas for the characteristic polynomial of a subdivision graph and for the generalized line graph.

We denote by P_k, as usual, the path with k vertices, and we denote by e_j the edge joining v_i and v_{i+1}. An edge e of a graph G is called a *bridge* if $G - e$ has more connected components than G.

DEFINITION 3.1: A graph Q_k belongs to the class $\mathcal{Q}_K$ if and only if it contains P_k as a subgraph with the edges e_j being bridges for $j = 1, \ldots k - 1$.

Hence every graph with at least one vertex belongs to $\mathcal{Q}_1$, every graph with a bridge belongs to $\mathcal{Q}_2$ and, in general, a graph from $\mathcal{Q}_k$ will have the structure of a graph as indicated in Figure 3.1.

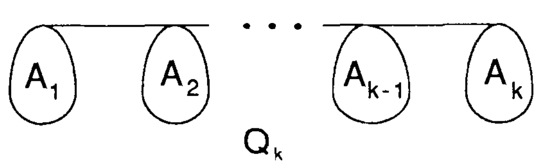

Figure 3.1 A graph in $\mathcal{Q}_k$

The graphs $A_1, A_2, \ldots A_k$ are arbitrary rooted, mutually disjoint graphs. Hence the graph Q_k can be thought of as having fragments $A_1, \ldots, A_k$ with the roots joined by a path. The path P_k itself belongs to $\mathcal{Q}_k$. The subgraph obtained by deleting v_j from A_j will be denoted B_j.

DEFINITION 3.2: A graph R_k belongs to the class $\mathcal{R}_k$ if and only if $R_k \in \mathcal{Q}_k$, $A_2 = A_3 = \cdots = A_{k-1}$, and $B_2 = B_3 = \cdots = B_{k-1}$. A graph R_k^* belongs to the class $\mathcal{R}_k^*$ if and only if $R_k^* \in \mathcal{R}_k$, $A_k = A_{k-1}$, and

$B_k = B_{k-1}$. A graph S_k belongs to the class $\mathcal{S}_k$ if and only if $S_k \in \mathcal{R}_k^*$, $A_1 = A_2$, and $B_1 = B_2$.

It is clear that $\mathcal{S}_k \subseteq \mathcal{R}_k^* \subseteq \mathcal{R}_k \subseteq \mathcal{Q}_k$. In Figure 3.2 the structure of R_k, R_k^*, and S_k are displayed.

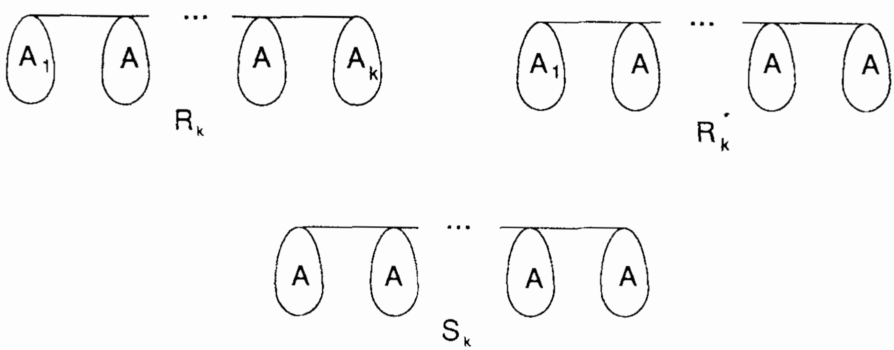

Figure 3.2 Graphs in $\mathcal{R}_k$, $\mathcal{R}_k^*$, and $\mathcal{S}_k$

Since e_k is a bridge, Q_k satisfies the following equality (see, for example, [CvDSa1], p. 59):

$$\Phi(Q_k) = \Phi(A_k)\Phi(Q_{k-1}) - \Phi(B_{k-1})\Phi(B_k)\Phi(Q_{k-2})$$

where $\Phi(G)$ is the characteristic polynomial of G.

This equation can be written in matrix form as

$$\begin{pmatrix} \Phi(Q_k) \\ \Phi(B_k)\Phi(Q_{k-1}) \end{pmatrix} = \begin{pmatrix} \Phi(A_k) & -\Phi(B_k) \\ \Phi(B_k) & 0 \end{pmatrix} \begin{pmatrix} \Phi(Q_{k-1}) \\ \Phi(B_{k-1})\Phi(Q_{k-2}) \end{pmatrix} \tag{3.2}$$

A repeated application of (3.2) yields

THEOREM 3.6 (I. GUTMAN [GUT11]): *Using definitions 3.1 and 3.2,*

$$\begin{pmatrix} \Phi(Q_k) \\ \Phi(B_k)\Phi(Q_{k-1}) \end{pmatrix} = T_k T_{k-1} \cdots T_2 T_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.3}$$

where

$$T_j = \begin{pmatrix} \Phi(A_j) & -\Phi(B_j) \\ \Phi(B_j) & 0 \end{pmatrix}, \quad j = 1, 2, \ldots k.$$

COROLLARY 3.6.1 (I. GUTMAN [GUT11]): *The characteristic polynomial of Q_k is completely determined by the characteristic polynomials of A_j and B_j, $j = 1, \ldots, k$.*

This implies the existence of many cospectral pairs of graphs. The two graphs in Figure 3.3, for example, are both in $\mathcal{S}_2$.

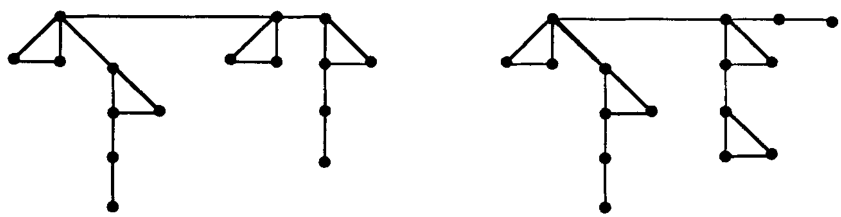

Figure 3.3 Cospectral graphs in $\mathcal{S}_2$

COROLLARY 3.6.2 (I. GUTMAN [GUT11]): *Using the definitions 3.1 and3.2,*

$$\begin{pmatrix} \Phi(R_k) \\ \Phi(B_k)\Phi(R^*_{k-1}) \end{pmatrix} = T_k T^{k-2} T_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \Phi(S_k) \\ \Phi(B_k)\Phi(S_{k-1}) \end{pmatrix} = T^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where

$$T = \begin{pmatrix} \Phi(A) & -\Phi(B) \\ \Phi(B) & 0 \end{pmatrix}.$$

COROLLARY 3.6.3 (I. GUTMAN [GUT11]):

$$\begin{pmatrix} \Phi(P_k) \\ \Phi(P_{k-1}) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Results of the form (3.3) can also be found in [KACH1]. Similar results are also contained in [GUT33].

Let G_x denote a graph with rooted vertex x. A composition $G_u \circ H_v$ has been considered in [GOMK1]. The graph $G_u \circ H_v$ consists of disjoint copies of $G - u$ and $H - v$ with additional edges joining each vertex adjacent to u in G to every vertex adjacent to v in H. The the following formula holds:

$$P_{G_u \circ H_v}(\lambda) = P_{G-u}(\lambda)P_{H-v}(\lambda) - (\lambda P_{G-u}(\lambda) - P_G(\lambda))(\lambda P_{H-v}(\lambda) - P_H(\lambda))$$

The spectrum of the corona of two graphs is determined in [CvGS1]. Let G be a graph on n vertices, and let H be a regular graph on m vertices with degree r. The characteristic polynomial of the corona of G and H, $G * H$, is the determinant of the matrix

$$\begin{pmatrix} \lambda I - A & -J_1 & -J_2 & \cdots & -J_n \\ -J_1^T & \lambda I - B & \cdots & & \\ -J_2^T & & \lambda I - B & \cdots & \\ \vdots & & & \ddots & \\ -J_n^T & & & & \lambda I - B \end{pmatrix}$$

where A and B are the adjacency matrices of G and H respectively and J_k is an $n \times m$ matrix with every entry in the k-th row equal to one and all other entries equal to zero. By elementary transformations this matrix can be changed to

$$\begin{pmatrix} (\lambda - \frac{m}{\lambda - r})I - A & 0 & 0 & \cdots & 0 \\ -J_1^T & \lambda I - B & & & \\ -J_2^T & & \lambda I - B & & \\ \vdots & & & \ddots & \\ -J_n^T & & & & \lambda I - B \end{pmatrix}$$

and so

$$P_{G*H} = P_G(\lambda - \frac{m}{\lambda - r})(P_H(\lambda))^n.$$

A special case of this result appears in [CvDSa1], p. 60. These results have been further generalized by C. Godsil and B. D. McKay. Given a graph H with n vertices and a family $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ of rooted graphs, they define the *rooted product* $H(\mathcal{G})$ to be the graph obtained by identifying, for $k = 1, 2, \ldots, n$, the root of G_k with the k-th vertex of H.

THEOREM 3.7 (C. GODSIL, B. D. MCKAY [GOMK2]): *If* (a_{ij}) *is the adjacency matrix of* H, *and* $A_\lambda(H, \mathcal{G}) = (\alpha_{ij})$ *is defined by* $\alpha_{ii} = P_{G_i}(\lambda)$ *and* $\alpha_{ij} = -a_{ij}P'_{G_i}(\lambda)$ *for* $i \neq j$. *Then*

$$P_{H(\mathcal{G})}(\lambda) = \det A_\lambda(H, \mathcal{G}).$$

Next we consider the following recursive definition of a class of trees which stems from [GUT28].

Let R be a rooted graph, *i.e.*, a graph with a particular vertex labelled by v_0. Let $d = (d_1, d_2, \ldots, d_m)$ be an m-tuple of positive integers. K_1 will denote the graph consisting of a single vertex. Then we define a graph $G_m = G_m(R, d)$ recursively in the following manner.

DEFINITION 3.3: (i) $G_0 = R$, and (ii) for $k = 0, 1, \ldots, m-1$, the graph G_{k+1} is obtained by taking d_{k+1} copies of G_k and joining each v_k to a new vertex which is labeled v_{k+1}.

Hence if G_k possesses n_k vertices then G_{k+1} possesses $n_{k+1} = d_{k+1}n_k + 1$ vertices. A recursive formula for the characteristic polynomial of G_m is derived in [GUT28]. Some special cases are treated there in further detail.

The spectrum of *complete k-ary tree* is obtained in [ROU1] by the use of a recursive formula along similar lines.

The following theorem is a graph-theoretic reformulation of a well known matrix-theoretic result (Jacobi's formula). It will be used in Chapter 5.

THEOREM 3.8: *Let* u *and* v *be vertices of a graph* G. *Let* $\mathcal{P}_{uv}$ *be the set of all paths which connect* u *and* v. *Then*

$$P_{G-u}(\lambda)P_{G-v}(\lambda) - P_G(\lambda)P_{G-u-v}(\lambda) = \Big(\sum_{T \in \mathcal{P}_{uv}} P_{G-T}(\lambda)\Big)^2.$$

See also [TUT2] from [CvDSA1].

We now turn to the subdivision graph and other related graph operations. Let us subdivide each edge of a graph by adding k new vertices. The resulting graph is called the *k-th subdivision graph* of the original graph. Let $S_k(G)$ be the k-th subdivision graph of a graph G and $L_{k+1}(G)$ be $L(S_k(G))$, where $L(H)$ denotes the line graph of a graph H. Further, Let $R_k(G)$ be the graph obtained from G by adding for each edge u of G a copy of K_k and by joining each endpoint of u with each vertex of the corresponding K_k.

THEOREM 3.9 (V. B. MNUHIN [MNU1]): *We have*

$$P_{S_k(G)}(\lambda) = (U_k(\frac{\lambda}{2}))^{m-n} \det(\lambda U_k(\frac{\lambda}{2})I - U_{k-1}(\frac{\lambda}{2})D - A)$$

where A is the adjacency matrix of G, D is the (diagonal) degree matrix of G, and $U_k(x)$ is the Chebyshev polynomial of the second kind.

Similar formulas are obtained for $L_{k+1}(G)$ and $R_k(G)$, thus generalizing some results of D. Cvetković (cf. [CVDSA1], pp. 63–64). Results of D. Cvetković are special cases of $k = 1$. Those for $S_1(G)$ also appear in [SHI1], [SHI2]. All these results are extended to digraphs in [MNU2].

The definition of a generalized line graph $L(G; a_1, a_2, \ldots, a_n)$ of a graph G on n vertices (where $a_1, a_2, \ldots, a_n$ are nonnegative integers) is given in Chapter 1. *Generalized line graphs* were introduced by A. J. Hoffman, and they play an important role in spectral graph theory (see Chapter 1). It has been proved that the least eigenvalue of a generalized line graph is bounded from below by -2 just as it is for line graphs. Here we present a result giving, in some cases, the whole spectrum of a generalized line graph.

THEOREM 3.10: *Let G be a graph having vertex degrees $d_1, d_2, \ldots, d_n$. If $a_1, a_2, \ldots, a_n$ are nonnegative integers such that $d_i + 2a_i = d$, $i = 1, 2, \ldots, n$, then*

$$P_{L(G;a_1,a_2,\ldots,a_n)}(\lambda) = \lambda^{\sum_{i=1}^{n} a_i} (\lambda+2)^{m-n+\sum_{i=1}^{n} a_i} P_G(\lambda - d + 2).$$

PROOF: Consider the matrix

$$\begin{pmatrix} R & L_1 & L_2 & \cdots & L_n \\ 0 & M_1 & 0 & \cdots & 0 \\ 0 & 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_n \end{pmatrix}$$

where L_i, $i = 1, \ldots, n$ is an n by $2a_i$ matrix in which i-th row has all entries equal to 1, all other entries being equal to 0, and M_i, ($i = 1, \ldots, n$), is an a_i by $2a_i$ matrix of the form $(\, I_{a_i} \quad -I_{a_i}\,)$, I_m being a unit matrix of order m. The theorem follows from *Lemma 2.1* from [CVDSA1] and the fact that $S^T S = B + 2I$ where B is the adjacency matrix of $L(G; a_n, \ldots, a_n)$.

We continue with miscellaneous results.

THEOREM 3.11 (M. BOROWIECKI, T. JÓŹWIAK [BoJó1]): *Let G be a multidigraph, $V(G)$ its vertex set, and let x be one of its vertices.*

Let $\mathcal{C}(x)$ be the set of all cycles of G containing x. Then

$$P_G(\lambda) = \lambda P_{G-x}(\lambda) - \sum_{C \in \mathcal{C}(x)} P_{G-V(C)}(\lambda).$$

This formula extends a previous result of A. J. Schwenk (see [CvDSa1], p. 78) to multidigraphs. The authors proceed in [BoJó1] in the same spirit and give generalized versions of the known reduction formula when one deletes an edge (arc) from a multidigraph as well as a formula for the coalescence of rooted multidigraphs (cf. [CvDSa1], p. 159). The proof is carried out easily by the use of the Sachs theorem.

These results have also been presented in [Józ1].

Similar technical generalizations of the Schwenk formula are given in [GiAc2] for the characteristic polynomial of sigraphs, defined in Section 3.1, and in [GiL2] for the characteristic polynomial of an arbitrary matrix. See also [Wat1], where a reformulation of the Sachs theorem has been used to generalize to multigraphs two results of A. J. Schwenk ([Schw3] from [CvDSa1]).

A disadvantage of formulas like the one in Theorem 3.11 is that one needs to construct the set of cycles $\mathcal{C}(x)$. This is avoided in the recent formula by P. Rowlinson [Row5]. Suppose that G is a multigraph with m edges connecting vertices u and v, $G - uv$ is obtained from G by deleting the edges between u and v, and G^* is constructed from $G - uv$ by identifying the vertices u and v (with vertices adjacent to both u and v producing multiple edges). We then have

$$\begin{aligned} P_G(\lambda) = P_{G-uv}(\lambda) + P_{G^*}(\lambda) + m(\lambda - m)P_{G-u-v}(\lambda) \\ - mP_{G-u}(\lambda) - mP_{G-v}(\lambda). \end{aligned}$$

Spectra or characteristic polynomials for graphs obtained by different compositions of graphs are given in papers [GoMK1], [Schw2], and [ScHE1]. Since the main goal of these papers is the construction of cospectral graphs, they are described in Section 1.4.

A *path polynomial* of a graph with respect to an initial and a final vertex has been introduced in [SiSr1]. The path polynomial is used to compute characteristic polynomials of some particular graphs. Results are more or less in terms of previously known facts.

Let $G_{k,m}$ be a graph obtained by attaching the end points of the paths P_k and P_m at a vertex of a nontrivial connected graph G. It is proved in [LIFE1] that for $1 \leq m \leq k$, the largest eigenvalue of $G_{k,m}$ is greater than the largest eigenvalue of $G_{k+1,m-1}$. As a corollary, the largest eigenvalue of P_n is less than the largest eigenvalue of any other connected graph on n vertices. It also follows that among unicyclic graphs obtained by attaching a tree on n vertices to a vertex of a circuit C_m, the index is smallest when the tree attached is a path. However, the authors' conjecture that the last graph has the smallest index among all unicyclic graphs with the same number of vertices has been disproved in [CVE6] using an counter example constructed by a computer.

Graph transformations arising from the application of formal grammars to graphs have been studied in [MIC1]. The characteristic polynomials of the resulting graphs have been expressed in terms of algebraic operations on the polynomials of the initial graphs and their subgraphs.

The definition of a very general n-ary graph operation, called NEPS (noncomplete extended p-sum of graphs), including one of its special cases, the product of graphs, is reproduced in the next section. Here we note that the same operation can be defined for digraphs (cf. [CVPE2], where eigenvalues and the strong connectedness of the digraph obtained as a NEPS of other digraphs have been studied, [ESHA2], and a review of that paper in **Mathematical Reviews** (MR 81m: 05096) for some bibliographical data).

The product $G \times K_2$ of a graph G and the graph K_2 is called the *bipartite square* $G \circ G$ of G in [CVDSA1]. It is noted in [POR1] that $P_{G \circ G}(\lambda) = (-1)^n P_G(\lambda) P_G(-\lambda)$. If G_1 and G_2 are nonisomorphic cospectral graphs then $G_1 \circ G_1$ and $G_2 \circ G_2$ are also nonisomorphic and cospectral, $G \circ G$ is always a bipartite graph, and it is disconnected if G is bipartite. These results appeared in [CVDSA1], p. 70.

Section 3.3: Constructions of Graphs Using Spectra

The idea of using graph spectra for the construction of graphs with given properties has been outlined in [CVDSA1], p. 190, with some examples being given there. Suppose that properties of the desired graph determine the spectrum or a spectral property of the graph. Now the problem is to construct a graph with the given spectrum or spectral property. One could start with graphs with known spectra and perform graph operations on them in order to obtain a new graph with the desired spectrum. Examples of such constructions of strongly regular graphs have been given in [CvE4] and some of them are generalized in [CvE2].

We shall reproduce these constructions here. First we follow [CvE4].

There is an n-ary composition on graphs which is called NEPS (noncomplete extended p-sum of graphs).

DEFINITION 3.4: Let B be a set of n-tuples $(\beta_1, \dots, \beta_n)$ of symbols 0 and 1 which does not contain the n-tuple $(0, \dots, 0)$. The *NEPS* with basis B of the graphs $G_1, \dots, G_n$ is the graph, whose vertex set is equal to the Cartesian product of the vertex sets of graphs $G_1, \dots, G_n$ in which two vertices $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are adjacent if and only if there is an n-tuple $(\beta_1, \dots, \beta_n)$ in B such that $x_i = y_i$ holds when $\beta_i = 0$ and x_i is adjacent to y_i in G_i when $\beta_i = 1$.

If B contains all n-tuples having exactly p coordinates equal to 1, then the NEPS is called the *p-sum*.

The spectrum of a NEPS can be determined by the spectra of the original factor graphs (see [CvDSa1], p. 69).

Let $\lambda_{i1}, \dots, \lambda_{in_i}$ be the spectrum of the graph G_i, $i = 1, \dots, n$. Let $\beta = (\beta_1, \dots, \beta_n)$. Then the spectrum of the NEPS with basis B of the graphs $G_1, \dots, G_n$ consists of all possible values of $\Lambda_{i_1,\dots,i_n}$, where

$$\Lambda_{i_1,\dots,i_n} = \sum_{\beta \in B} \lambda_{1i_1}^{\beta_1} \lambda_{2i_2}^{\beta_2} \dots \lambda_{ni_n}^{\beta_n}, \quad i_k = 1, 2, \dots n_k, \; k = 1, 2, \dots n.$$

In particular, the spectrum of the p-sum of graphs $G_1, \dots, G_n$ consists of all the values of the elementary symmetric function of order p in variables $x_1, \dots, x_n$, where the variable x_i runs through the eigenvalues of G_i, $i = 1, \dots, n$.

We shall now construct some regular connected graphs with 3 distinct eigenvalues. Such graphs are known to be strongly regular (see, for example, [CvDSa1], p. 103). The difficulties in our constructions arise from the fact that the NEPS generally contains many more distinct eigenvalues than do the starting graphs, and that only in exceptional cases do some eigenvalues become equal. We start with some examples.

EXAMPLE 3.1 Since the spectrum of K_n consists of the simple eigenvalue $n-1$ as well as $n-1$ eigenvalues equal to -1 we can readily check the following statements.

- The 1-sum of two complete graphs K_n yields the graph $L(K_{n,n})$ which is strongly regular with distinct eigenvalues $2n-2$, $n-2$, and -2.
- The 2-sum of three copies of the graph K_4 results in a strongly regular graph on 64 vertices with the distinct eigenvalues 27, 3, -5.
- The 2-sum of four copies of K_3 gives a graph on 81 vertices with distinct eigenvalues 24, 6, -3.
- The 3-sum of four copies of K_3 gives a strongly regular graph on 81 vertices having distinct eigenvalues 32, 5, -4.
- The 4-sum of five copies of K_2 yields a disconnected graph with two components, each being the complement of the Clebsch graph.

We proceed to more general constructions. The next few constructions have been given according to [CVE2].

DEFINITION 3.5: The *odd (even) sum* of graphs is the NEPS with the basis containing all the n-tuples with an odd (even) number of 1's.

DEFINITION 3.6: The *mixed sum* of graphs is the NEPS with the basis containing all the n-tuples in which the number of 1's is congruent to 1 or 2 modulo 4.

Note that the odd, the even, and the mixed sum of two graphs is called the sum, the product, and the strong product, respectively.

In Theorems 3.12 and 3.14 we construct two more infinite series of strongly regular graphs by means of the NEPS.

THEOREM 3.12 (D. CVETKOVIĆ [CVE2]): *For all $n \geq 2$ the odd sum F_n of n copies of the graph K_4 is a strongly regular graph with the eigenvalues $2^{2n-1}+(-1)^{n-1}2^{n-1}, 2^{n-1}, -2^{n-1}$.*

PROOF: The distinct eigenvalues of K_4 are 3 and -1. Let S_p be the elementary symmetric function of order p on the variables $x_1, \ldots, x_n$, and let these variables take the values 3 or -1. If k variables take the value 3 and if the remaining $n-k$ ones take value -1, the value of S_i is equal to the coefficient of x^{n-i} in the polynomial $P_k(x) = (x-3)^k(x+1)^{n-k}$. The eigenvalues of the odd sum are given by $\sum_i S_i$

where the summation goes over all odd numbers i not greater than n. If $P_k(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then we have

$$\Lambda_k = -(a_1 + a_3 + \cdots) = \frac{1}{2}((-1)^n P_k(-1) - P_k(1)), \quad k = 0, 1, \ldots n,$$

Λ_k are the eigenvalues of F_n, and we immediately get $\Lambda_k = (-1)^{k+1}2^{n-1}$, $k = 0, 1, \ldots, n-1$, and $\Lambda_n = 2^{2n-1} + (-1)^{n-1}2^{n-1}$, which proves the theorem.

F_n is a regular graph of degree $2^{2n-1} + (-1)^{n-1}2^{n-1}$ on 4^n vertices. Any two distinct vertices have $2^{2n-2} + (-1)^{n-1}2^{n-1}$ common neighbours. F_n can be visualized in the following way. The 4^n n-tuples of 4 distinct symbols are the vertices and two n-tuples are adjacent if they differ in an odd number of coordinates.

A result on spectral characterizations of block designs (cf. [CvDSa1], p. 167) can be reformulated in the following way.

THEOREM 3.13: *A symmetric BIBD with the parameters (v, k, λ) exists if and only if there exists a graph with the spectrum consisting of eigenvalues k, $(k-\lambda)^{1/2}$, $-(k-\lambda)^{1/2}$, $-k$ with the multiplicities $1, v-1, v-1, 1$, respectively.*

The even sum of graphs F_n and K_2 is a regular bipartite graph with four distinct eigenvalues $\pm(2^{2n-1} + (-1)^{n-1}2^{n-1})$, $\pm 2^{n-1}$. Hence, by Theorem 3.13 we have constructed a family of symmetric BIBDs with the parameters $v = b = 4^n$, $r = k = 2^{2n-1} + (-1)^{n-1}2^{n-1}$, $\lambda = 2^{2n-2} + (-1)^{n-1}2^{n-1}$.

The graphs F_n and the corresponding block designs have been constructed in the literature in many different ways; the construction given above is a spectral one.

THEOREM 3.14 (D. CVETKOVIĆ [CvE2]): *For $s \geq 1$, the mixed sum H_s of $4s$ copies of the graph K_2 is a strongly regular graph with eigenvalues $2^{4s-1} - (-1)^s 2^{2s-1}$, 2^{2s-1}, and -2^{2s-1}.*

PROOF: The eigenvalues of K_2 are 1 and -1. In order to obtain the eigenvalues of the mixed sum of $n = 4s$ copies of K_2 consider the polynomial

$$Q(x) = (x-1)^k(x+1)^{n-k} = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n. \quad (3.4)$$

Define α_0, α_1, α_2, α_3 by the equations

$$\begin{aligned}
\alpha_0 &= b_0 + b_4 + b_8 + \cdots + b_n, \\
\alpha_1 &= b_1 + b_5 + b_9 + \cdots + b_{n-1}, \\
\alpha_2 &= b_2 + b_6 + b_{10} + \cdots + b_{n-2}, \\
\alpha_3 &= b_3 + b_7 + b_{11} + \cdots + b_{n-3}.
\end{aligned}$$

The values of $-\alpha_1 + \alpha_2$ for $k = 0, 1, \ldots, n$ represent the eigenvalues of H_s. First we have

$$\begin{aligned}
Q_k(1) &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, \\
Q_k(i) &= \alpha_0 - i\alpha_1 - \alpha_2 + i\alpha_3, \\
Q_k(-1) &= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3, \\
Q_k(-i) &= \alpha_0 + i\alpha_1 - \alpha_2 - i\alpha_3.
\end{aligned}$$

Further we obtain

$$-\alpha_1 + \alpha_2 = \frac{1}{4}(2Q_k(-1) - (i+1)Q_k(i) - (1-i)Q_k(-i)) = \Lambda_k \quad (3.5)$$

Using (3.4) and (3.5) we get $\Lambda_k = (-1)^{s-1+\binom{k}{2}} 2^{2s-1}$ for $k = 0, 1, \ldots 4s - 1$ and $\lambda_{4s} = 2^{4s-1} - (-1)^s 2^{2s-1}$, which proves the theorem.

The *Seidel spectrum* of H_s contains the eigenvalues $2^{2s} - 1$ and $-2^{2s} - 1$, and again we have a regular two-graph with the same eigenvalues as with F_{2s} of Theorem 3.12.

We now mention constructions of another type following [CvE2].

Distance-regular graphs are defined in Chapter 2.

If a graph G is distance-regular with adjacency matrix A, we can find a linear combination of matrices A^2, A and I which represents the adjacency matrix of graph G^2 defined on the vertex set of G, with two vertices being adjacent in G^2 if they are at distance 2 in G.

If G is the cubic lattice graph (see [CvDSa1], p. 178), with characteristic 4, the graph G^2 coincides with the graph of the second type in Example 3.1. If we take for G the exceptional graph in the characterization of the cubic lattice graph of characteristic 4, then G^2 represents a graph with the same spectrum as above.

Starting with G equal to the tetrahedral graph (see [CvDSa1],p. 180) with characteristic 10, we get for G^2 a strongly regular graph on 120 vertices with eigenvalues 56, 8, -4 which is related to the exceptional root system E_8 (see Chapter 1).

$L(K_n)$ is a strongly regular graph. In particular, the complement of $L(K_5)$ is the Petersen graph. The 2-sum of the Petersen graph and the graph K_2 yields Desargues graph. Tutte's 8-cage can be constructed as a certain square root of a graph. Namely, if G is the 8-cage, then G^2 is isomorphic to the complement of $2L(K_6)$.

If G is the Hoffman-Singleton graph, then $(L(G))^2$ is a strongly regular graph on 175 vertices having the eigenvalues 72, 2, and -18.

Spectral constructions of graphs have been used by H. Sachs and M. Stiebitz [SaSt2] to obtain transitive graphs which satisfy an upper bound for the number of simple eigenvalues (see Section 3.4). It is easy to see that the NEPS of transitive graphs is again a transitive graph. The construction starts again with complete graphs which are transitive and whose spectrum is known.

Let $\tau(G)$ be the number of simple eigenvalues of a graph G. A graph is called an I-graph if the following holds:

1. The eigenvalues λ_i of G are integers and $\lambda_i \neq -2, 0$, $i = 1, \ldots, n$, and
2. $\lambda_i - \lambda_j \neq 2$, $i, j = 1, \ldots, n$.

Let $G_1 + G_2$ and $G_1 \cdot G_2$ be the sum and the strong product of graphs G_1 and G_2. The sum and the strong product are special cases of the NEPS. If λ_i are eigenvalues of G_1 and if μ_j are eigenvalues of G_2 then $\lambda_i + \mu_j$ are eigenvalues of $G_1 + G_2$ and $\lambda_i\mu_j + \lambda_i + \mu_j$ are eigenvalues of $G_1 \cdot G_2$ (see Section 3.6 and [CvDSa1], p. 70).

It is proved in [SaSt2] that if G is an I-graph, then for any $m \geq 3$ the graph $(G + K_2) \cdot K_m$ is also an I-graph.

Let us define $\mathcal{G}_0 = \{K_{2m+1} \mid m \geq 1\}$, and $\mathcal{G}_{k+1} = \{(G + K_2) \times K_{2m+1} \mid G \in \mathcal{G}_k, m \geq 1\}$.

Theorem 3.15 (H. Sachs, M. Stiebitz [SaSt2]): *For any $G \in \mathcal{G}_k$ we have $\tau(G) = 2^k$ and $\tau(G + K_2) = 2^{k+1}$.*

With these graphs, the upper bound for $\tau(G)$ described in Section 3.4 is attained.

See [Row2] for an extension of these constructions.

Spectral constructions of graphs are used to construct integral graphs and Gaussian digraphs (see Section 3.12). A suitable operation for such purposes is the NEPS. Since the eigenvalues of a NEPS